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Heron's Triangles, Golden Section and Quantization of Decays of Scalar, Strange Mesons and Δ, N Baryons in the Hyperbolic Lobachevsky Velocity Space

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Abstract

The ends of the velocity vectors of the decay particles of resonance represent material points-velocities in the hyperbolic Lobachevsky velocity space of negative curvature $k = -1/C^2$ ($C = 1$ is the speed of light, the rest masses of the decay particles are assigned to the points-velocities). Two points-velocities of the decay particles can be connected by a line segment and an arc of a line of constant curvature 0, called the oricycle. Archimedes' leverage laws define a 3rd point on the arc of the oricycle to which an additive mass (sum of rest masses of particles) is assigned. Connecting 3 points-velocities by line segments, we obtain isosceles triangles of decays of resonances in the Beltrami model of the Lobachevsky velocity space. In the decay triangles of resonances, the **golden section** is found and the Stewart, Brettschneider theorems on oricyclic arcs are satisfied. Near the decay triangles of scalar, strange mesons and Δ, N baryons, isosceles triangles-satellites with integer values of their characteristics were found. On the satellite triangles, the Lorentz invariant function—the product of the length of the arc of the oricycle subtending the base and the cotangent of half the angle at the vertex opposite the base—takes integer values. The function is called the oricyclic cotangent of a triangle (OCT). In addition to the integer values of OCT, these satellite triangles also have the sum of the hyperbolic cosines of the lengths of the lateral sides and the hyperbolic cosines of the base lengths equal to integers. These satellite triangles are called Heron triangles. On Heron triangles, the generalized cosines of the angles between the tangent to the oricycle at the point-velocity of the additive mass and the tangent at the point-velocity of the base of the triangle take multiples of 1/2 values.

Keywords

Lobachevsky Velocity Space, Resonance Decay Triangles, Oricyclic Cotangent Triangle, Heron's Hyperbolic Triangle, Golden Section, Stewart's Theorem, Bretschneider's Theorem, Quantization Resonance Decays

1. Introduction

In inelastic reactions at high energies, the particle velocity vectors are measured in some frame of reference. The ends of the velocity vectors represent material points-velocities in the velocity space located inside a sphere of radius C (C is the speed of light, the points-velocities are assigned rest masses of the particles) [1]-[4]. The Lorentz group defines the Lobachevsky geometry of negative curvature $k = -1/C^2$ in the velocity space [5]. The material points-velocities inside a sphere of radius C represent the Lorentz invariant geometric image of inelastic reaction kinematics in hyperbolic Lobachevsky velocity space (HLVS) (further everywhere the speed of light $C = 1$) [6] [7]. Two material points-velocities of resonance decay particles can be connected by a line segment and an arc of a line of constant zero curvature, called the oricycle [8]. Archimedes' lever laws (3) and (8) define the 3rd point-velocity on the oricycle arc, to which an additive mass is assigned (the sum of the rest masses of the decay particles) [4] [9]-[11]. Connecting 3 velocity points with straight line segments, we obtain isosceles triangles of resonance decays in the Beltrami model of the Lobachevsky velocity space (**Figure 1**) [9]-[11]. The effective mass of the resonance corresponds to the length of the base of its decay triangle (6). In these triangles, the **golden section** is discovered and the Stewart and Bretschneider theorems are fulfilled on the arcs of the oricycle (**Appendix A**).

Near the decay triangles of scalar, strange mesons and Δ, N baryons, satellite triangles with integer values of their characteristics were found (**Tables 1-3**). Namely, on these satellite triangles, the function (4), called the oricyclic cotangent of a triangle (OCT), takes integer values. The dimensionless Lorentz invariant function OCT is constructed based on the arc length of the oricycle and the opposite base angle. In addition to the integer values of OCT, for these satellite triangles, the sum of the hyperbolic cosines of the lengths of the lateral sides and the hyperbolic cosines of the lengths of the bases is also equal to integers. These satellite triangles are called Heron's triangles in HLVS [9]-[11]. On Heron's triangles, another Lorentz invariant function (5), called the orcyclic tangent of a triangle (OTT), is constant and equal to 4. Also, **Tables 1-3** give the values of the generalized cosines of the angles between the tangent to the oricycle at the point-velocity of the additive mass and the tangent at the point-velocity of the base of Heron's triangles located near the known scalar, strange mesons and Δ, N baryons (**Appendix B**).

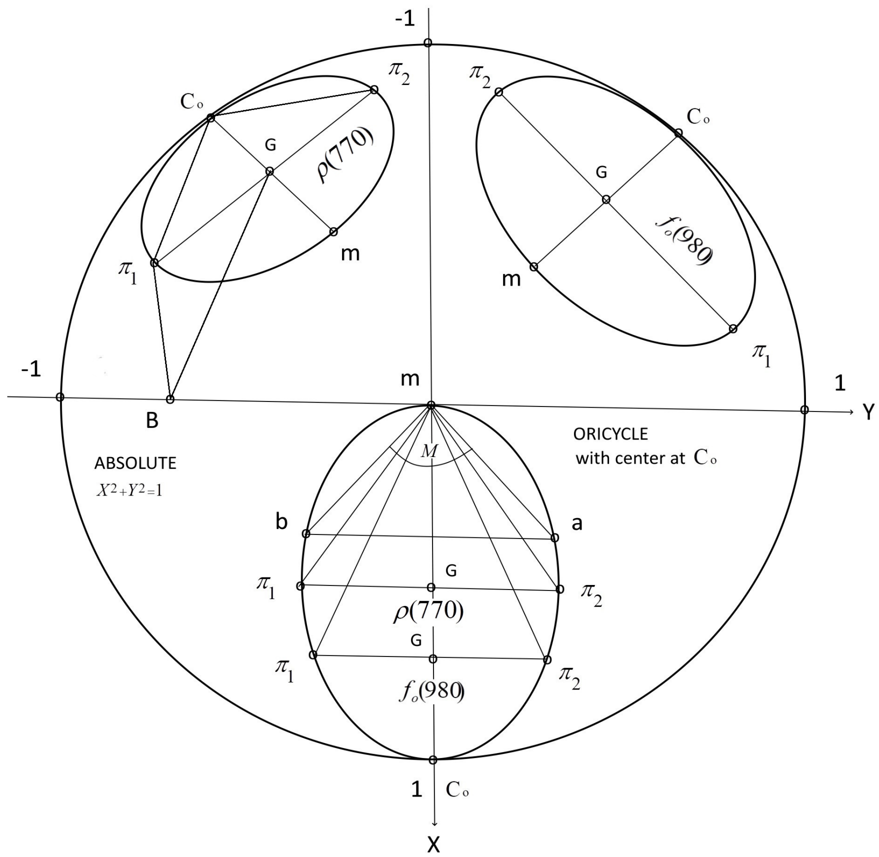

Figure 1. Decays of scalar mesons in the Beltrami model of the Lobachevsky velocity space. The separate ellipses of decay oricycles of $\rho(770)$, $f_o(980)$ scalar mesons with centers in the "C_0" points of the ***Absolute*** and $\Delta\pi_1 \mathrm{m}\pi_2$ tringles of $\rho(770)$, $f_o(980)$ scalar mesons decays, combined into one oricycle with the center at the point "C_0" (1, 0) on the ***Absolute***. The point-velocity "G" represents the center of inertia of the pair particles π_1, π_2.

Table 1. Heron's triangle characteristics near decay triangles of scalar mesons.

Name Scalar Mezon	Effective Mass m_{eff} (Mev)	Effective Mass m_{eff}^{Her} (Mev)	OCT	ch S_{ma} ch S_{mb}	ch S_{ab}	$l_{b\pi_1}/l_{m\pi_1}$	GCosTang(b, m)
			Scalar Mezon $\to \pi_1 + \pi_2$				
$f_o(500)$	495.0	483.5	2	2	5	0.034	0.0
$\rho(770)$	775.3	789.5	7	4.5	15	0.022	2.5
$f_o(980)$	980.0	967.0	11	6.5	23	0.014	4.5
$f_2(1270)$	1275.5	1279.2	20	11	41	0.040	9.0
$f_o(1500)$	1504.5	1503.2	28	15	57	0.000	13.0
$\rho_3(1690)$	1688.8	1697.9	36	19	73	0.060	17.0
$f_4(2050)$	2018.8	2012.9	51	26.5	103	0.020	24.5

Table 2. Heron's triangle characteristics near decay triangles Δ, N baryons.

Name Δ, N Barion	Effective Mass m_{eff} (Mev)	Effective Mass m_{eff}^{Her} (Mev)	OCT	ch S_{ma} ch S_{mb}	ch S_{ab}	$l_{b\pi_1}/l_{m\pi_1}$	GCosTang(b, m)
			N Barion $\to P+\pi_1$				
N (1440)	1370.0	1361.1	4	3	9	0.016	1.0
N (1520)	1515.0	1539.9	7	4.5	15	0.044	2.5
N (1535)	1530.0	1539.9	7	4.5	15	0.012	2.5
			Δ Barion $\to P+\pi_1$				
Δ (1232)	1232.0	1227.8	2	2	5	0.014	0.0
Δ (1600)	1510.0	1482.8	6	4	13	0.038	2.0
Δ (1620)	1600.0	1595.1	8	5	17	0.060	3.0

Table 3. Heron's triangle characteristics near decay triangles of strange mesons.

Name Strange Mezon	Effective Mass m_{eff} (Mev)	Effective Mass m_{eff}^{Her} (Mev)	OCT	ch S_{ma} ch S_{mb}	ch S_{ab}	$l_{b\pi_1}/l_{m\pi_1}$	GCosTang(b, m)
			Strange Mezon $K^* \to K493+\pi_1$				
$K^*(892)$	891.8	860.9	3	2.5	7	0.072	0.5
$K^*(1410)$	1421.1	1410.1	14	8	29	0.010	6.0
$K_0^*(1430)$	1425.6	1410.1	14	8	29	0.020	6.0
$K^*(1680)$	1718.0	1734.5	23	12.5	47	0.014	10.5
$K_4^*(2045)$	2045.0	2035.3	33	17.5	67	0.060	15.5

2. Decay of Scalar Mesons in the Beltrami Model of Lobachevsky Velocity Space

Consider an inelastic reaction with the birth of 2 pi mesons:

$$\mathrm{B}+\mathrm{T}\to \pi_1+\pi_2+\mathrm{all} \quad (1)$$

in which the velocities of particles B, π_1, π_2 in some reference frame "0" are measured. The ends of the velocity vectors of the particles of reaction (1) represent material points-velocities "π_1", "π_2", "B" in the hyperbolic Lobachevsky velocity space (HLVS) located inside a sphere of radius C (hereinafter everywhere the speed of light $C=1$, the rest masses $m_B, m_{\pi_1}, m_{\pi_2}$ of particles B, π_1, π_2 are assigned to the points-velocities) [4]. Let's consider the reaction (1) in the Beltrami model of the Lobachevsky velocity space [6]. In the plane "π_1", "π_2", "B", we introduce a rectangular coordinate system $X0Y$ with the origin at the point "0" (**Figure 1**). The orthogonal projections (X_{π_1}, Y_{π_1}) of the velocity vector of the particle π_1 on the axes $0X$, $0Y$ are called the Beltrami coordinates of the point "π_1" in HLVS. The length $S_{\pi_1\pi_2}$ of the line segment ($\pi_1-\pi_2$) with the Beltrami coordinates of the ends "π_1" (X_{π_1}, Y_{π_1}), "π_2" (X_{π_2}, Y_{π_2}) is represented by the formula [8]:

$$\mathrm{ch}\left(S_{\pi_1\pi_2}\right)=\left(1-X_{\pi_1}*X_{\pi_2}-Y_{\pi_1}*Y_{\pi_2}\right)/\left(R_{\pi_1}*R_{\pi_2}\right) \quad (2)$$

$$R_{\pi_2}=\sqrt{1-X_{\pi_2}^2-Y_{\pi_2}^2} \qquad R_{\pi_1}=\sqrt{1-X_{\pi_1}^2-Y_{\pi_1}^2}$$

The length $S_{\pi_1\pi_2}$ of a line segment ($\pi_1-\pi_2$) is called the rapidity [12]. In addition to the straight line ($\pi_1-\pi_2$), through the points-velocities "π_1", "π_2" passes a single pair of symmetric arcs of lines of curvature 0, called oricycles (the ellipses in **Figure 1** represent oricycles whose lengths S_{0C_0} of radii are infinitely large and whose centers "C_0" are located on the circle $x^2+y^2=1$, called the ***Absolute*** of HLVS) [8]. All oricycles in HLVS are congruent as straight lines in Euclidean space are congruent [8]. Therefore, the ellipse with semimajor axis (C_0 – 0) in **Figure 1** represents an oricycle, into which the oricycles of decays of separate scalar mesons are combined.

The point "m" on the oricycle with additive mass $m_{\pi_1\pi_2}=m_{\pi_1}+m_{\pi_2}$ is determined by Archimedes' laws of levers (3). The roles of forces in the levers are played by the masses m_{π_1}, m_{π_2}, and the arms of the levers are equal to the Euclidean lengths $l_{\pi_1\pi_2}, l_{m\pi_1}, l_{m\pi_2}$ of the arcs of the oricycle [9]-[11]:

$$l_{\pi_1\pi_2}=l_{m\pi_1}+l_{m\pi_2},\quad m_{\pi_1\pi_2}=m_{\pi_1}+m_{\pi_2}$$

$$m_{\pi_1}l_{m\pi_1}=m_{\pi_2}l_{m\pi_2}=m_{\pi_1}\left(l_{\pi_1\pi_2}-l_{m\pi_1}\right) \quad (3)$$

$$l_{m\pi_1}=m_{\pi_2}l_{\pi_1\pi_2}/\left(m_{\pi_1}+m_{\pi_2}\right)$$

$$l_{m\pi_2}=m_{\pi_1}l_{\pi_1\pi_2}/\left(m_{\pi_1}+m_{\pi_2}\right)$$

Connecting the points "π_1", "m", "π_2" with each other by line segments, we obtain an isosceles triangle $\Delta\pi_1\mathrm{m}\pi_2$ of scalar meson decays inscribed in the oricycle (**Figure 1**). Let us consider an arbitrary Δamb triangle, for which we introduce a dimensionless Lorentz invariant function:

$$\mathrm{OCT}=l_{ab}*\mathrm{ctg}\left(\mathrm{M}/2\right),\ l_{ab}=2*\mathrm{sh}\left(S_{ab}/2\right) \quad (4)$$

where l_{ab} is the length of the oricycle arc subtending the base of length S_{ab}, M is the angle at the vertex "m". The function (4) is named oricyclic cotangent of triangle (OCT). The calculations have shown that when OCT = N, where N is an integer, then the lengths S_{ma}, S_{mb}, S_{ab} of sides and bases of the triangle Δamb are related by the relations $\mathrm{ch}\,S_{ma}=\mathrm{ch}\,S_{mb}=\left(\mathrm{N}+2\right)/2$, $\mathrm{ch}\,S_{ab}=2\mathrm{N}+1$. The triangles Δamb with values OCT = N are called Heron triangles in HLVS [8]-[10]. Another Lorentz invariant function called the oricyclic tangent of a triangle (OTT):

$$\mathrm{OTT}=l_{ab}/\mathrm{ctg}\left(\mathrm{M}/2\right) \quad (5)$$

takes one value = 4 on Heron triangles. From the set of Heron triangles, we should especially note the tangent and right triangles (see **Appendix A**). Tangent (OCT =

1) and right (OCT = 4) triangles with absolute characteristics in the form of integers and the golden ratio are constructed on the tangent to an arbitrary point of the oricycle.

Table 1 shows the effective masses m_{eff} of scalar mesons from [13]. According to Formula (6), the values of m_{eff} correspond to the lengths $S_{\pi_1\pi_2}$ of the bases of the triangles $\Delta\pi_1 \mathrm{m}\pi_2$ of the decays of scalar mesons [4]:

$$m_{eff}^2 = m_{\pi_1}^2 + m_{\pi_2}^2 + 2m_{\pi_1}m_{\pi_2}\operatorname{ch}\left(S_{\pi_1\pi_2}\right) \quad (6)$$

Next, the base (a – b) of the triangle Δamb was shifted along the radius (C_0 – m) in small steps. At each shift, the OCT function was calculated using (4) through the length l_{ab} of the arc and the angle M at the vertex "m" using the formulas [14]:

$$l_{mb} = l_{ab}/2 = \operatorname{sh}\left(S_{ab}/2\right) = 2*\operatorname{sh}\left(S_{mb}/2\right), \sin\left(\mathrm{M}/2\right) = \operatorname{sh}\left(S_{ab}/2\right)/\operatorname{sh}\left(S_{mb}\right)$$

We determined the characteristics of Heron's triangles Δamb—integer values OCT = N, $\operatorname{ch} S_{ma} = \operatorname{ch} S_{mb} = (\mathrm{N}+2)/2$, $\operatorname{ch} S_{ab} = 2\mathrm{N}+1$ and Beltrami coordinates of points "a" (X_a, Y_a), "b"(X_b, Y_b), which were "close" to triangles $\Delta\pi_1 \mathrm{m}\pi_2$ (the OCT function was calculated with an accuracy of 3 decimal places) (Figure 1). The "closeness" of triangles was determined by the ratio of arc lengths $l_{b\pi_1}/l_{m\pi_1}$ ($l_{b\pi_1}$ is the length of the arc of the oricycle between points "b" and "π_1" of the bases of triangles Δamb and $\Delta\pi_1 \mathrm{m}\pi_2$, $l_{m\pi_1}$ is the length of the arc of the oricycle between points "m" and "π_1"). The effective mass m_{eff}^{Her} is calculated using Formula (6) through the length S_{ab} of the base of Heron's triangle Δamb (the rest masses $m_{\pi_1} = m_{\pi_2}$ are assigned to points "a" and "b"). The last column of Table 1 contains the values of GCosTang(b_m) = (N − 2)/2—generalized cosines of the decay angle θ between the tangent to the oricycle at point "m" of the additive mass and the tangent at point "b" of the base of the Heron triangles Δamb (**Appendix B**).

Table 1 shows that the triangles $\Delta\pi_1 \mathrm{m}\pi_2$ of scalar meson decays almost coincide with Heron's triangles Δamb (very small values of the ratios of arc lengths $l_{b\pi_1}/l_{m\pi_1}$). It is also interesting that the values of the GCosTang(b, m) are multiples of 1/2 and nearby values of the effective masses m_{eff}^{Her} and m_{eff}.

It should be noted that Archimedes' levers in HLVS were first used by N.A. Chernikov, who used the following expressions for the momenta P_{π_1}, P_{π_2} and kinetic energies T_{π_1}, T_{π_2} of particles π_1, π_2 in the system of their center of mass "G" (Figure 1) [4]:

$$P_{\pi_1} = m_{\pi_1}\operatorname{sh}\left(S_{G\pi_1}\right)$$

$$P_{\pi_2} = m_{\pi_2}\operatorname{sh}\left(S_{G\pi_2}\right)$$

$$T_{\pi_1} = m_{\pi_1}\left(\operatorname{ch}\left(S_{G\pi_1}\right)-1\right)$$

$$T_{\pi_2} = m_{\pi_2}\left(\operatorname{ch}\left(S_{G\pi_2}\right)-1\right)$$

$$S_{\pi_1\pi_2} = S_{G\pi_1} + S_{G\pi_2}$$

Since in the reference frame "G" the momenta $P_{\pi_1} = P_{\pi_2}$ are equal, then:

$$m_{\pi_1} \operatorname{sh}\left(S_{G\pi_1}\right) = m_{\pi_2} \operatorname{sh}\left(S_{G\pi_2}\right)$$

$$\frac{m_{\pi_1}}{2\pi} 2\pi \operatorname{sh}\left(S_{G\pi_1}\right) = \frac{m_{\pi_2}}{2\pi} 2\pi \operatorname{sh}\left(S_{G\pi_2}\right)$$

The expression $2\pi \operatorname{sh}\left(S_{G\pi_1}\right)$ represents the length of a circle of radius $S_{G\pi_1}$ in HLVS. Therefore, N.A. Chernikov used the lengths of circles of radii $S_{G\pi_1}$, $S_{G\pi_2}$ as the lever arms (point "G" is assigned an effective mass m_{eff}) (**Figure 1**). However, the expression $\operatorname{sh}\left(S_{G\pi_1}\right)$ represents the length $l_{m\pi_1}$ of the oricycle arc and Archimedes' laws of levers can be represented in the form (3) [9]-[11]. Accordingly, the particles π_1, π_2 fly apart along the tangent to the oricycle at the point "m", where the additive mass $m_{\pi_1} + m_{\pi_2}$ is concentrated. With such a fly-off, self-oscillations arise, caused by the gravitational force of the particle masses m_{π_1}, m_{π_2}. The integer values of OCT and the hyperbolic cosines of the sides of Heron's triangles reflect this self-oscillating process.

3. Decay of Δ, N Barions in the Beltrami Model of Lobachevsky Velocity Space

Let us consider an inelastic reaction with the birth of proton P and π_1 meson:

$$\mathrm{B} + \mathrm{T} \to \mathrm{P} + \pi_1 + \mathrm{all} \tag{7}$$

in which the velocities of particles B, π_1, P in some reference frame "0" are measured (**Figure 2**). The bottom part of **Figure 2** shows the oricycle centered at the point "C_0" (+1, 0), inscribed in it an isosceles triangle Δamb and the different-sided triangles $\Delta \mathrm{Pm}\pi_1$ of the decays of Δ(1232) and Δ(1600) baryons. The point "m" of the additive mass $m_{P\pi_1} = m_{\pi_1} + m_P$ of triangles $\Delta \mathrm{Pm}\pi_1$ is determined from Archimedes' leverage laws (8) (m_P—rest mass of a proton, m_{π_1}—rest mass of a pi meson). For the case of different rest masses $m_P > m_{\pi_1}$, the point "m" shifts along the arc of the oricycle to the point-velocity "P" of the particle with larger rest mass:

$$\begin{gathered} l_{P\pi_1} = l_{m\pi_1} + l_{mP},\ m_{P\pi_1} = m_P + m_{\pi_1} \\ m_P l_{mP} = m_{\pi_1} l_{m\pi_1} = m_{\pi_1}\left(l_{P\pi_1} - l_{mP}\right) \\ l_{m\pi_1} = m_P l_{P\pi_1} \big/ \left(m_P + m_{\pi_1}\right) \\ l_{mP} = m_{\pi_1} l_{P\pi_1} \big/ \left(m_P + m_{\pi_1}\right) \end{gathered} \tag{8}$$

Based on the values of m_{eff}, m_P, m_{π_1} and Formulas (6) and (8), the lengths $S_{P\pi_1}$, $S_{m\pi_1}$, S_{mP} of the sides $\Delta \mathrm{Pm}\pi_1$ were calculated. According to Formula (6), the values of m_{eff} correspond to the lengths $S_{P\pi_1}$ of the side (π_1 – P) of the triangles $\Delta \mathrm{Pm}\pi_1$ of the baryon decays. To the triangle $\Delta \mathrm{Pm}\pi_1$ of Δ1600 baryon decay from the condition of equality of the lengths of the sides $S_{m\pi_1} = S_{m\pi_2}$, corresponds an isosceles rotary triangle $\Delta\pi_1 m\pi_2$. We determined the characteristics of Heron's triangles Δamb—integer values OCT = N, $\text{ch}\,S_{ma} = \text{ch}\,S_{mb} = (N+2)/2$, $\text{ch}\,S_{ab} = 2N+1$, which were "close" to the rotary triangles $\Delta\pi_1 m\pi_2$ (the OCT function was calculated with an accuracy of 3 decimal places) (**Figure 2**). The "closeness" of triangles was determined by the ratio of arc lengths $l_{b\pi_1}/l_{m\pi_1}$ ($l_{b\pi_1}$ is the length of the arc of the oricycle between points "b" and "π_1" of the bases of triangles Δamb and $\Delta\pi_1 m\pi_2$, $l_{m\pi_1}$ is the length of the arc of the oricycle between points "m" and "π_1"). The effective mass m_{eff}^{Her} is calculated using Formula (6) based on the length S_{bP_b} of side (b – P_b) of triangle $\Delta P_b mb$ (point "P_b" is found from relations (8) for Δamb, points "P_b", "b" are assigned rest masses m_p, m_{π_1}). The last column of **Table 2** shows the values of GCosTang(b_m) = (N – 2)/2.

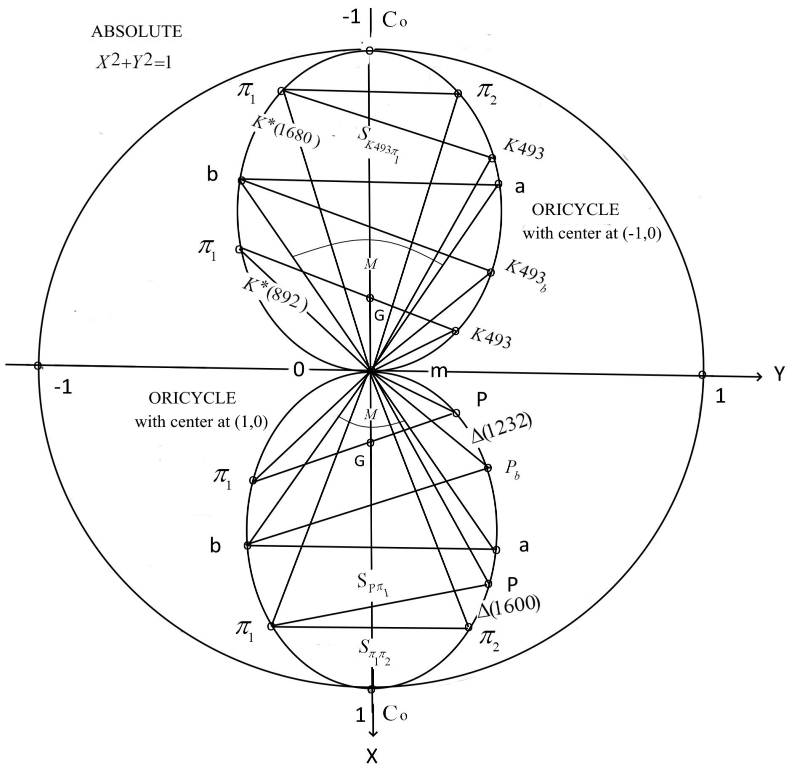

Figure 2. (Bottom part). Decays of Δ, N baryons and K^* strange mesons in the Beltrami model of the Lobachevsky velocity space. Isosceles triangles Δamb, the different-sided triangles $\Delta Pm\pi_1$ of baryon decays Δ(1232), Δ(1600), isosceles rotary triangle $\Delta\pi_1 m\pi_2$ of the decay Δ(1600) baryon, inscribed in an oricycle with the center at the point "C_0" (+1, 0) on the ***Absolute***. The point-velocity "G" represents the center of inertia of the pair of particles P, π_1. (Top part). Isosceles triangles Δamb, the different-sided triangles $\Delta K493m\pi_1$ of the decays of strange mesons $K^*(892) \to K493 + \pi_1$, $K^*(1680) \to K493 + \pi_1$, isosceles rotary triangle $\Delta\pi_1 m\pi_2$ of the decay $K^*(1680)$ of strange meson, inscribed in an oricycle with the center at the point "C_0" $(-1,0)$ on the ***Absolute***. The point-velocity "G" represents the center of inertia of the pair particles $K493$, π_1.

Table 2 shows the effective masses m_{eff} of the Δ, N of baryons from [13]. As can be seen from Table 2, the rotary triangles $\Delta\pi_1 m\pi_2$ of decays Δ, N baryons almost coincide with the Heron triangles Δamb. The close values of effective masses m_{eff}^{Her} and m_{eff} also indicate the "closeness" of the different-sided triangles $\Delta P_b \mathrm{mb}$ and $\Delta \mathrm{Pm}\pi_1$. It is also interesting that the values of the GCosTang(b, m) are multiples of 1/2.

4. Decay of Strange Mesons in the Beltrami Model of Lobachevsky Velocity Space

Let us consider an inelastic reaction with the birth of *K*493 and π_1 mesons:

$$\mathrm{B}+\mathrm{T}\to K493+\pi_1+\mathrm{all} \tag{9}$$

in which the velocities of particles B, π_1, *K*493 in some reference frame "0" are measured (Figure 2). The top part of Figure 2 shows an oricycle centered at the point "C_0"(−1, 0), the isosceles triangle Δamb, the different-sided triangles $\Delta K493\mathrm{m}\pi_1$ of the decays of $K^*(1680)\to K493+\pi_1$, $K^*(892)\to K493+\pi_1$. The rest mass m_{K493} of the strange meson *K*493 is assigned to the point "*K*493". The point "π_1" is assigned to the rest mass m_{π_1} pi meson. The point "m" of additive mass $m_{K493\pi_1}=m_{K493}+m_{\pi_1}$ is determined from Archimedes' laws of levers (8). Based on the values of m_{eff}, m_{K493}, m_{π_1} and Formulas (6) and (8), the lengths $S_{K493\pi_1}$, $S_{m\pi_1}$, S_{mK493} of the sides $\Delta K493\mathrm{m}\pi_1$ were calculated. To the triangle $\Delta K493\mathrm{m}\,\pi_1$ of the decay K^*(1680) from the condition of equality of the lengths of the sides $S_{m\pi_1}=S_{m\pi_2}$, corresponds an isosceles rotary triangle $\Delta\pi_1 m\pi_2$. We determined the characteristics of Heron's triangles Δamb—integer values OCT = N, $\mathrm{ch}\,S_{ma}=\mathrm{ch}\,S_{mb}=(\mathrm{N}+2)/2$, $\mathrm{ch}\,S_{ab}=2\mathrm{N}+1$, which were "close" to triangles $\Delta\pi_1 m\pi_2$ (the OCT function was calculated with an accuracy of 3 decimal places) (Figure 2). The "closeness" of triangles was determined by the ratio of arc lengths $l_{b\pi_1}/l_{m\pi_1}$ ($l_{b\pi_1}$ is the length of the arc of the oricycle between points "b" and "π_1" of the bases of triangles Δamb and $\Delta\pi_1 m\pi_2$, $l_{m\pi_1}$ is the length of the arc of the oricycle between points "m" and "π_1"). The effective mass m_{eff}^{Her} is calculated using Formula (6) based on the length S_{bK493_b} of side (b − $K493_b$) of triangle $\Delta K493_b\mathrm{mb}$ (point "$K493_b$" is found from relations (8) for Δamb, points "$K493_b$" and "b" are assigned rest masses m_{K493}, m_{π_1}).

Table 3 shows the effective masses m_{eff} of the strange meson K^* decays from [13]. The effective masses m_{eff} correspond to the lengths $S_{K493\pi_1}$ of the side (π_1 − *K*493) of the triangles $\Delta K493\mathrm{m}\pi_1$. As can be seen from Table 3, the rotary triangles $\Delta\pi_1 m\pi_2$ of the strange meson K^* decays almost coincide with Heron's triangles Δamb. The close values of effective masses m_{eff}^{Her} and m_{eff} also indicate the "closeness" of the different-sided triangles $\Delta K493_b\mathrm{mb}$ and $\Delta K493\mathrm{m}\pi_1$. It is also interesting that the values of the GCosTang(b, m) are multiples of 1/2.

5. Conclusions

In the hyperbolic Lobachevsky velocity space, the Heron triangles near the decay triangles of scalar, strange mesons and Δ, N baryons are found. The set of Heron's triangles with OCT values of 1, 2, 3, 4, 5, ... from a series of natural numbers and integer side lengths forms a crystalline structure. From this set, a subset of Heron triangles with integer values of other characteristics can be identified (expressions (A1), (A3), (A4), (A8), (A10), and (A11) in **Appendix A**). This subset of Heron triangles will then correspond to crystals with different types of symmetry. Additional discrete characteristics of this subset of Heron's triangles may be somehow related to the quantum numbers of the resonances.

It would be very interesting to process real data from reactions (1), (7) and (9) using the Heron's triangle method. The detection of discrete hadron spectra in real data will open the connection of resonance physics with the theory of integers. Namely, from a number of natural numbers of OCT values, it will be possible to distinguish a number of integers corresponding to resonances. This series may turn out to be a series of primes, composite numbers, Pythagorean numbers, Fibonacci numbers, etc. Finding such a series for OCT values > 200 will contribute to the discovery of previously unknown resonances with very large effective masses (>6000 Mev).

Further development of the described approach will consist of:

- Identifying resonances using the Heron triangle method and analyzing the angular distributions of their decays, using the parameterization of the dynamic spin quantization axis by Lobachevsky straight line beams [15].
- Processing reactions (1), (7) and (9) with the production of more than 2 particles and searching for 3-particle decays of resonances based on a 3-dimensional analog of Heron triangles.

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Data Availability Statement

The data used in the article are taken from open sources [13].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A

In the Beltrami model of the HLVS plane, we introduce a rectangular coordinate system X0Y. The Beltrami coordinates (x_u , y_u) of the point "*u*" will be the orthogonal projections of its velocity vector in the reference frame "0" onto the coordinate axes 0X, 0Y (**Figure A1**). The circle $x^2 + y^2 = 1$ represents the ***Absolute*** HLVS, the ellipses with the semi-axes (C_0 – 0) represent the oricycles with the centers at the points "C_0". The line segment connecting an arbitrary point "*m*" of the oricycle with the center "C_0" is its radius.

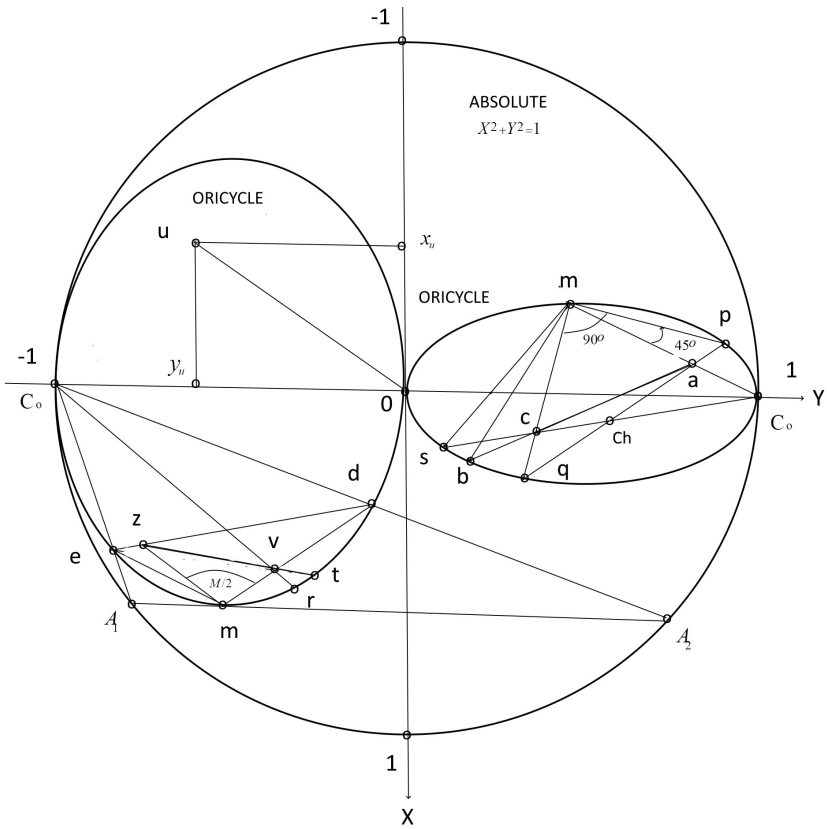

Figure A1. Right Δpmq and tangent Δdme Heron triangles constructed at an arbitrary point "m" of the oricycle.

Let us draw 2 lines from the point "m" of the oricycle at an angle of 45˚ on both sides of the radius (C_0 – m), which will cross the oricycle at the points "p", "q" (**Figure A1**). Connecting the points "p", "q", "m" by line segments, we obtain an isosceles right Heron's triangle Δpmq with an angle 90˚ at the vertex "m". Through the midpoint "c" of the lateral side (m – q) of triangle Δpmq and the midpoint "a" of the base (p – q), we draw a line (a – c) cutting the oricycle at point "b". Through point "a" and the center of the oricycle "C_0" we draw a line (C_0 – c), cutting the oricycle at point "s". The ratio of arc lengths l_{mq}/l_{mb} of the oricycle is equal to the **large golden section 1.61803...**, the ratio of arcs lengths l_{mq}/l_{ms} is equal to 2.

The lengths of arcs l_{mp}, l_{pq}, l_{mq} of the oricycle and the lengths of the sides $S_{pq}, S_{mp}, S_{mq}, S_{ma}$ of the right Heron's triangle Δpmq are related by the relations:

$$l_{pq} = 2 * \text{sh}\left(S_{pq}/2\right) = 4 \quad (A1)$$

$$\text{ch}\left(S_{mp}\right) = \text{ch}\left(S_{mq}\right) = 3, \text{ch}\left(S_{pq}\right) = 9 \quad (A2)$$

$$\text{sh}\left(S_{ma}\right) * \text{sh}\left(S_{pq}\right) = 8 \quad (A3)$$

$$\text{ctg}\left(\frac{\text{M}}{2}\right) = \text{ctg}\left(\pi/4\right) = 1 \quad (A4)$$

$$\text{OCT} = l_{pq} * \text{ctg}\left(\frac{\text{M}}{2}\right) = 4 \quad (A5)$$

$$\text{OTT} = l_{pq} \Big/ \text{ctg}\left(\frac{\text{M}}{2}\right) = 4 \quad (A6)$$

$$\text{GCosTang}\left(\text{p}, \text{q}\right) = 7 \quad (A7)$$

GCosTang(p_q)—Generalized cosine of the angle between the tangents to the oricycle at points "p" and "q" (**Appendix B**).

The isosceles tangent triangle Δdme is constructed on the points "d", "e" of the intersections of the lines (C_0 – A_1), (C_0 – A_2) with the oricycle, where "A_1", "A_2" are the points of intersection of the tangent at point "m" with the ***Absolute*** (**Figure A1**). Through the midpoint "v" of the side (m – d) of the triangle Δdme, the midpoint "z" of the base (d – e), we draw a line (z –v), intersecting the oricycle at points "t". The arc length l_{mt} of the oricycle is equal to the **small golden section 0.61803..**. Through the midpoint "v" and the center "C_0", we draw a line (C_0 – v), intersecting the oricycle at points "r". The length of the arc l_{mr} of the oricycle is equal to 0.5. The ratio of the arc lengths l_{md}/l_{mt} is equal to the **large golden section 1.61803...**, the ratio of the arc lengths l_{md}/l_{mr} is equal to 2 (**Figure A1**). The lengths of the arcs l_{de}, l_{md}, l_{me} of the oricycle and the lengths of the side S_{de}, S_{md}, S_{me} of the tangent Heron's triangle Δdme are related by the relations:

$$l_{de} = 2 * \text{sh}\left(S_{de}/2\right) = 2 \quad (A8)$$

$$\text{ch}\left(S_{me}\right) = \text{ch}\left(S_{md}\right) = 1.5, \text{ch}\left(S_{de}\right) = 3 \quad (A9)$$

$$\text{sh}\left(S_{mz}\right) * \text{sh}\left(S_{de}\right) = 4 \quad (A10)$$

$$\text{ctg}\left(\frac{\text{M}}{2}\right) = \text{ctg}\left(\frac{\pi}{2\sqrt{2}}\right) = 0.5 \quad (A11)$$

$$\text{OCT} = l_{de} * \text{ctg}\left(\frac{\text{M}}{2}\right) = 1 \quad (A12)$$

$$\text{OTT} = l_{pq} \Big/ \text{ctg}\left(\frac{\text{M}}{2}\right) = 4 \quad (A13)$$

$$\text{GCosTang}\left(d, e\right) = 1 \quad (A14)$$

GCosTang(d_e)—Generalized cosine of the angle between the tangents to the oricycle at points "d" and "e". Relations (A1) – (A14) are absolute, since they are satisfied for any point of tangency to the oricycle.

If the cevian (m – Ch) of the triangle Δpmq is projected from the center "C_0" onto the oricycle, then for the corresponding arc lengths of the oricycle, Stewart's theorem will be satisfied (**Figure A1**):

$$l_{mq}^2 = \left(l_{pm}^2 * l_{qs} + l_{qm}^2 * l_{ps}\right) / \left(l_{qs} + l_{ps}\right) - l_{qs} * l_{ps}$$

Bretschneider's theorem will be valid in the quadrilateral (pmbq), if instead of the lengths of the sides and diagonals of the quadrilateral, we take the corresponding arc lengths of the oricycle:

$$l_{pb} * l_{mq} = l_{pq} * l_{mb} + l_{bq} * l_{mp}$$

Appendix B

Figure A2 shows the angles θ between the tangent at point "m" of the additive mass and the tangent at points "a", "s", "q" of the bases of Heron's triangles for various cases of the location of the point "p" (x_p, y_p) of the intersection of these tangents with respect to the ***Absolute***. Generalized cosine of the angle between the Tangents to the oricycle at points ("a" and "m"), ("s" and "m"), ("q" and "m"):

$$\text{GCosTang}(\text{a},\text{m}) = \cos(\theta) < 1, \; x_p^2 + y_p^2 < 1 \quad \text{(B1)}$$

$$\text{GCosTang}(\text{s},\text{m}) = \cos(0) = 1, \; x_p^2 + y_p^2 = 1 \quad \text{(B2)}$$

$$\text{GCosTang}(\text{q},\text{m}) = \text{ch}(S_{cd}) > 1, \; x_p^2 + y_p^2 > 1 \quad \text{(B3)}$$

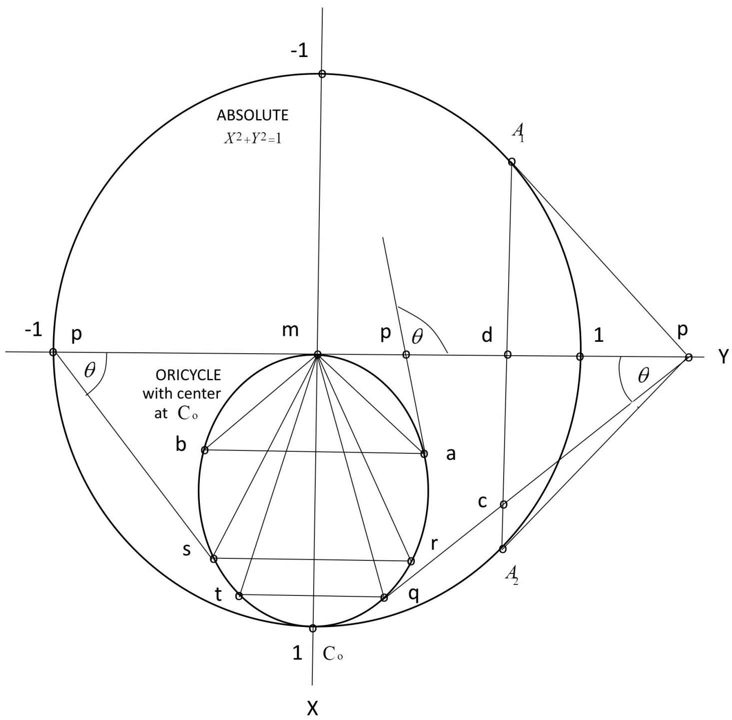

Figure A2. Angle θ between the tangent to the oricycle at point "m" of the additive mass and the tangents at points "a", "q", "s" of the bases of Heron's triangles. The point "p" of the intersection of the tangents is located inside/on/outside the ***Absolute***.

Formula (B1) corresponds to the case where the tangents (m – p) and (a – p) intersect inside the ***Absolute***. Formula (B2) corresponds to the case where the tangents (m – p), (s – p) intersect on the ***Absolute***, then the angle θ between them is 0˚. Formula (B3) corresponds to the case where the tangents (m − p), (q − p) intersect outside the ***Absolute*** at the point "p", then the decay θ between them corresponds to a segment (c – d) of length S_{cd} which the tangents cut off on the line (A_1 − A_2). The lines (A_1 − p), (A_2 − p) are tangents to the ***Absolute***, drawn from the point "p".