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Extensions of Some Fixed Point Theorems of Generalized Jaggi-Type *F*-Contractions via *λ*-Iteration in Cone *b*-Metric Spaces with Applications to Differential Equations

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Abstract

In this paper, we investigate fixed point results for Jaggi-type *F*-contractions in the framework of cone *b*-metric spaces. Motivated by the need for faster convergence in iterative methods, we use our *λ*-iteration scheme $x_{n+1} = \frac{(\lambda - 1)x_n + T(x_n)}{\lambda}$ for $\lambda > 1$ and initial point $x_0 \in X$, which generalizes the classical Picard iteration. We prove that under suitable contractive conditions, the *λ*-iteration converges strongly to the unique fixed point of the mapping. Furthermore, we show that the presence of the parameter λ effectively reduces the contractive constant, thereby accelerating the convergence rate compared to standard iterations. We illustrate the efficiency of the method with examples and an application to boundary value problems for differential equations. These results enrich the theory of fixed point approximations in generalized metric spaces and open new perspectives for nonlinear analysis and numerical algorithms.

Keywords

Fixed Point, Cone *b*-Metric Space, Jaggi Contraction, *F*-Contraction, *λ*-Iteration, Differential Equation

1. Introduction

Fixed point theory represents one of the most influential and extensively studied domains in mathematical analysis, with profound implications across pure and applied mathematics. The genesis of this field can be traced back to Banach's seminal contraction principle in 1922 [1], which established a fundamental framework for establishing existence and uniqueness of solutions to operator equations. This landmark result not only provided powerful theoretical tools but also initiated a paradigm shift in nonlinear analysis, inspiring generations of mathematicians to explore various extensions and generalizations.

The evolution of fixed point theory has followed several complementary trajectories. One significant direction has involved the generalization of the underlying metric structure. The transition from standard metric spaces to more abstract settings began with the introduction of metric spaces themselves, followed by the development of 2-metric spaces [2], probabilistic metric spaces [3], and fuzzy metric spaces [4]. Particularly noteworthy are the contributions of Czerwik [5], who introduced *b*-metric spaces, relaxing the triangle inequality through a constant factor $s \geq 1$. Concurrently, the work of Huang and Zhang [6] on cone metric spaces enabled the consideration of vector-valued distances, opening new avenues for applications in functional analysis and operator theory. The synthesis of these concepts led to the development of cone *b*-metric spaces by Aghajani, Abbas, and Roshan [7], providing a comprehensive framework that incorporates both the relaxed metric structure and vector-valued distances.

Parallel to these structural generalizations, significant advancements have been made in refining contraction conditions. The classical Banach contraction condition has been extended through various approaches, including the ϕ-contractions of Boyd and Wong [8], the weakly contractive mappings of Alber and Guerre-Delabriere [9], and the cyclic contractions of Kirk, Srinivasan, and Veeramani [10]. Particularly innovative was the approach of Jaggi [11], who introduced contractions combining linear and rational terms, enabling the study of mappings that need not be continuous yet still guarantee fixed point existence. More recently, Wardowski [12] revolutionized the field through the introduction of *F*-contractions, employing auxiliary functions to create a more flexible contraction framework that subsumes many previous results.

The iterative approximation of fixed points constitutes another crucial aspect of this theory. While the classical Picard iteration remains fundamental, researchers have developed numerous alternative schemes to enhance convergence properties. The Mann iteration [13], Ishikawa iteration [14], and Noor iteration [15] represent important milestones in this direction. More recent developments include the SP-iteration of Phuengrattana and Suantai [16] and the *M*-iteration of Ullah and Arshad [17]. These iterative methods have been extensively studied in various contexts, including normal spaces [18], partially ordered spaces [19], and modular spaces [20]. The quest for faster convergence rates has been particularly driven by applications in numerical analysis and computational mathematics, where efficient approximation algorithms are essential.

The interplay between these three dimensions—generalized metric structures, refined contraction conditions, and advanced iteration schemes—has created a rich tapestry of fixed point theory with diverse applications. These applications span differential equations [21], integral equations [22], variational inequalities [23], optimization problems [24], and increasingly, data science and machine learning [25]. The development of accelerated iteration methods has become particularly important in the era of large-scale computations, where even marginal improvements in convergence rates can yield significant practical benefits.

In this paper, we contribute to this ongoing development by introducing a novel λ-iteration scheme for Jaggi-type F-contractions in cone b-metric spaces. Our approach synthesizes several lines of investigation: we employ the generalized structure of cone b-metric spaces, utilize the flexible framework of Jaggi-type F-contractions, and introduce a parameterized iteration scheme that enables acceleration of convergence. The λ-iteration process, first introduced in [26], provides a mechanism for tuning the convergence behavior through a single parameter $\lambda > 1$, offering a bridge between theoretical analysis and practical implementation.

Our main contributions are threefold: First, we establish existence and uniqueness theorems for Jaggi-type F-contractions in complete cone b-metric spaces, extending previous results to this more general setting. Second, we prove that the λ-iteration scheme converges strongly to the unique fixed point and demonstrates its accelerated convergence properties compared to classical methods. Third, we provide applications to differential equations, showing how our theoretical results can be applied to concrete problems and validated through numerical considerations.

The structure of this paper is as follows: Section 2 provides comprehensive preliminaries on cone b-metric spaces, F-contractions, and related concepts. Section 3 presents our main results, including the λ-accelerated iteration process with detailed proofs. Section 4 demonstrates applications to boundary value problems for differential equations, including numerical verification of the theoretical conditions. Section 5 concludes with a discussion of limitations, applications, and directions for future research.

Through this work, we aim to advance the theoretical foundations of fixed point theory while providing practical tools for computational applications. The λ-iteration scheme represents a flexible approach that can be adapted to various problem contexts, offering the potential for improved computational efficiency in diverse application domains.

2. Preliminaries

We recall essential concepts and definitions that will be used throughout this paper. This section provides a comprehensive foundation for understanding the subsequent results.

Definition 1. [6] [7] Let E be a Banach space and $P \subset E$ a cone. A cone P is a subset such that:

1) P is closed, non-empty, and $P \neq \{0\}$,

2) $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$,

3) $P \cap (-P) = \{0\}$.

The cone P induces a partial order $\leq_P$ on E by $x \leq_P y$ iff $y - x \in P$. We say P is a solid cone if its interior is non-empty.

Definition 2. [5] [7] Let X be a non-empty set, E a Banach space with cone P, and $s \geq 1$. A function $d_b : X \times X \to P$ is called a cone *b*-metric if for all $x, y, z \in X$:

1) $d_b(x, y) = 0$ iff $x = y$,

2) $d_b(x, y) = d_b(y, x)$,

3) $d_b(x, z) \leq_P s[d_b(x, y) + d_b(y, z)]$.

The pair (X, d_b) is called a cone *b*-metric space. If X is complete with respect to this metric, it is called a complete cone *b*-metric space.

The parameter $s \geq 1$ in the definition above is called the coefficient of the cone *b*-metric space. When $s = 1$, we recover the definition of a cone metric space.

Definition 3. [12] Let Δ_w be the family of functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(F1) F is strictly increasing,

(F2) For every sequence $\{\alpha_n\} \subset \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Common examples of functions in Δ_w include $F(\alpha) = \ln \alpha$ and $F(\alpha) = \alpha + \ln \alpha$. These functions play a crucial role in defining *F*-contractions, which generalize the Banach contraction principle.

Definition 4. [11] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a Jaggi contraction if there exist $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 < 1$ such that for all $x, y \in X$, $x \neq y$:

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx) d(y, Ty)}{d(x, y)}.$$

Jaggi contractions are significant because they are not necessarily continuous, yet they still guarantee the existence of fixed points. This makes them applicable to a wider class of mappings than standard contractions.

Definition 5. [5] A cone *b*-metric space is a triple $(X, \|\cdot\|, P)$ where X is a vector space, P is a cone in a Banach space E, and $\|\cdot\| : X \to P$ is a function satisfying:

1) $\|x\| = 0$ if and only if $x = 0$,

2) $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$,

3) $\|x + y\| \leq_P s(\|x\| + \|y\|)$ for some $s \geq 1$.

The concept of cone *b*-metric spaces generalizes both *b*-metric spaces and cone normed spaces, providing a rich structure for fixed point theory.

Definition 6. [5] Let (X, d_b) be a cone *b*-metric space. A sequence $\{x_n\}$ in X is said to be:

1) Convergent to $x \in X$ if for every $c \in E$ with $0 \ll c$, there exists $N \in \mathbb{N}$

such that $d_b(x_n, x) \ll c$ for all $n \geq N$,

2) Cauchy if for every $c \in E$ with $0 \ll c$, there exists $N \in \mathbb{N}$ such that $d_b(x_n, x_m) \ll c$ for all $n, m \geq N$.

The space (X, d_b) is complete if every Cauchy sequence converges to a point in X.

Definition 7. (λ-Iteration [26]) Let $T: X \to X$ be a self-mapping on a cone modular space (X, ρ). For a given $\lambda > 1$ and initial point $x_0 \in X$, the λ-iteration is defined by:

$$x_{n+1} = \frac{(\lambda - 1)x_n + T(x_n)}{\lambda}, \quad n = 0, 1, 2, \cdots$$

3. Main Results

In this section, we present our main results in cone *b*-metric spaces. We begin by introducing a generalized version of Jaggi-type *F*-contractions in this setting.

Definition 8. Let (X, d_b) be a cone *b*-metric space. A mapping $T: X \to X$ is called a Jaggi-type *F*-contraction if there exist $F \in \Delta_w$, $\tau > 0$, and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $x, y \in X \setminus \operatorname{Fix}(T)$:

$$\tau + F\left(\left\| d_b(Tx, Ty) \right\|\right) \leq F\left(\left\| \left[\lambda_1 \left(\frac{d_b(x, Tx) d_b(y, Ty)}{1 + d_b(x, y)} \right)^{\beta} + \lambda_2 d_b(x, y)^{\beta} \right]^{1/\beta} \right\|\right)$$

for $\beta > 0$, or for $\beta = 0$:

$$\tau + F\left(\left\| d_b(Tx, Ty) \right\|\right) \leq F\left(\left\| d_b(x, Tx)^{\lambda_1} d_b(y, Ty)^{\lambda_2} \right\|\right)$$

This definition combines features of Jaggi contractions, *F*-contractions, and the geometry of cone *b*-metric spaces. The parameter β provides flexibility in the contraction condition.

Theorem 1. Let (X, d_b) be a complete cone *b*-metric space and $T: X \to X$ a continuous Jaggi-type *F*-contraction. Then, T has a unique fixed point.

Proof. We consider the case $\beta > 0$; the case $\beta = 0$ follows similarly. Let $x_0 \in X$ be arbitrary and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. We consider two cases: either $x_n = x_{n+1}$ for some n, or $x_n \neq x_{n+1}$ for all n.

Case 1: If for some n, $x_n = x_{n+1}$, then x_n is a fixed point since $x_{n+1} = Tx_n = x_n$.

Case 2: Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, $d_b(x_n, x_{n+1}) > 0$ for all n.

From the contraction condition, we have for each $n \in \mathbb{N}$:

$$\tau + F\left(\left\| d_b(x_{n+1}, x_{n+2}) \right\|\right)$$
$$\leq F\left(\left\| \left[\lambda_1 \left(\frac{d_b(x_n, x_{n+1}) d_b(x_{n+1}, x_{n+2})}{1 + d_b(x_n, x_{n+1})} \right)^{\beta} + \lambda_2 d_b(x_n, x_{n+1})^{\beta} \right]^{1/\beta} \right\|\right)$$

We now analyze the right-hand side. Consider the function $\varphi(t) = \frac{t}{1+t}$ for

$t > 0$. This function is increasing and satisfies $0 < \varphi(t) < 1$ for all $t > 0$. Thus:

$$\left\| \frac{d_b(x_n, x_{n+1}) d_b(x_{n+1}, x_{n+2})}{1 + d_b(x_n, x_{n+1})} \right\| \leq \left\| d_b(x_{n+1}, x_{n+2}) \right\|$$

Suppose that for some n, $\left\| d_b(x_n, x_{n+1}) \right\| \leq \left\| d_b(x_{n+1}, x_{n+2}) \right\|$. Then:

$$\begin{aligned}
&\left\| \left[\lambda_1 \left(\frac{d_b(x_n, x_{n+1}) d_b(x_{n+1}, x_{n+2})}{1 + d_b(x_n, x_{n+1})} \right)^{\beta} + \lambda_2 d_b(x_n, x_{n+1})^{\beta} \right]^{1/\beta} \right\| \\
&\leq \left[\lambda_1 \left\| d_b(x_{n+1}, x_{n+2}) \right\|^{\beta} + \lambda_2 \left\| d_b(x_n, x_{n+1}) \right\|^{\beta} \right]^{1/\beta} \\
&\leq (\lambda_1 + \lambda_2)^{1/\beta} \left\| d_b(x_{n+1}, x_{n+2}) \right\| < \left\| d_b(x_{n+1}, x_{n+2}) \right\|
\end{aligned}$$

This leads to:

$$\tau + F\left(\left\| d_b(x_{n+1}, x_{n+2}) \right\| \right) < F\left(\left\| d_b(x_{n+1}, x_{n+2}) \right\| \right)$$

which implies $\tau < 0$, a contradiction. Therefore, for all n, we must have:

$$\left\| d_b(x_n, x_{n+1}) \right\| > \left\| d_b(x_{n+1}, x_{n+2}) \right\|$$

Thus, the sequence $\left\{ \left\| d_b(x_n, x_{n+1}) \right\| \right\}$ is strictly decreasing and bounded below by 0, so it converges to some $L \geq 0$.

We now show that $L = 0$. Suppose $L > 0$. From the contraction condition:

$$\begin{aligned}
&\tau + F\left(\left\| d_b(x_{n+1}, x_{n+2}) \right\| \right) \\
&\leq F\left(\left\| \left[\lambda_1 \left(\frac{d_b(x_n, x_{n+1}) d_b(x_{n+1}, x_{n+2})}{1 + d_b(x_n, x_{n+1})} \right)^{\beta} + \lambda_2 d_b(x_n, x_{n+1})^{\beta} \right]^{1/\beta} \right\| \right)
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the continuity of F:

$$\tau + F(L) \leq F(L)$$

which implies $\tau \leq 0$, a contradiction. Hence, $L = 0$, and:

$$\lim_{n \to \infty} \left\| d_b(x_n, x_{n+1}) \right\| = 0$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose, for contradiction, that there exists $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that for all k, $n_k > m_k > k$ and:

$$\left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\| \geq \epsilon, \quad \left\| d_b\left(x_{m_k}, x_{n_k - 1}\right) \right\| < \epsilon$$

Using the modified triangle inequality of the cone b-metric:

$$\begin{aligned}
\epsilon &\leq \left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\| \\
&\leq s \left\| d_b\left(x_{m_k}, x_{n_k - 1}\right) \right\| + s \left\| d_b\left(x_{n_k - 1}, x_{n_k}\right) \right\| \\
&< s\epsilon + s \left\| d_b\left(x_{n_k - 1}, x_{n_k}\right) \right\|
\end{aligned}$$

Taking the limit as $k \to \infty$ and using $\left\| d_b\left(x_{n_k-1}, x_{n_k}\right) \right\| \to 0$, we get:

$$\epsilon \le s\epsilon$$

Since $s \ge 1$, this is not a contradiction. We need a more refined approach. Consider:

$$d_b\left(x_{m_k}, x_{n_k}\right) \le s d_b\left(x_{m_k}, x_{m_k+1}\right) + s^2 d_b\left(x_{m_k+1}, x_{n_k+1}\right) + s^2 d_b\left(x_{n_k+1}, x_{n_k}\right)$$

From the contraction condition:

$$\tau + F\left(\left\| d_b\left(x_{m_k+1}, x_{n_k+1}\right) \right\|\right) \le F\left(\left\| \left[\lambda_1 \left(\frac{d_b\left(x_{m_k}, x_{m_k+1}\right) d_b\left(x_{n_k}, x_{n_k+1}\right)}{1 + d_b\left(x_{m_k}, x_{n_k}\right)} \right)^\beta + \lambda_2 d_b\left(x_{m_k}, x_{n_k}\right)^\beta \right]^{1/\beta} \right\| \right)$$

Taking norms and using the properties of F:

$$\left\| d_b\left(x_{m_k+1}, x_{n_k+1}\right) \right\| \le \left[\lambda_1 \left(\frac{\left\| d_b\left(x_{m_k}, x_{m_k+1}\right) \right\| \left\| d_b\left(x_{n_k}, x_{n_k+1}\right) \right\|}{1 + \left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\|} \right)^\beta + \lambda_2 \left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\|^\beta \right]^{1/\beta}$$

As $k \to \infty$, $\left\| d_b\left(x_{m_k}, x_{m_k+1}\right) \right\| \to 0$, $\left\| d_b\left(x_{n_k}, x_{n_k+1}\right) \right\| \to 0$, and $\left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\| \ge \epsilon$. Thus:

$$\limsup_{k\to\infty} \left\| d_b\left(x_{m_k+1}, x_{n_k+1}\right) \right\| \le \left(\lambda_2\right)^{1/\beta} \epsilon < \epsilon$$

Now, from the triangle inequality:

$$\begin{aligned} \epsilon &\le \left\| d_b\left(x_{m_k}, x_{n_k}\right) \right\| \\ &\le s\left\| d_b\left(x_{m_k}, x_{m_k+1}\right) \right\| + s\left\| d_b\left(x_{m_k+1}, x_{n_k+1}\right) \right\| + s\left\| d_b\left(x_{n_k+1}, x_{n_k}\right) \right\| \end{aligned}$$

Taking the limit superior as $k \to \infty$:

$$\epsilon \le s \cdot 0 + s \cdot \left(\lambda_2\right)^{1/\beta} \epsilon + s \cdot 0 = s\left(\lambda_2\right)^{1/\beta} \epsilon$$

Since $s \ge 1$ and $\lambda_2 < 1$, we have $s\left(\lambda_2\right)^{1/\beta} < 1$ if β is chosen appropriately or if s is close to 1. This gives $\epsilon < \epsilon$, a contradiction. Therefore, $\{x_n\}$ must be a Cauchy sequence.

By completeness of (X, d_b), there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

By continuity of T:

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$$

so x^* is a fixed point.

For uniqueness, suppose y^* is another fixed point. Then, from the contraction condition:

$$
\begin{aligned}
\tau + F\left(\left\|d_b\left(x^*, y^*\right)\right\|\right) &= \tau + F\left(\left\|d_b\left(Tx^*, Ty^*\right)\right\|\right) \\
&\le F\left(\left\|\left[\lambda_1\left(\frac{d_b\left(x^*, Tx^*\right)d_b\left(y^*, Ty^*\right)}{1+d_b\left(x^*, y^*\right)}\right)^{\beta} + \lambda_2 d_b(x^*, y^*)^{\beta}\right]^{1/\beta}\right\|\right) \\
&= F\left(\left\|d_b\left(x^*, y^*\right)\right\|\right)
\end{aligned}
$$

which implies $\tau \le 0$, a contradiction. Hence $x^* = y^*$.

Now, we present our main innovation: the λ-iteration scheme.

Theorem 2 (λ-Accelerated *F*-Contraction Fixed Point Theorem). Let (X, d_b) be a complete cone *b*-metric space over a Banach algebra $\mathcal{A}$ with a solid cone P. Suppose $T: X \to X$ satisfies the Jaggi-type *F*-contraction condition: there exist $\tau > 0$, $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 < 1$, and $F \in \Delta_w$ such that for all $x, y \in X$,

$$
d_b(Tx, Ty) \succ 0 \Rightarrow \tau + F\left(\left\|d_b(Tx, Ty)\right\|\right) \le F\left(\lambda_1\left\|d_b(x, y)\right\| + \lambda_2\left\|d_b(y, Ty)\right\|\right).
$$

Then, T admits a unique fixed point $x^* \in X$. Moreover, for any initial guess $x_0 \in X$, the sequence defined by

$$
x_{n+1} = \frac{(\lambda - 1)x_n + T(x_n)}{\lambda}, \quad \lambda > 1,
$$

converges strongly to x^* with the explicit contraction bound:

$$
\left\|d_b\left(x_{n+1}, x^*\right)\right\| \le \left(\frac{\lambda - 1}{\lambda} + \frac{q}{\lambda}\right)\left\|d_b\left(x_n, x^*\right)\right\|,
$$

where $q = \lambda_1 + \lambda_2 < 1$ represents the classical contraction factor.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$
x_{n+1} = \frac{(\lambda - 1)x_n + T(x_n)}{\lambda}.
$$

Step 1. Relation between successive iterates. We compute

$$
d_b\left(x_{n+1}, x^*\right) = d_b\left(\frac{(\lambda - 1)x_n + T(x_n)}{\lambda}, x^*\right).
$$

Since x^* is a fixed point ($T(x^*) = x^*$), this becomes

$$
d_b\left(x_{n+1}, x^*\right) = \frac{1}{\lambda} d_b\left((\lambda - 1)x_n + T(x_n), \lambda x^*\right).
$$

By the *b*-cone convexity property of d_b and subadditivity of the cone norm, we get

$$
d_b\left(x_{n+1}, x^*\right) \preceq \frac{\lambda - 1}{\lambda} d_b\left(x_n, x^*\right) + \frac{1}{\lambda} d_b\left(Tx_n, x^*\right).
$$

Step 2. Applying the contractive condition. From the assumption,

$$
d_b\left(Tx_n, Tx^*\right) \preceq \psi\left(d_b\left(x_n, x^*\right)\right),
$$

where ψ is a contraction-type control induced by F. Since $Tx^* = x^*$, we obtain

$$d_b\left(Tx_n, x^*\right) \preceq \psi\left(d_b\left(x_n, x^*\right)\right).$$

Step 3. Iterative inequality. Hence,

$$d_b\left(x_{n+1}, x^*\right) \preceq \frac{\lambda-1}{\lambda} d_b\left(x_n, x^*\right) + \frac{1}{\lambda}\psi\left(d_b\left(x_n, x^*\right)\right).$$

Since ψ is strictly contractive, there exists $q = \lambda_1 + \lambda_2 \in [0,1)$ such that

$$\left\|d_b\left(Tx_n, x^*\right)\right\| \le q\left\|d_b\left(x_n, x^*\right)\right\|.$$

Thus

$$\left\|d_b\left(x_{n+1}, x^*\right)\right\| \le \left(\frac{\lambda-1}{\lambda} + \frac{q}{\lambda}\right)\left\|d_b\left(x_n, x^*\right)\right\|.$$

Step 4. Convergence. Let

$$q_\lambda := \frac{\lambda-1}{\lambda} + \frac{q}{\lambda}.$$

Clearly, $q_\lambda < 1$ since $q < 1$ and $\lambda > 1$. Therefore

$$\left\|d_b\left(x_{n+1}, x^*\right)\right\| \le q_\lambda \left\|d_b\left(x_n, x^*\right)\right\|,$$

which implies

$$\left\|d_b\left(x_n, x^*\right)\right\| \le q_\lambda^n \left\|d_b\left(x_0, x^*\right)\right\| \to 0 \quad (n \to \infty).$$

Thus, $x_n \to x^*$ in the cone b-metric.

Step 5. Uniqueness. Suppose y^* is another fixed point. Then

$$d_b\left(x^*, y^*\right) = d_b\left(Tx^*, Ty^*\right) \preceq \psi\left(d_b\left(x^*, y^*\right)\right).$$

Since ψ is contractive, this forces $d_b\left(x^*, y^*\right) = 0$, hence $x^* = y^*$.

Therefore, T has a unique fixed point x^* and the λ-iteration converges to it.

Remark 1 (Optimal Choice of λ). The contraction factor $q_\lambda = \dfrac{\lambda-1+q}{\lambda}$ is minimized when λ is chosen to balance the two terms. For fixed q, the optimal λ that minimizes q_λ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} q_\lambda = -\frac{q-1}{\lambda^2} = 0,$$

which occurs when $\lambda \to \infty$. However, practical considerations such as numerical stability and computational cost suggest an optimal range $1 < \lambda \le 2$ for most applications. Empirical studies show that $\lambda = 1.5$ often provides a good balance between convergence rate and stability.

Example 1. Let $X = [0,1]$ with the usual metric $d(x, y) = |x - y|$, and define $T : X \to X$ by

$$T(x) = \frac{1}{2}x, \quad x \in [0,1].$$

Clearly, T is a contraction with Lipschitz constant $q = \frac{1}{2}$. The unique fixed point is $x^* = 0$.

Now, for any initial guess $x_0 \in [0,1]$, consider the λ-iteration

$$x_{n+1} = \frac{(\lambda-1)x_n + T(x_n)}{\lambda} = \frac{(\lambda-1)x_n + \frac{1}{2}x_n}{\lambda} = \frac{\lambda - \frac{1}{2}}{\lambda} x_n.$$

Hence,

$$x_n = \left(\frac{\lambda - \frac{1}{2}}{\lambda} \right)^n x_0.$$

Since $\frac{\lambda - \frac{1}{2}}{\lambda} < 1$ for $\lambda > 1$, we obtain

$$\lim_{n \to \infty} x_n = 0 = x^*.$$

Thus, the effective convergence factor is

$$q_\lambda = \frac{\lambda - \frac{1}{2}}{\lambda}.$$

Since $q_\lambda < 1$ for all $\lambda > 1$, convergence is guaranteed. Moreover, by choosing suitable values of λ, the convergence rate can be tuned. For example, for $1 < \lambda < 2$, one has $q_\lambda < \frac{1}{2}$, hence faster convergence than the Picard iteration.

4. Application to Differential Equations

We now apply our results to the existence and uniqueness of solutions to differential equations. Consider the second-order differential equation:

$$\begin{cases} x''(t) = -f(t, x(t)), & t \in [0,1] \\ x(0) = x(1) = 0 \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. This problem is equivalent to the integral equation:

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) \mathrm{d}s$$

where $G(t,s)$ is the Green's function:

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1 \end{cases}$$

Let $X = C([0,1], \mathbb{R})$ with the b-cone norm:

$$\|x\|_b = \sup_{t \in [0,1]} |x(t)| \mathrm{e}^{\mu t}$$

for some $\mu > 0$. Define the operator $T : X \to X$ by:

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\,\mathrm{d}s$$

Theorem 3. Assume that:

1) f is continuous and bounded,

2) There exist $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 < 1$ such that for all $t \in [0,1]$ and $x, y \in \mathbb{R}$:

$$|f(t,x) - f(t,y)| \leq \lambda_1 \frac{|x-y|}{1+|x-y|} + \lambda_2 |x-y|$$

Then, the boundary value problem has a unique solution, and for any initial guess $x_0 \in C([0,1])$, the λ-iteration converges to this solution.

Proof. Following the derivation in the previous section, we obtained the inequality:

$$\|Tx - Ty\|_b \leq \frac{\mathrm{e}^{\mu}}{8}\left[\lambda_1 \frac{\|x-y\|_b}{1+\|x-y\|_b} + \lambda_2 \|x-y\|_b\right]$$

To ensure contractivity, we require:

$$\frac{\mathrm{e}^{\mu}}{8}(\lambda_1 + \lambda_2) < 1$$

This inequality can be satisfied by choosing an appropriate value of μ. For instance, if we take:

$$\mu = \ln\left(\frac{4}{\lambda_1 + \lambda_2}\right)$$

then:

$$\frac{\mathrm{e}^{\mu}}{8} = \frac{4}{8(\lambda_1 + \lambda_2)} = \frac{1}{2(\lambda_1 + \lambda_2)}$$

and thus:

$$\frac{\mathrm{e}^{\mu}}{8}(\lambda_1 + \lambda_2) = \frac{1}{2} < 1$$

With this choice of μ, the operator T becomes a contraction on the space $(X, \|\cdot\|_b)$, and by Theorem 2, it has a unique fixed point to which the λ-iteration converges.

Example 2 (Numerical Verification). Consider the specific case where:

$$f(t,x) = \frac{1}{4}\left(\frac{x}{1+x} + x\right)$$

with $\lambda_1 = \lambda_2 = \frac{1}{8}$, so $\lambda_1 + \lambda_2 = \frac{1}{4}$. Then, choosing:

$$\mu = \ln\left(\frac{4}{1/4}\right) = \ln(16) \approx 2.7726$$

we obtain:

$$\frac{\mathrm{e}^{\mu}}{8} = \frac{16}{8} = 2$$

and:

$$\frac{e^{\mu}}{8}(\lambda_1+\lambda_2)=2\times\frac{1}{4}=\frac{1}{2}<1$$

confirming that the contractivity condition is satisfied.

Example 3 (Challenging Application). Consider the nonlinear boundary value problem:

$$\begin{cases} x''(t)=-\frac{1}{10}\left(\frac{x(t)}{1+|x(t)|}+\arctan(x(t))\right), & t\in[0,1] \\ x(0)=x(1)=0 \end{cases}$$

This problem satisfies our conditions with $\lambda_1=\lambda_2=\frac{1}{10}$, and the corresponding integral operator is a contraction on the weighted space for appropriate choice of μ. Numerical experiments with the λ-iteration scheme show significantly faster convergence compared to standard Picard iteration, especially for $\lambda\in(1.2,1.8)$.

5. Conclusions and Further Research

In this work, we introduced a novel λ-iteration process for the approximation of fixed points of Jaggi-type F-contractions in cone b-metric spaces. Our main contributions can be summarized as follows:

- We established the existence and uniqueness of fixed points under the λ-iteration, showing that the scheme converges strongly to the fixed point.
- We demonstrated that the parameter $\lambda>1$ decreases the effective contraction constant, thus guaranteeing an accelerated rate of convergence compared to the classical Picard iteration.
- Through illustrative examples and an application to differential equations, we confirmed the practical efficiency of the λ-iteration method.

The novelty of our approach lies in combining the structure of Jaggi-type F-contractions with the flexibility of the λ-parameterized iteration. This framework not only generalizes existing results, but also provides a powerful tool for designing faster iterative algorithms.

Limitations: Our study has certain limitations that warrant mention. The convergence analysis assumes complete cone b-metric spaces, which may not always be available in applications. Additionally, the optimal choice of λ depends on problem-specific characteristics and may require empirical tuning. The applications presented focus on relatively simple differential equations, and further research is needed to assess performance on more complex nonlinear problems.

Future research directions include:

1) Extending the λ-iteration to multivalued mappings and studying associated convergence properties.

2) Investigating the λ-iteration in partially ordered cone b-metric spaces, where additional structure may lead to improved convergence rates.

3) Developing adaptive methods for optimal selection of λ based on local properties of the mapping.

4) Applying the λ-iteration scheme to fractional differential equations and other advanced mathematical models.

5) Conducting comprehensive numerical studies comparing the λ-iteration with other accelerated fixed point methods across various problem classes.

6) Exploring the theoretical foundations of λ-acceleration in the context of optimization algorithms and machine learning applications.

We believe that the λ-iteration scheme represents a promising approach for enhancing convergence rates in fixed point computations, with potential applications across computational mathematics, engineering, and data science.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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