

Regularity and Asymptotic Behavior of Solutions for 3D Incompressible MHD Equations with Damping

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Abstract

In this paper, we mainly investigate the Cauchy problem for the 3D incompressible magnetohydrodynamic (MHD) equations with damping terms. Under a certain smallness assumption on the initial data, this paper not only establishes the global existence of strong solutions to the equations when parameters satisfy $1 \leq \alpha, \beta < 3$, but also obtains the decay estimates of the solutions to the MHD equations.

Keywords

Global Strong Solutions, Incompressible MHD Equations, Decay, Damping

1. Introduction and the Main Results

In this paper, we consider the following 3D incompressible MHD equations with damping terms:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + |u|^{\alpha-1} u + \nabla P = b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b - \Delta b + |b|^{\beta-1} b = b \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(0, x) = u_0, \quad b(0, x) = b_0. \end{cases} \quad (1.1)$$

here, $x \in \mathbb{R}^3, t > 0$, and $u = (u^1, u^2, u^3)(x, t)$, $b = (b^1, b^2, b^3)(x, t)$, $P = P(x, t)$ denote the density, velocity, magnetic and pressure of the fluid, respectively. In the damping terms, $\alpha, \beta \geq 1$ are real exponents and if $\alpha = 1$ (resp. $\beta = 1$), we actually mean there is no velocity (resp. magnetic) damping. The damping terms are used to describe nonlinear dissipative mechanisms. Their core role is to regulate the energy dissipation rate through the magnitude of velocity or magnetic field,

reflecting the nonlinear relationship between resistance and motion intensity in physical systems. The damping arises from the resistance opposing the flow motion. It accounts for a variety of physical phenomena, such as porous media flow, drag forces or frictional effects, and certain dissipative mechanisms. In MHD flow within porous media (such as subsurface magnetohydrodynamic exploration or oil and gas reservoir development), the damping terms can describe the nonlinear seepage resistance of fluids in complex pore structures (e.g., nonlinear correction of Darcy's law). Specifically, a larger α value indicates a stronger retarding effect of pores on high-speed flow, while a larger β value signifies more significant nonlinear variation in the permeability of the medium under strong magnetic fields. When $b = 0$, problem (1.1) reduces to Navier-Stokes with damping,

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + |u|^{\alpha-1} u + \nabla P = 0, \\ u(0, x) = u_0. \end{cases} \tag{1.2}$$

The physical background of problem (1.2) was introduced firstly by Cai and Jiu [1], they obtained the global existence of strong solutions when $\alpha \geq \frac{7}{2}$, and the strong solution is unique for $\frac{7}{2} \leq \alpha \leq 5$. In [2], Zhang, Wu and Lu proved for $\alpha > 3$ and $u_0 \in H^1 \cap L^{\alpha+1}$ that problem (1.2) has a global strong solution and the strong solution is unique when $3 < \alpha \leq 5$. Zhou [3] proved that the constant 3 was the critical in some sense and established regularity criteria for $1 \leq \alpha < 3$ as follows: assume that $u(t, x)$ satisfies $u \in L^s(0, T; L^r)$ with $\frac{2}{s} + \frac{3}{r} \leq 1$, $3 < r < \infty$ or $\nabla u \in L^{\tilde{s}}(0, T; L^{\tilde{r}})$ with $\frac{2}{\tilde{s}} + \frac{3}{\tilde{r}} \leq 1$, $3 < \tilde{r} < \infty$, then the solution remains smooth on $[0, T]$. Later, Zhong [4] proved that the strong solutions exist globally for $1 \leq \alpha < 3$. In addition, we also refer to [5]-[7] for related works. In the presence of magnetic field, Ye [8] first presented the definition of weak solution, concerned the preliminary L^2 decay for weak solution for $\alpha, \beta > \frac{7}{3}$, and showed the global regularity of problem (1.1) if one of the following five conditions verifies:

$$\begin{aligned} &\alpha \geq 4, \beta \geq 4; \\ &3 < \alpha \leq \frac{7}{2}, \frac{7}{2\alpha-5} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}; \\ &\frac{7}{2} < \alpha < 4, \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}; \\ &4 \leq \alpha \leq \frac{17}{3}, \frac{5\alpha+7}{2\alpha} \leq \beta < 4; \\ &\frac{17}{3} < \alpha \leq 7, \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{\alpha+5}{\alpha-3}. \end{aligned}$$

Later, this result was improved by Zhang *et al.* [9], the unique global strong solution to problem (1.1) was obtained for one of the four conditions: $3 \leq \alpha \leq \frac{27}{8}$,

$\beta \geq 4$; $\frac{27}{8} < \alpha \leq \frac{7}{2}$, $\beta \geq \frac{7}{2\alpha-5}$; $\frac{7}{2} < \alpha < 4$, $\beta \geq \frac{5\alpha+7}{2\alpha}$; $\alpha \geq 4$, $\beta \geq 1$. Subsequently, Ma and Zhang [10] proved that the global regularity of 2D generalized magnetohydrodynamics equations with magnetic damping provided that $0 < \alpha < 1$, $\beta > 3$. Recently, Li and Xiao [11] proved the existence and uniqueness of global strong solution when one of the following two conditions is verified: 1) $3 \leq \alpha < 4$, $\beta \geq \frac{6}{\alpha-1} + 1$; 2) $\alpha \geq 4$, $\beta > 1$. However, the global existence of strong solution to problem (1.1) for $1 \leq \alpha, \beta < 3$ is still unknown, and the L^2 decay solution for $u_0, b_0 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ is an interesting problem when $1 \leq \alpha, \beta < 3$. Motivated by [8] [9], the aim of this paper is to investigate the global existence of strong solution to system (1.1).

As regards the damped MHD equations (1.1), the definition of strong solution is as follows.

Definition 1.1 The triplet (u, b, P) is called a strong solution to (1.1) in $\mathbb{R}^3 \times (0, T)$ if (1.1) holds almost everywhere in $\mathbb{R}^3 \times (0, T)$ and

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)), \quad b \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)).$$

Now, we can state the first main theorem of the present paper:

Theorem 1.1 Let $u_0, b_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Assume that $1 \leq \alpha, \beta < 3$, then there exists a positive constant ε_0 independent of u_0, b_0, α, β such that if

$$\|(u_0, b_0)\|_{L^2}^2 \left(\|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) \leq \varepsilon_0, \quad (1.3)$$

the problem (1.1) has a unique global strong solution.

In the following, we consider time decay rate of solutions to problem (1.1). The motivation is to understand how the α, β affect the time decay rate of its solutions. Here, we contrast with the heat equations and study the decay rate of solutions for problem (1.1), and give the L^2 decay of solutions for $u_0, b_0 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. We present the second main theorem as follows:

Theorem 1.2 Let $u_0, b_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Assume that $1 \leq \alpha, \beta < 3$, $u_0, b_0 \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, then for the solution $(u, b)(x, t)$ of problem (1.1), there exists a positive constant $C = C(\alpha, \beta, \|u_0\|_{L^1}, \|u_0\|_{L^2}, \|b_0\|_{L^1}, \|b_0\|_{L^2})$ such that

$$\|(u, b)(x, t)\|_{L^2}^2 \leq C(1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}}. \quad (1.4)$$

Remark 1.1 The time decay problem of solutions to the dissipative equations implies that the trivial solution is asymptotically stable. It is an interesting problem to study the time decay rate of solutions to dissipative equations. We improve this decay result obtained by Ye [8] in Theorem 1.2.

Remark 1.2 It is worth noting that $1 \leq \alpha, \beta < 3$, thus using Hölder's and Sobolev's inequalities yields

$$\|u_0\|_{L^{\alpha+1}}^{\alpha+1} \leq C \|u_0\|_{L^2}^{\frac{5-\alpha}{2}} \|\nabla u_0\|_{L^2}^{\frac{3\alpha-3}{2}}, \quad \|b_0\|_{L^{\beta+1}}^{\beta+1} \leq C \|b_0\|_{L^2}^{\frac{5-\beta}{2}} \|\nabla b_0\|_{L^2}^{\frac{3\beta-3}{2}}. \quad (1.5)$$

Consequently, it follows from (1.4) that the problem (1.1) admits a unique global strong solution when the norm $\|(u_0, b_0)\|_{L^2} \|(\nabla u_0, \nabla b_0)\|_{L^2}$ is sufficiently small.

The rest of this paper is organized as follows. In Section 2, we state some elementary facts and inequalities that will be used later. In Section 3, we are devoted to the local existence and uniqueness of solutions. In Section 4, we give the proof of Theorem 1.1. Finally, we establish the L^2 decay of solutions to problem (1.1).

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used later. Now, we begin with the following Gagliardo-Nirenberg inequality, which can be obtained by [12].

Lemma 2.1 *Let $1 \leq p, q, r \leq \infty$, and j, m are arbitrary integers satisfying $0 \leq j < m$. Assume that $u \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\|D^j u\|_{L^p} \leq C \|u\|_{L^q}^{1-a} \|D^m u\|_{L^r}^a, \quad \text{where } -j + \frac{n}{p} = (1-a)\frac{n}{q} + a\left(-m + \frac{n}{r}\right),$$

and

$$a \in \begin{cases} \left[\frac{j}{m}, 1\right], & \text{if } m - j - \frac{n}{r} \text{ is an nonnegative integer,} \\ \left[\frac{j}{m}, 1\right], & \text{otherwise.} \end{cases} \quad (2.1)$$

Then, constant C depends only n, m, j, q, r, a .

Next, we give the following Gronwall's inequality (see [13]), which plays a key role in the proof of the priori estimates on strong solutions (u, b, P) .

Lemma 2.2 *Suppose that h and r are integrable on (a, b) and nonnegative a.e. in (a, b) . Further assume that $y \in C[a, b]$, $y' \in L(a, b)$, and*

$$y'(t) \leq h(t) + r(t)y(t) \quad \text{for a.e. } t \in (a, b).$$

Then

$$y(t) \leq \left[y(a) + \int_a^t h(s) \exp\left(-\int_a^t r(\tau) d\tau\right) ds \right] \exp\left(\int_a^t r(s) ds\right), \quad t \in [a, b].$$

Finally, in order to obtain the uniform bounds in next section, we present the following lemma, which can be found in [14].

Lemma 2.3 *Let $g \in W^{1,1}(0, T)$ and $k \in L^1(0, T)$ satisfy*

$$\frac{dg}{dt} \leq F(g) + k \quad \text{in } [0, T], \quad g(0) \leq g_0,$$

where F is bounded on bounded sets from \mathbb{R} into \mathbb{R} . Then, for every $\varepsilon > 0$, there exists T_ε independent of g such that

$$g(t) \leq g_0 + \varepsilon, \quad \forall t \leq T_\varepsilon.$$

Next, we consider the Cauchy problem of heat equations

$$u_t - \Delta u = 0, \quad u(0, x) = u_0(x), \tag{2.2}$$

we have the following space-time estimates:

Lemma 2.4 (see [15]) *Let $1 \leq p \leq q \leq \infty$ and $u_0(x) \in L^p(\mathbb{R}^3)$. Then, for $\mu > 0$, problem (2.2) satisfies the following estimates:*

$$\|u(x, t)\|_{L^q} \leq Ct^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p}, \tag{2.3}$$

$$\left\| (-\Delta)^{\frac{\mu}{2}} u(x, t) \right\|_{L^q} \leq Ct^{\frac{\mu}{2} - \frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p}. \tag{2.4}$$

3. Local Existence and Uniqueness of Solutions

In this section, we will prove the following local existence and uniqueness of strong solutions to the Cauchy problem (1.1) by giving the a priori estimates later.

Theorem 3.1 *Let $u_0, b_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Assume that $1 \leq \alpha, \beta < 3$, then there exists a small positive time $T_0 > 0$ and a unique strong solution (u, b, P) to the Cauchy problem (1.1) in $\mathbb{R}^3 \times (0, T_0]$.*

In order to give the proof of Theorem 3.1, we first derive the following key a priori estimates on $E_2(t)$ defined by

$$E_2(t) \triangleq \|(u, b)(t)\|_{H^1}^2 + 1.$$

More precisely,

Proposition 3.1 *Let $u_0, b_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ and (u, b, P) be a solution to problem (1.1) on $\mathbb{R}^3 \times (0, T]$. Then, there exists a small time $T_0 \in (0, T]$ and a positive constant C depending only on α, β and E_2 such that*

$$\sup_{t \in (0, T_0]} E_2(t) \leq C.$$

Proof. Testing (1.1)_{1,2} by u, b , respectively, and adding the resultant equations, it follows from the divergence theorem that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u, b)(t)\|_{L^2}^2 + \|u\|_{L^{\alpha+1}}^{\alpha+1} + \|b\|_{L^{\beta+1}}^{\beta+1} + \|(\nabla u, \nabla b)\|_{L^2}^2 \\ &= -\int (u \cdot \nabla) u \cdot u \, dx + \int (b \cdot \nabla) b \cdot u \, dx - \int (u \cdot \nabla) b \cdot b \, dx + \int (b \cdot \nabla) u \cdot b \, dx \\ &= \int (b \cdot \nabla)(b \cdot u + u \cdot b) \, dx = 0, \end{aligned} \tag{3.1}$$

thus integrating with respect to t , we get

$$\|(u, b)(t)\|_{L^2}^2 + \int_0^t \|u\|_{L^{\alpha+1}}^{\alpha+1} \, ds + \int_0^t \|b\|_{L^{\beta+1}}^{\beta+1} \, ds + \int_0^t \|(\nabla u, \nabla b)\|_{L^2}^2 \, ds \leq \|(u_0, b_0)\|_{L^2}^2. \tag{3.2}$$

Multiplying (1.1)_{1,2} by u_t, b_t , respectively, and adding the resultant equations, then applying Cauchy-Schwartz inequality gives

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\nabla u, \nabla b)\|_{L^2}^2 + \frac{1}{\alpha+1} \frac{d}{dt} \|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \frac{d}{dt} \|b\|_{L^{\beta+1}}^{\beta+1} + \|(u_t, b_t)\|_{L^2}^2 \\ & = -\int (u \cdot \nabla) u \cdot u_t dx + \int (b \cdot \nabla) b \cdot b_t dx - \int (u \cdot \nabla) b \cdot b_t dx + \int (b \cdot \nabla) u \cdot b_t dx \quad (3.3) \\ & \leq \frac{1}{2} \|(u_t, b_t)\|_{L^2}^2 + \frac{1}{2} \int |u \cdot \nabla u|^2 + |b \cdot \nabla b|^2 + |u \cdot \nabla b|^2 + |b \cdot \nabla u|^2 dx. \end{aligned}$$

Similarly, testing (1.1)_{1,2} by $-\Delta u, -\Delta b$ and integrating the resulting equations over \mathbb{R}^3 , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\nabla u, \nabla b)\|_{L^2}^2 + \alpha \int |u|^{\alpha-1} |\nabla u|^2 dx + \beta \int |b|^{\beta-1} |\nabla b|^2 dx + \|(\Delta u, \Delta b)\|_{L^2}^2 \\ & = \int (u \cdot \nabla) u \cdot \Delta u dx - \int (b \cdot \nabla) b \cdot \Delta u dx + \int (u \cdot \nabla) b \cdot \Delta b dx - \int (b \cdot \nabla) u \cdot \Delta b dx \quad (3.4) \\ & \leq \frac{1}{2} \|(\Delta u, \Delta b)\|_{L^2}^2 + \frac{1}{2} \int |u \cdot \nabla u|^2 + |b \cdot \nabla b|^2 + |u \cdot \nabla b|^2 + |b \cdot \nabla u|^2 dx. \end{aligned}$$

Gathering the above equation with (3.3) and applying the Gagliardo-Nirenberg inequality, Sobolev's inequality, we get

$$\begin{aligned} & \frac{d}{dt} \|(\nabla u, \nabla b)\|_{L^2}^2 + \frac{1}{\alpha+1} \frac{d}{dt} \|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \frac{d}{dt} \|b\|_{L^{\beta+1}}^{\beta+1} + \|(u_t, b_t)\|_{L^2}^2 \\ & + \alpha \int |u|^{\alpha-1} |\nabla u|^2 dx + \beta \int |b|^{\beta-1} |\nabla b|^2 dx + \|(\Delta u, \Delta b)\|_{L^2}^2 \\ & \leq \int |u \cdot \nabla u|^2 + |b \cdot \nabla b|^2 + |u \cdot \nabla b|^2 + |b \cdot \nabla u|^2 dx \quad (3.5) \\ & \leq C \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}^3 + C \|\Delta b\|_{L^2} \|\nabla b\|_{L^2}^3 + C \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\Delta b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{3}{2}} \\ & \leq \frac{1}{2} \|(\Delta u, \Delta b)\|_{L^2}^2 + C \|(\nabla u, \nabla b)\|_{L^2}^6. \end{aligned}$$

Note that $E_2(t) \triangleq \|(u, b)(t)\|_{H^1}^2 + 1$, now integrating (3.5) with respect to t and combining (3.2), we have

$$E_2(t) \leq C + C \exp\left(C \int_0^t E_2^2(s) ds\right). \quad (3.6)$$

Let $E(t) \triangleq \int_0^t E_2^2(s) ds$, then (3.6) can be rewritten as

$$\frac{d}{dt} E(t) \leq \left[C + C \exp(CE(t)) \right]^2.$$

Hence, combining the above inequality and Lemma 2.3, we obtain the desired result. This completes the proof of Proposition 3.1.

After giving a priori estimates in higher norms, using a standard Galerkin method (see [16]), we obtain the local existence of strong solutions. And the uniqueness of strong solutions follows from the weak strong unique result in [8]. Here, we omit the details. This completes the proof of Theorem 3.1.

4. Proof of Theorem 1.1

According to the local well-posedness obtained by Theorem 3.1, we derive the proof of Theorem 1.1 in this section. Throughout this section, we define

$$E_0 \triangleq \|(u_0, b_0)\|_{L^2}^2, \quad E_1(t) \triangleq \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^2. \quad (4.1)$$

Before giving the proof of Theorem 1.1, we need the following lemma, which plays a key role in the proof.

Lemma 4.1 *Let (u, b, P) be the strong solution to the problem (1.1) on $\mathbb{R}^3 \times (0, T)$. Then, there exists an absolute constant C independent of T, α , and β such that for any $t \in (0, T)$, there holds*

$$\begin{aligned} \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^2 &\leq \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} \\ &+ \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1} + CE_0 \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^4. \end{aligned} \tag{4.2}$$

Proof. It follows from (3.5) that

$$\begin{aligned} \frac{d}{dt} \|(\nabla u, \nabla b)\|_{L^2}^2 &+ \frac{1}{\alpha + 1} \frac{d}{dt} \|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \frac{d}{dt} \|b\|_{L^{\beta+1}}^{\beta+1} + \|(u_t, b_t)\|_{L^2}^2 \\ &+ \alpha \int |u|^{\alpha-1} |\nabla u|^2 \, dx + \beta \int |b|^{\beta-1} |\nabla b|^2 \, dx + \|(\Delta u, \Delta b)\|_{L^2}^2 \\ &\leq C \|(\nabla u, \nabla b)\|_{L^2}^6. \end{aligned} \tag{4.3}$$

Now, integrating (4.3) with respect to t and applying Hölder inequality, (3.3), we arrive at

$$\begin{aligned} \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^2 &+ \frac{1}{\alpha + 1} \sup_{s \in [0, t]} \|u\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \sup_{s \in [0, t]} \|b\|_{L^{\beta+1}}^{\beta+1} + \int_0^t \|(\Delta u, \Delta b)\|_{L^2}^2 \, ds \\ &\leq C \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \sup_{s \in [0, t]} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \sup_{s \in [0, t]} \|b_0\|_{L^{\beta+1}}^{\beta+1} \\ &+ \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^4 \int_0^t \|(\nabla u, \nabla b)\|_{L^2}^2 \, ds \\ &\leq C \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \sup_{s \in [0, t]} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \sup_{s \in [0, t]} \|b_0\|_{L^{\beta+1}}^{\beta+1} \\ &+ CE_0 \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^4. \end{aligned} \tag{4.4}$$

This completes the proof of Lemma 4.1.

Lemma 4.2 *Let (u, b, P) be the strong solution to the problem (1.1) on $\mathbb{R}^3 \times (0, T)$. Then, there exists a positive constant ε_0 independent of T, α , and β such that*

$$\sup_{t \in [0, T]} \|(\nabla u, \nabla b)\|_{L^2}^2 \leq 2 \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1}, \tag{4.5}$$

provided that

$$E_0 \left(\|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) \leq \varepsilon_0. \tag{4.6}$$

Proof. As stated in (4.2), $E_1(t) \triangleq \sup_{s \in [0, t]} \|(\nabla u, \nabla b)\|_{L^2}^2$, which implies that $E_1(t)$ is a continuous function on $[0, T]$. By (4.3), there is an absolute constant M such that

$$E_1(t) \leq \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1} + ME_0 E_1^2(t). \tag{4.7}$$

We define

$$T_* \triangleq \max \left\{ t \in [0, T] : E_1(s) \leq 4 \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1}, \forall s \in [0, t] \right\}, \tag{4.8}$$

and assume that

$$ME_0 \left(\left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) \leq \frac{1}{8}. \tag{4.9}$$

Now, we claim that

$$T_* = T.$$

Otherwise, which implies $T_* \in (0, T)$. Applying the continuity of $E_1(t)$ and (4.7) - (4.9), one has

$$\begin{aligned} E_1(T_*) &\leq \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} + ME_0 E_1^2(T_*) \\ &\leq \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \\ &\quad + ME_0 E_1(T_*) \left(4 \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) \\ &\leq \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} + \frac{1}{2} E_1(T_*), \end{aligned}$$

which implies that

$$E_1(T_*) \leq 2 \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{2}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{2}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1},$$

this makes a contradiction with (4.8).

Taking $\varepsilon_0 = \frac{1}{8M}$, we deduce that

$$E_1(t) \leq 2 \left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{2}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{2}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1}, \quad 0 < t < T,$$

provided that (4.6) holds true. Thus completes the proof of Lemma 4.2.

Next, we present the proof of Theorem 1.1.

Proof of Theorem 1.1 As stated in Lemma 4.2, ε_0 is constant and assume that $u_0, b_0 \in H^1(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, and

$$E_0 \left(\left\| (\nabla u_0, \nabla b_0) \right\|_{L^2}^2 + \frac{1}{\alpha+1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) \leq \varepsilon_0.$$

In Theorem 3.1, we obtain a unique local strong solution (u, b, P) to the Cauchy problem (1.1). Assume that T_* is the maximal existence time to solution. Now, we shall prove that $T_* = \infty$ by contradiction. According to the regularity criteria obtained by Zhou in [3], we can deduce similarly that problem (1.2) has regularity criteria as follows: if $u(t, x), b(t, x)$ satisfies $u, b \in L^s(0, T; L^r)$ with $\frac{2}{s} + \frac{3}{r} \leq 1, 3 < r < \infty$ or $\nabla u, \nabla b \in L^{\tilde{s}}(0, T; L^{\tilde{r}})$ with $\frac{2}{\tilde{s}} + \frac{3}{\tilde{r}} \leq 1, 3 < \tilde{r} < \infty$. As-

sume that $T_* < \infty$, it follows that for any (s, r) with $\frac{2}{s} + \frac{3}{r} \leq 1, 3 < r < \infty$,

$$\int_0^{T_*} \|(u, b)\|_{L^r}^s dt = \infty,$$

using Sobolev inequality gives

$$\int_0^{T_*} \|(\nabla u, \nabla b)\|_{L^2}^4 dt = \infty. \tag{4.10}$$

For any $0 < T < T_*$, it holds from Lemma 4.2 that

$$\sup_{t \in [0, T]} \|(\nabla u, \nabla b)\|_{L^2}^2 \leq 2 \|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1}. \tag{4.11}$$

Thus, we deduce from (1.5) that

$$\begin{aligned} & \int_0^{T_*} \|(\nabla u, \nabla b)\|_{L^2}^4 dt \\ & \leq 4 \left(\|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta + 1} \|b_0\|_{L^{\beta+1}}^{\beta+1} \right) T_* \\ & \leq 4 \left(\|(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \frac{1}{\alpha + 1} \|u_0\|_{L^2}^{\frac{5-\alpha}{2}} \| \nabla u_0 \|_{L^2}^{\frac{3\alpha-3}{2}} + \frac{1}{\alpha + 1} \|b_0\|_{L^2}^{\frac{5-\beta}{2}} \| \nabla b_0 \|_{L^2}^{\frac{3\beta-3}{2}} \right) T_* \\ & < +\infty, \end{aligned}$$

this contradicts with (4.10). This contradiction implies that $T_* = \infty$ and we obtain the global strong solution. This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

In this section, we are devoted to the proof of Theorem 1.2 by using property of the heat equation. Assume that v, h are the solution of the following equation:

$$v_t - \Delta v = 0, v(0, x) = u_0, h_t - \Delta h = 0, h(0, x) = b_0. \tag{5.1}$$

If $u_0(x), b_0(x) \in L^1(\mathbb{R}^3)$, we obtain from (2.3) that

$$\|v\|_{L^2} \leq Ct^{-\frac{3}{4}} \|u_0\|_{L^1}, \|h\|_{L^2} \leq Ct^{-\frac{3}{4}} \|b_0\|_{L^1}. \tag{5.2}$$

Let $w = u - v, z = b - h$, then w, z satisfy the following equations:

$$\begin{cases} \partial_t w - \Delta w + u \cdot \nabla u + |u|^{\alpha-1} u + \nabla P = b \cdot \nabla b, \\ \partial_t z - \Delta z + u \cdot \nabla b + |b|^{\beta-1} b = b \cdot \nabla u, \\ \operatorname{div} w = \operatorname{div} z = 0, \\ w(0, x) = 0, z(0, x) = 0. \end{cases} \tag{5.3}$$

Proof of Theorem 1.2 Testing (5.3)_{1,2} by w, z , respectively, and adding the resultant equations, it follows by integrating over \mathbb{R}^3 that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(w, z)\|_{L^2}^2 + \|(\nabla w, \nabla z)\|_{L^2}^2 \\ & = - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u^j w^j dx + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i b^j z^j dx - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i b^j z^j dx \\ & \quad + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i u^j z^j dx - \int_{\mathbb{R}^3} |u|^{\alpha-1} u \cdot w dx - \int_{\mathbb{R}^3} |b|^{\beta-1} b \cdot z dx + \int_{\mathbb{R}^3} \nabla P \cdot w dx \\ & \leq C \|\nabla v\|_{L^\infty} \|(u, b)\|_{L^2}^2 + C \|\nabla h\|_{L^\infty} \|u\|_{L^2}^2 \|b\|_{L^2}^2 - \frac{1}{2} \|u\|_{L^{\alpha+1}}^{\alpha+1} - \frac{1}{2} \|b\|_{L^{\beta+1}}^{\beta+1} + C \|v\|_{L^{\alpha+1}}^{\alpha+1} + C \|h\|_{L^{\beta+1}}^{\beta+1}, \end{aligned} \tag{5.4}$$

where we have used the facts that

$$\begin{aligned} & \left| -\sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u^j w^j dx - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i b^j z^j dx \right| + \left| \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i b^j w^j dx + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i u^j z^j dx \right| \\ &= \left| -\sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u^j (u^j - v^j) dx - \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i b^j (b^j - h^j) dx \right| \\ & \quad + \left| \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i b^j (u^j - v^j) dx + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i u^j (b^j - h^j) dx \right| \\ &\leq \left| \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i v^j u^j dx + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i h^j b^j dx \right| + \left| \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i v^j b^j dx + \sum_{i,j=1}^3 \int_{\mathbb{R}^3} b_i \partial_i h^j u^j dx \right| \\ &\leq C \|\nabla v\|_{L^\infty} \|(u, b)\|_{L^2}^2 + C \|\nabla h\|_{L^\infty} \|u\|_{L^2}^2 \|b\|_{L^2}^2, \end{aligned}$$

and by the definition of w, z gives

$$\begin{aligned} & -\int_{\mathbb{R}^3} |u|^{\alpha-1} u \cdot w dx - \int_{\mathbb{R}^3} |b|^{\beta-1} b \cdot z dx \\ &= -\int_{\mathbb{R}^3} |u|^{\alpha-1} u \cdot (u - v) dx - \int_{\mathbb{R}^3} |b|^{\beta-1} b \cdot (b - h) dx \\ &= -\|u\|_{L^{\alpha+1}}^{\alpha+1} - \|b\|_{L^{\beta+1}}^{\beta+1} + \int_{\mathbb{R}^3} |u|^{\alpha-1} u \cdot v dx + \int_{\mathbb{R}^3} |b|^{\beta-1} b \cdot h dx \\ &\leq -\frac{1}{2} \|u\|_{L^{\alpha+1}}^{\alpha+1} - \frac{1}{2} \|b\|_{L^{\beta+1}}^{\beta+1} + C \|v\|_{L^{\alpha+1}}^{\alpha+1} + C \|h\|_{L^{\beta+1}}^{\beta+1}. \end{aligned}$$

Applying Lemma 2.4, one has

$$\|\nabla v\|_\infty, \|\nabla h\|_\infty \leq Ct^{-2}, \|v\|_{L^{\alpha+1}} \leq Ct^{\frac{3}{2}(1-\frac{1}{\alpha+1})}, \|h\|_{L^{\beta+1}} \leq Ct^{\frac{3}{2}(1-\frac{1}{\beta+1})}.$$

Then, it follows from (5.4) that

$$\begin{aligned} & \frac{d}{dt} \|(w, z)\|_{L^2}^2 + 2\|(\nabla w, \nabla z)\|_{L^2}^2 + \|u\|_{L^{\alpha+1}}^{\alpha+1} + \|b\|_{L^{\beta+1}}^{\beta+1} \\ &\leq C(1+t)^{-2} \|(u, b)\|_{L^2}^2 + C(1+t)^{-\min\{\frac{3\alpha}{2}, \frac{3\beta}{2}\}}, \quad \forall t > 1. \end{aligned} \tag{5.5}$$

Since $\|(u, b)\|_{L^2} \leq C$, using Fourier splitting method (see [6]), we have

$$\|(w, z)\|_{L^2}^2 \leq C(1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}}, \quad \forall t > 1. \tag{5.6}$$

According to the analysis above, it directly follows that

$$\begin{aligned} \|(u, b)\|_{L^2}^2 &= \|(v + w, h + z)\|_{L^2}^2 \leq C \|v\|_{L^2}^2 + C \|w\|_{L^2}^2 + C \|h\|_{L^2}^2 + C \|z\|_{L^2}^2 \\ &\leq C(1+t)^{\frac{3}{2}} + C(1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}} \\ &\leq C(1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}}, \quad \forall t > 1. \end{aligned} \tag{5.7}$$

Next, we shall find sharper estimates by using this first preliminary decay to bootstrap. Summing up (5.5) and (5.7) gives

$$\begin{aligned} & \frac{d}{dt} \|(w, z)\|_{L^2}^2 + 2\|(\nabla w, \nabla z)\|_{L^2}^2 + \|u\|_{L^{\alpha+1}}^{\alpha+1} + \|b\|_{L^{\beta+1}}^{\beta+1} \\ &\leq C(1+t)^{-2} (1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}} + C(1+t)^{-\min\{\frac{3\alpha}{2}, \frac{3\beta}{2}\}}, \quad \forall t > 1, \end{aligned} \tag{5.8}$$

from which we infer that

$$\begin{aligned} \|(w, z)\|_{L^2}^2 &\leq C(1+t)^{-1} (1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}} + C(1+t)^{-\min\{\frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}} \\ &\leq C(1+t)^{-\min\{1, \frac{3\alpha-2}{2}, \frac{3\beta-2}{2}\}}, \quad \forall t > 1. \end{aligned} \quad (5.9)$$

By the definition of w, z and the bootstrap argument, we finally get the same decay rate to (5.7). This completes the proof of Theorem 1.2.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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