Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(11), 3739-3764
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Constitutive Theories for Linear Micromorphic Polymeric Solids

Karan S. Surana[1*], Sri Sai Charan Mathi[2]

[1]Department of Mechanical Engineering, University of Kansas, Lawrence, KS, USA
[2]Trane Technologies, La Crosse, WI, USA
Email: *kssurana@ku.edu

How to cite this paper: Surana, K.S. and Mathi, S.S.C. (2025) Constitutive Theories for Linear Micromorphic Polymeric Solids. *Journal of Applied Mathematics and Physics*, **13**, 3739-3764.
https://doi.org/10.4236/jamp.2025.1311209

Received: September 3, 2025
Accepted: November 3, 2025
Published: November 6, 2025

Abstract

In this paper, we consider the derivation of constitutive theories for a linear micromorphic polymeric solid medium in which the microconstituents, the solid medium and the interaction of the microconstituents with the solid medium have mechanisms of elasticity, dissipation and rheology. Thermodynamically and mathematically consistent conservation and balance laws derived by Surana *et al.* in a recent paper for linear micromorphic solids are utilized in the present work. The conjugate pairs in the entropy inequality, in conjunction with the axiom of causality, are used in establishing constitutive tensors and the initial choice of argument tensors. These are modified or augmented to incorporate a more comprehensive ordered rate mechanism of dissipation and rheology for the microconstituents, the medium, and the interaction of the microconstituents with the solid medium. The constitutive theories presented in the paper provide spectra of viscosities and relaxation times. Constitutive theories and the material coefficients are derived using the representation theorem based on integrity. Simplified forms of the constitutive theories are also presented. It is shown that the complete mathematical model, consisting of the conservation and balance laws and the constitutive theories, has closure without the use of the conservation of microinertia law advocated and used by Eringen and another additional balance law also used by Eringen to obtain six equations needed for closure; both of these laws are outside the thermodynamic framework.

Keywords

Linear Micromorphic, Microcontinuum, Conservation and Balance Laws, Constitutive Theories, Integral-Average, Integrity, Representation Theorem, Dissipation, Memory, Rheology, Polymeric Solid

1. Introduction and Scope of Work

Surana *et al.* in reference [1] presented the derivation of conservation and balance laws for a linear micromorphic elastic solid, including constitutive theories. The authors also presented a review of pertinent published works on 3M microcontinuum theories [2]-[32]. Discussions of the works in these references are omitted in this paper for the sake of brevity; the interested reader can refer to [1]. The authors in reference [1] concluded that the works of Eringen and Eringen *et al.* [7]-[24] are viewed as the most prominent and complete works on 3M theories. There are many other works besides these, but all such works primarily follow Eringen's approach with minor derivations. Surana *et al.* in references [33]-[55] have considered many aspects of micropolar theory, in which the authors have pointed out issues and concerns related to similar theories published by Eringen and Eringen *et al.* The authors have presented many model problem studies in their works to demonstrate the validity of their published theories. The issues and concerns pointed out by Surana *et al.* in connection with micropolar theories become far more serious in the case of micromorphic theories. The authors showed that the derivation of the conservation and balance laws for linear micromorphic theory presented in reference [1] is thermodynamically and mathematically consistent. Surana *et al.* also compared the derivation presented in reference [1] with the works of Eringen on linear micromorphic theory for a solid medium and provided details of the differences between the two theories. It was clearly established in reference [1] that Eringen's work on micromorphic theories has many issues and concerns: incorrect definitions and concepts, incorrect derivation of some balance laws, incorrect integral-average definitions, unjustifiable use of weighted integrals in deriving balance laws; the use of constitutive tensors and their argument tensors and the approach of deriving constitutive theories that cannot be supported by the well-established representation theorem, and the use of conservation of micro inertia, etc.

The conservation and balance laws presented in reference [1] do not have any of these issues and concerns. Their derivation abides by the laws of thermodynamics and well-established concepts in applied mathematics, such as the theory of isotropic tensors. The linear micromorphic theory [1] is shown to be thermodynamically and mathematically consistent.

In this paper, we utilize the conservation and balance laws derived by Surana *et al.* [1] for a linear micromorphic elastic solid medium to present the derivation of constitutive theories for a linear micromorphic polymeric solid medium with elasticity, dissipation and rheology. The microconstituents, the solid medium, and the interaction of the microconstituents with the solid medium all possess elasticity, dissipation, and rheology mechanisms. All three dissipation and rheology mechanisms are described by ordered rate theories in which the constitutive tensors and/or their argument tensors can consist of rates of up to any desired orders. This approach leads to spectra of dissipation coefficients and spectra of relaxation times for the microconstituents, the solid medium, and the interaction of the microconstituents with the solid medium. The initial determination of constitutive tensors and their argument tensors is made using the conjugate pairs in the entropy inequality, which are modified to incorporate more comprehensive dissipation and rheology mechanisms. Constitutive theories are derived using the representation theorem based on integrity (the complete basis of the spaces of constitutive tensors). Material coefficients are derived in each case. Simplified forms of the constitutive theories are also presented. It is shown that the complete mathematical model, consisting of conservation and balance laws and the constitutive theories, has closure without the use of the conservation of micro inertia conservation law advocated and used by Eringen and is thermodynamically and mathematically consistent.

2. Micro and Macro Deformations, Preliminary Considerations and Various Measures

In an isotropic, homogeneous solid matter containing microconstituents, the macro deformation of the solid medium is influenced by the micro deformation of the microconstituents. Thus, there needs to be a mechanism through which macro deformation can be modified depending upon the specific nature of micro deformation. In 3M microcontinuum theories, we precisely try to accomplish this. Consideration of each microconstituent with different position coordinates within the volume of matter and establishing its deformation physics is a formidable task. Instead, we consider a more practical approach. Since material points in continuum mechanics are finite volumes, we consider subdivision of the volume of matter in material points. We further assume that each material point has finite number of microconstituents that have their own position coordinates. Consideration of each microconstituent deformation within the material point is no simpler than the original problem of the entire volume of matter with the microconstituents. To make this problem tractable, we assume that the deformation of all microconstituents at the material point is some statistically averaged deformation. We further assume that there exists a surrogate configuration of microconstituents in which each microconstituent has the same response at the material point; furthermore, this response is also the same as statically averaged response of the original configuration of microconstituents in the volume of the material point. With these assumptions, we only need to consider the micro deformation of one surrogate microconstituent and its influence at the center of mass of the material point.

Referring to **Figure 1**, $V+\partial V$ and $\bar{V}+\partial\bar{V}$ are undeformed and deformed volumes of a material with the center of mass at P and $\bar{P}$ in the reference and the current configurations. $\boldsymbol{x}$ locates the center of mass of the material point in the fixed x-frame and $\boldsymbol{x}^{(\alpha)}$ and $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ locate the microconstituent "α" with respect to the fixed x-frame and with respect to the center of mass of the material point in the reference configuration. Likewise, $\bar{\boldsymbol{x}},\bar{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ are the corresponding quantities in the current configuration. $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$ and $\underset{\sim}{\bar{\boldsymbol{x}}}^{(\alpha)}$ are called directors in the reference and the current configurations. The deformation of $\underset{\sim}{\boldsymbol{x}}^{(\alpha)}$

(or $\bar{\underset{\sim}{x}}^{(\alpha)}$) characterizes the microdeformation of the microconstituent α. Since in this theory there is only director in the material point, based on Eringen [23] [24], the microcontinuum theory is referred to as a microcontinuum theory of grade one. A microcontinuum theory of grade $\tilde{n}$ will have $\tilde{n}$ directors in each material point. The rationale presented for one director applies to $\tilde{n}$ groups of microconstituents leading to $\tilde{n}$ directors. Using details in **Figure 1**, Surana *et al.* [33] have presented the derivation of nonlinear deformation measures for 3M theories. These measures have been used by the authors to present: a nonlinear micropolar theory for thermoelastic solid matter, a nonlinear micropolar theory for thermoviscoelastic solids [56], and for polymeric solids [56]. More recently, the authors have also presented a linear micromorphic theory for thermoelastic solid matter [1].

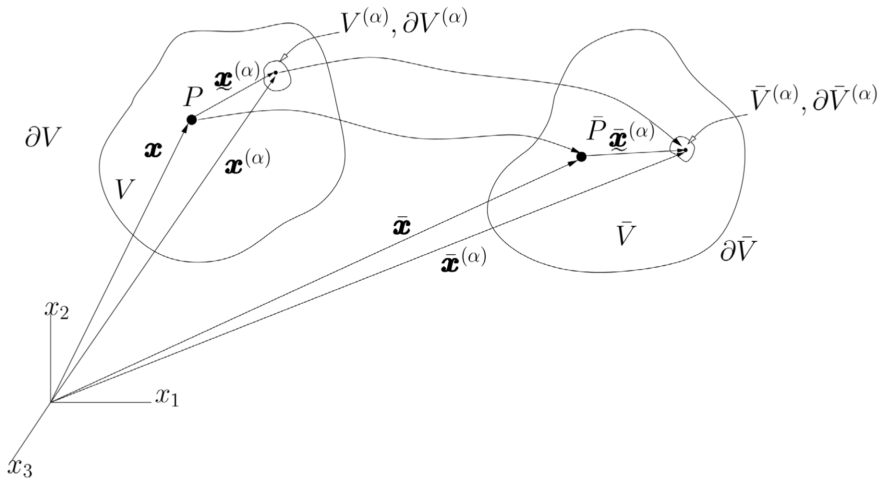

Figure 1. Undeformed and deformed configurations of material point volume.

In all of these works by Surana *et al.* and also the works [33]-[55], there are some common concepts used that are different from those in the published works on 3M theories. These are described here: 1) It is now quite well established in the published works of Surana *et al.* that classical rotations ${}_c\boldsymbol{\Theta}$ (due to $\nabla \times \boldsymbol{u} = \boldsymbol{e}_i \times {}_c\Theta_i$) describe rigid rotations of the microconstituents in the 3M theories, thus eliminating the need for ${}_\alpha\boldsymbol{\Theta}$ rotations as unknown rotational degrees of freedom for the microconstituents. 2) In microcontinuum theories, rotations and moments constitute a new kinematically conjugate pair in addition to force and displacement. It has been shown [45] [55] [57] that this new kinematic pair requires a new balance law, the "balance of moment of moments" that is not used in any of the published works. This has led to spurious constitutive theories. 3) Mathematical consistency of the constitutive theories is always ensured when the constitutive theories are derived using the representation theorem. 4) In the present work, the degrees of freedom for microconstituents are known rigid rotations ${}_c\boldsymbol{\Theta}$ and six independent components of ${}^d_s\boldsymbol{J}^{(\alpha)}$ (unknown), constituting a total of nine degrees of freedom in which only six are unknown. The rationale for this choice and a comparison with Eringen's work in which all nine components of $\boldsymbol{J}^{(\alpha)}$ (unknown) together with unknown rigid rotations ${}_{\alpha}\boldsymbol{\Theta}$ are microconstituent degrees of freedom, is presented in reference [1]. 5) Additive decomposition of the stress tensor essential to establish correct constitutive tensors are never considered in Eringen's works. A consequence of this is a wrong choice of stress constitutive tensor(s).

3. Conservation and Balance Laws for Linear Micromorphic Continua

The derivations of the conservation and balance laws for a linear micromorphic microcontinuum solid medium have been presented by Surana *et al.* [1]. The derivation is initiated by applying the conservation and balance laws of classical continuum mechanics to the microconstituents. The resulting equations for the microconstituents are used to define "integral-average" definitions for the macro continua and then used to derive the macro conservation and balance laws. The derivation presented in reference [1] shows major differences between the approach used by Surana *et al.* compared to Eringen [7]-[24]. A discussion of these differences has been presented in reference [1] to point out major weaknesses in the micromorphic theory presented by Eringen [7]-[24]. In this paper, we follow the conservation and balance laws for a linear micromorphic solid medium presented in reference [1] in the Lagrangian description. These are given in the following.

Conservation of mass, balance of linear momenta, balance of angular momenta, first and second law of thermodynamics and balance of moment of moments are given in the following.

$$\rho_0\left(\boldsymbol{x}\right)=\left|\boldsymbol{J}\right|\rho\left(\boldsymbol{x},t\right) \tag{1}$$

$$\rho_0 a_k-\rho_0^b F_k-\sigma_{lk,l}=0 \tag{2}$$

$$\epsilon_{nmk}\left(\sigma_{mk}\pm S_{mk}\right)+m_{lk,l}=0 \tag{3}$$

$$\rho_0\dot{e}-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}-\nabla\cdot\boldsymbol{q}-\left({}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}\right)=0 \tag{4}$$

$$\rho_0\left(\dot{\phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\left({}_{c}\dot{\boldsymbol{\Theta}}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}\right)\leq 0 \tag{5}$$

$$\epsilon_{ijk}m_{ij}=0 \tag{6}$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3) and the energy equation (1) in thirty-four dependent variables: $\boldsymbol{u}(3),\boldsymbol{\sigma}(9),\boldsymbol{S}(6),\boldsymbol{m}(6),\boldsymbol{q}(3),\theta(1),{}_{s}\boldsymbol{J}^{(\alpha)}(6)$. Thus, additional twenty-seven equations are needed for closure. Constitutive theories provide twenty-one equations: $\boldsymbol{\sigma}(6),\boldsymbol{S}(6),\boldsymbol{m}(6),\boldsymbol{q}(3)$. Thus, additional six equations are needed for closure of the mathematical model. These are discussed in the following.

Eringen and Eringen *et al.* [7]-[24] proposed a new balance law to obtain additional six equations needed for closure of the mathematical model. He proposed that the sum of the symmetric part of stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ must be equal to the gradients of the symmetric part of the moment tensor. First, this balance statement is not part of thermodynamics, hence cannot be used in this thermodynamic framework. Secondly, the moment tensor in Eringen's work is non-symmetric due to omission of balance of moment of moment balance law necessitated by the new kinematically conjugate pair of rotations and moments [45] [55] [57] in all 3M microcontinuum theories. The moment tensor definition itself is not valid as it is based on $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$, the microconstituent Cauchy stress (due to classical continuum mechanics). Thus, both balance of angular momenta as well as the proposed new balance law are in error as moment tensor is invalid due to incorrect definition. These details and more explanation are provided by the authors in reference [1].

Surana *et al.* [1] showed that balance of angular momenta in fact contains nine equations, unfortunately, six of them related to $\boldsymbol{S}$ and symmetric part of $\boldsymbol{S}$ are eliminated due to the presence of permutation on the left hand side of the balance of angular momenta. The authors showed that by premultiplying balance of angular momenta with the inverse of the permutation tensor the following six equations can be obtained in Eulerian description.

$$_s\bar{\sigma}_{mk} - \bar{S}_{mk} = 0 \tag{7}$$

In the Lagrangian description, we have

$$_s\sigma_{mk} - S_{mk} = 0 \tag{8}$$

Refer to reference [1] for the details of the derivation.

Thus, we see that in this approach, a new balance law is not needed for obtaining additional equations. Equation (8) are part of the thermodynamic framework as these are derived from the balance of angular moment. It is worth noticing that balance of angular momenta contains quantities purely related to nonclassical physics, whereas (8) contains classical physics only. Classical and nonclassical physics are not intermixed in the balance of angular momenta and the additional equations (8) derived from the balance of angular momenta. This is not the case in Eringen's work, a cause for concern.

4. Constitutive Theories for Linear Micromorphic Polymeric Solids

In deriving the constitutive theories, we consider comprehensive material behavior. We assume that the medium is elastic, and has a dissipation mechanism, as well as rheology. The microconstituents naturally have elasticity but are assumed to have their own dissipation and rheology mechanisms. Additionally, rigid rotations of the microconstituents in a viscous medium with long chain molecules create dissipation and relaxation physics, *i.e.*, upon cessation of disturbance, the rotated microconstituents are not able to return to their original state immediately due to viscous drag and interference with long chain molecules. This physics results in a dissipation mechanism and relaxation for the Cauchy moment tensor. Thus, in the derivation of the constitutive theory for $\boldsymbol{\sigma}, \boldsymbol{S}$ and $\boldsymbol{m}$, elasticity, dissipation and rheology physics are considered. Furthermore, all three dissipation and relaxation mechanisms are due to ordered rate constitutive theories, hence resulting in dissipation and relaxation time spectra for each of the three constitutive theories corresponding to the rates considered.

4.1. Initial Determination of Constitutive Tensors and Their Argument Tensors

In deriving constitutive theories, we always begin with the rate of work or otherwise conjugate pairs in the entropy inequality for determination of constitutive tensors based on the causality axiom of constitutive theory and their possible argument tensors. The choice of constitutive tensors can be altered or changed if the physics requires it, and the argument tensors of the constitutive tensors can be augmented with additional tensors if the physics requiring this has not been considered while deriving the entropy inequality. We follow the details and the guidelines presented in references [58] [59]. Once the constitutive tensors and their argument tensors are established, we follow the theory of isotropic tensors or the representation theorem in deriving the constitutive theories and the standard procedure of Taylor series expansion of the coefficients used in the linear combination of the basis of the space of the constitutive tensor [58] [59] to determine material coefficients.

Consider entropy inequality (5)

$$\rho_0\left(\dot{\phi}+\eta\dot{\theta}\right)-\boldsymbol{\sigma}:\dot{\boldsymbol{J}}-\boldsymbol{S}:\dot{\boldsymbol{J}}^{(\alpha)}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}-\left({}_c\dot{\Theta}\cdot\left(\nabla\cdot\boldsymbol{m}\right)+\boldsymbol{m}:{}^{c\Theta}\dot{\boldsymbol{J}}\right)\leq 0 \tag{9}$$

The macro stress tensor $\boldsymbol{\sigma}$ is nonsymmetric, and hence cannot be a constitutive tensor due to the representation theorem [60]-[71]. Thus, we need additive decomposition of $\boldsymbol{\sigma}$ into the symmetric tensor ${}_s\boldsymbol{\sigma}$ and the skew-symmetric tensor ${}_a\boldsymbol{\sigma}$. There cannot be a constitutive theory for ${}_a\boldsymbol{\sigma}$, as it is defined by the gradients of the Cauchy moment tensor due to the balance of angular momenta. Thus, ${}_s\boldsymbol{\sigma}$ is the constitutive tensor and not $\boldsymbol{\sigma}$ or ${}_a\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma}={}_s\boldsymbol{\sigma}+{}_a\boldsymbol{\sigma} \tag{10}$$

Secondly

$$\dot{\boldsymbol{J}}={}^d\dot{\boldsymbol{J}}={}^d_s\dot{\boldsymbol{J}}+{}^d_a\dot{\boldsymbol{J}}=\dot{\boldsymbol{\varepsilon}}+{}^d_a\dot{\boldsymbol{J}} \tag{11}$$

in which ${}^d\boldsymbol{J}$ is displacement gradient tensor and ${}^d_s\boldsymbol{J}$ and ${}^d_a\boldsymbol{J}$ are symmetric and skew-symmetric tensors due to ${}^d\boldsymbol{J}$ obtained by additive decomposition of ${}^d\boldsymbol{J}$.

$${}^d\boldsymbol{J}={}^d_s\boldsymbol{J}+{}^d_a\boldsymbol{J} \tag{12}$$

Likewise, additive decomposition of ${}^{c\Theta}\boldsymbol{J}$ and $J^{(\alpha)}$ into symmetric and skew-symmetric tensors gives:

$${}^{c\Theta}\boldsymbol{J}={}^{c\Theta}_s\boldsymbol{J}+{}^{c\Theta}_a\boldsymbol{J} \tag{13}$$

Also

$$\dot{\boldsymbol{J}}^{(\alpha)}={}^d\dot{\boldsymbol{J}}^{(\alpha)} \tag{14}$$

$$\text{and}\quad {}^{d}\boldsymbol{J}^{(\alpha)} = {}^{d}_{s}\boldsymbol{J}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} \tag{15}$$

in which ${}^{d}\boldsymbol{J}^{(\alpha)}$ is micro displacement gradient tensor and ${}^{d}_{s}\boldsymbol{J}^{(\alpha)}$ and ${}^{d}_{a}\boldsymbol{J}^{(\alpha)}$ are symmetric and skew-symmetric tensors due to additive decomposition of ${}^{d}\boldsymbol{J}^{(\alpha)}$. Furthermore,

$${}^{d}\boldsymbol{J}^{(\alpha)} = {}^{d}_{s}\boldsymbol{J}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} = \boldsymbol{\varepsilon}^{(\alpha)} + {}^{d}_{a}\boldsymbol{J}^{(\alpha)} \tag{16}$$

Also

$${}^{c\Theta}\dot{\boldsymbol{J}} = {}^{c\Theta}_{s}\dot{\boldsymbol{J}} + {}^{c\Theta}_{a}\dot{\boldsymbol{J}} \tag{17}$$

Substituting (10) - (17) as needed in the entropy inequality (5) and noting that

$${}_{s}\boldsymbol{\sigma} : {}^{d}_{a}\dot{\boldsymbol{J}} = 0;\ \ {}_{a}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = 0;\ \ \boldsymbol{S} : {}^{d}_{a}\dot{\boldsymbol{J}}^{(\alpha)} = 0;\ \ \boldsymbol{m} : {}^{c\Theta}_{a}\dot{\boldsymbol{J}} = 0 \tag{18}$$

We can write (5) as follows:

$$\rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{a}\boldsymbol{\sigma} : {}^{d}_{a}\dot{\boldsymbol{J}} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} + {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\nabla \cdot \boldsymbol{m}\right) - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{19}$$

From balance of angular momenta

$$\nabla \cdot \boldsymbol{m} = -\epsilon : \boldsymbol{\sigma} \tag{20}$$

Substituting (20) in (19)

$$\rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_{a}\boldsymbol{\sigma} : \left({}^{d}_{a}\dot{\boldsymbol{J}}\right) - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\epsilon : \boldsymbol{\sigma}\right) - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{21}$$

A simple calculation shows that

$${}_{a}\boldsymbol{\sigma} : {}^{c\Theta}_{a}\dot{\boldsymbol{J}} = {}_{c}\dot{\boldsymbol{\Theta}} \cdot \left(\epsilon : \boldsymbol{\sigma}\right) \tag{22}$$

Using (22) in (21), (22) reduces to

$$\rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{23}$$

Further additive decomposition of ${}_{s}\boldsymbol{\sigma}$ into equilibrium ${}^{e}_{s}\boldsymbol{\sigma}$ and deviatoric stress ${}^{d}_{s}\boldsymbol{\sigma}$ is needed to derive constitutive theory for volumetric and distortional deformation physics that are mutually exclusive

$${}_{s}\boldsymbol{\sigma} = {}^{e}_{s}\boldsymbol{\sigma} + {}^{d}_{s}\boldsymbol{\sigma} \tag{24}$$

Substituting (24) in (23)

$$\rho_0\left(\dot{\phi} + \eta\dot{\theta}\right) - {}^{e}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}^{d}_{s}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : {}^{c\Theta}_{s}\dot{\boldsymbol{J}} - \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{25}$$

The rate of work conjugate pairs and the last term in (25) suggest in conjunction with the axiom of causality [58] [59], that ${}^{e}_{s}\boldsymbol{\sigma}, {}^{d}_{s}\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ are valid choices of constitutive tensors. The initial choice of constitutive tensors and their argument tensors is as follows (θ is included as an argument tensor in all constitutive tensors because of non-isothermal physics):

$${}^{e}_{s}\boldsymbol{\sigma} = {}^{e}_{s}\boldsymbol{\sigma}\left(\rho, \theta\right) \tag{26}$$

$${}^{d}_{s}\boldsymbol{\sigma} = {}^{d}_{s}\boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}, \theta\right) \tag{27}$$

$$\boldsymbol{S} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \theta\right) \tag{28}$$

$$\boldsymbol{m} = \boldsymbol{m}\left({}_{s}^{c}\Theta \boldsymbol{J}, \theta\right) \tag{29}$$

$$\boldsymbol{q} = \boldsymbol{q}\left(\boldsymbol{g}, \theta\right) \tag{30}$$

Even though we do not need a constitutive theory for Φ, its argument tensors are essential to establish as it is used to simplify the entropy inequality (25) as well as to derive the constitutive theory for ${}_{s}^{e}\boldsymbol{\sigma}$. Argument tensors of ${}_{s}^{e}\boldsymbol{\sigma}$ are not based on conjugate pair in (25), but are based on physics of volumetric deformation. As shown, (25) is not helpful in deriving constitutive theory for ${}_{s}^{e}\boldsymbol{\sigma}$. We need to use (25) in Eulerian description for this purpose. The presence of η in (25) must be addressed as well. The Helmholtz free energy density must depend on ρ and θ. In the Lagrangian description, ρ is not admissible as an argument tensor, but we use it in (26), (31), (32) in a symbolic sense. Other argument tensors of Φ and η are chosen based on the principle of equipresence. However, the principle of equipresence is not used in (26) - (30), as the conjugate pairs in the entropy inequality (25) clearly dictate their choices:

$$\Phi = \Phi\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}_{s}^{c}\Theta \boldsymbol{J}, \boldsymbol{q}, \theta\right) \tag{31}$$

$$\eta = \eta\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}_{s}^{c}\Theta \boldsymbol{J}, \boldsymbol{q}, \theta\right) \tag{32}$$

4.2. Constitutive Theory for Equilibrium Cauchy Stress Tensor ${}_{s}^{e}\boldsymbol{\sigma}$

In the Lagrangian description, density $\rho(\boldsymbol{x},t)$ is deterministic from the conservation of mass $\rho(\boldsymbol{x},t) = \dfrac{\rho_0}{|\boldsymbol{J}|}$ once the deformation gradient tensor $\boldsymbol{J}$ is known. Hence, the density $\rho(\boldsymbol{x},t)$ cannot be an argument tensor of the constitutive tensors [59]. However, compressibility and incompressibility physics are related to density and temperature. Thus, the constitutive theory for ${}_{s}^{e}\boldsymbol{\sigma}$ cannot be derived using the entropy inequality (25) in the Lagrangian description, instead we must consider an entropy inequality similar to (25) in the Eulerian description. The derivation that follows has been presented in references [1] [58] [59], but is necessary to include here for the sake of completeness of the constitutive theories.

$$\bar{\rho}\left(\dot{\bar{\Phi}} + \eta\dot{\bar{\theta}}\right) - {}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - {}_{s}^{d}\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}} : \left({}_{s}^{r}{}_{c}\bar{\Theta}\bar{\boldsymbol{J}}\right) + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{33}$$

In this case, $\bar{\rho}$ is unknown, and hence is a dependent variable in the mathematical model. Following the same procedure as in the Lagrangian description, the constitutive tensors and their argument tensors (including $\bar{\Phi}$ and $\bar{\eta}$) are given by:

$${}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)} = {}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho}, \bar{\theta}\right) \tag{34}$$

$${}_{s}^{d}\bar{\boldsymbol{\sigma}}^{(0)} = {}_{s}^{d}\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\theta}\right) \tag{35}$$

$$\bar{\boldsymbol{S}} = \bar{\boldsymbol{S}}\left(\bar{\boldsymbol{D}}^{(\alpha)}, \bar{\theta}\right) \tag{36}$$

$$\bar{\boldsymbol{m}} = \bar{\boldsymbol{m}}\left(\bar{\rho}, {}_{s}^{r}{}_{c}\bar{\Theta}\bar{\boldsymbol{J}}, \bar{\theta}\right) \tag{37}$$

$$\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}\left(\bar{\rho}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{38}$$

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{39}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}, \bar{\boldsymbol{g}}, \bar{\theta}\right) \tag{40}$$

We have used principle of equipresence for the argument tensors of $\bar{\Phi}$ and $\bar{\eta}$.

Using (39), we can write

$$\dot{\bar{\Phi}} = \frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\dot{\bar{\rho}} + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}} : \dot{\bar{\boldsymbol{D}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}^{(\alpha)}} : \dot{\bar{\boldsymbol{D}}}^{(\alpha)} + \frac{\partial\bar{\Phi}}{\partial\, {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}} : {}_{s}^{c}{}^{r}\dot{\bar{\Theta}}\boldsymbol{J} + \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\dot{\bar{\theta}} \tag{41}$$

From conservation of mass in Eulerian description

$$\dot{\bar{\rho}} = -\bar{\rho}\left(\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}}\right) = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\bar{D}_{kk}\delta_{lk} = \bar{\rho}\bar{\boldsymbol{D}} : \boldsymbol{\delta} \tag{42}$$

substituting from (42) for $\dot{\bar{\rho}}$ in (41) and then substituting (41) in (33), we obtain the following after regrouping the terms

$$\begin{aligned} &\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\boldsymbol{\delta} - {}_{s}^{e}\bar{\boldsymbol{\sigma}}^{(0)}\right) : \bar{\boldsymbol{D}} + \bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}} : \dot{\bar{\boldsymbol{D}}} + \bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}^{(\alpha)}} : \dot{\bar{\boldsymbol{D}}}^{(\alpha)} + \bar{\rho}\frac{\partial\bar{\Phi}}{\partial\left({}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}\right)} : {}_{s}^{c}{}^{r}\dot{\bar{\Theta}}\boldsymbol{J} \\ &+ \frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}} \cdot \dot{\bar{\boldsymbol{g}}} + \bar{\rho}\left(\bar{\eta} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\right)\dot{\bar{\theta}} - {}_{s}^{d}\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}} : {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \leq 0 \end{aligned} \tag{43}$$

The entropy inequality (43) holds for arbitrary but admissible choices of $\dot{\bar{\boldsymbol{D}}}, \dot{\bar{\boldsymbol{D}}}^{(\alpha)}, {}_{s}^{c}{}^{r}\dot{\bar{\Theta}}\boldsymbol{J}, \dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$ if the following conditions hold:

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}\left(\bar{\boldsymbol{D}}\right) \tag{44}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{D}}^{(\alpha)}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}\left(\bar{\boldsymbol{D}}^{(\alpha)}\right) \tag{45}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\, {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}\left({}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}\right) \tag{46}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{\boldsymbol{g}}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}\left(\bar{\boldsymbol{g}}\right) \tag{47}$$

$$\bar{\rho}\left(\bar{\eta} + \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}\right) = 0 \Rightarrow \bar{\eta} = -\frac{\partial\bar{\Phi}}{\partial\bar{\theta}} \tag{48}$$

Equations (44) - (48) imply that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{D}}, \bar{\boldsymbol{D}}^{(\alpha)}, {}_{s}^{c}{}^{r}\bar{\Theta}\boldsymbol{J}$ and $\bar{\boldsymbol{g}}$. Equation (48) implies that $\bar{\eta}$ is deterministic from $\bar{\Phi}$, hence $\bar{\eta}$ is not a constitutive or dependent variable. Using (44) - (48), the constitutive tensor and its argument tensors in (34) - (38) remain the same, but the argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ can be modified:

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\theta}\right) \tag{49}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\Phi}\right) \tag{50}$$

and the entropy inequality (43) reduces to

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_s^e\boldsymbol{\sigma}\right) : \bar{\boldsymbol{D}} - {}_s^d\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}}^{(0)} : {}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{51}$$

Constitutive theory for ${}_s^e\bar{\boldsymbol{\sigma}}^{(0)}$ for compressible matter can be obtained by setting coefficient of $\bar{\boldsymbol{D}}$ in the first term of (51) to zero.

$$ {}_s^e\bar{\boldsymbol{\sigma}}^{(0)}\left(\bar{\rho},\bar{\theta}\right) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} = \bar{p}\left(\bar{\rho},\bar{\theta}\right)\boldsymbol{\delta} \tag{52}$$

$$\bar{p}\left(\bar{\rho},\bar{\theta}\right) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{53}$$

in which $\bar{p}\left(\bar{\rho},\bar{\theta}\right)$ is thermodynamic pressure or equation of state for compressible matter. When the deforming matter is incompressible, there is no change in volume. Thus, for a fixed mass, the density is constant, *i.e.*, $\bar{\rho}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t) = \bar{\rho} = \rho_0$. For this case, from conservation of mass, we have:

$$\dot{\bar{\rho}} = -\bar{\rho}\left(\nabla \cdot \bar{\boldsymbol{v}}\right) = 0 \tag{54}$$

and

$$\frac{\partial \bar{\Phi}\left(\bar{\rho},\bar{\theta}\right)}{\partial \bar{\rho}} = \frac{\partial \bar{\Phi}\left(\rho_0,\theta\right)}{\partial \bar{\rho}} = 0 \tag{55}$$

Hence, for incompressible solid, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ cannot be derived using (52) and (53). First, using (55), the entropy inequality (51) reduces to

$$-{}_s^e\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}}^{(0)} : {}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{56}$$

In order to derive constitutive theory for ${}_s^e\bar{\boldsymbol{\sigma}}^{(0)}$ for incompressible solid matter, we must introduce incompressibility condition in (56). From continuity equation, the velocity field for incompressible matter is divergence free, *i.e.*,

$$\bar{\nabla} \cdot \bar{\boldsymbol{v}} = \bar{D}_{kk} = \bar{D}_{kk}\delta_{lk} = \boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0. \tag{57}$$

If (57) holds, then the following holds too:

$$\bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} : \bar{\boldsymbol{D}} = 0 \tag{58}$$

in which, $\bar{p}\left(\bar{\theta}\right)$ is a Lagrange multiplier. Adding (58) to (56) and regrouping terms

$$\left(\bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} - {}_s^e\bar{\boldsymbol{\sigma}}^{(0)}\right) : \bar{\boldsymbol{D}} - {}_s^d\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} - \bar{\boldsymbol{m}}^{(0)} : {}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}} \le 0 \tag{59}$$

Entropy inequality (59) holds for arbitrary but admissible $\bar{\boldsymbol{D}}$, if the coefficient of $\bar{\boldsymbol{D}}$ in the first term in (59) is set to zero, giving:

$$ {}_s^e\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}\left(\bar{\theta}\right)\boldsymbol{\delta} \tag{60}$$

The reduced form of entropy inequality is given by:

$$-{}_s^d\bar{\boldsymbol{\sigma}}^{(0)} : \bar{\boldsymbol{D}} - \bar{\boldsymbol{S}} : \bar{\boldsymbol{D}}^{(\alpha)} - \bar{\boldsymbol{m}}^{(0)} : {}_s^r{}_c^{\Theta}\bar{\boldsymbol{J}} + \frac{\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{g}}}{\bar{\theta}} \le 0 \tag{61}$$

In the Lagrangian description, the constitutive theory for ${}_s^e\boldsymbol{\sigma}$ can be obtained directly from (52), (53) and (60).

$$ {}_s^e\boldsymbol{\sigma}^{(0)} = p(\rho,\theta)\boldsymbol{\delta};\;\; p(\rho,\theta) = -\rho^2 \frac{\partial \Phi}{\partial \rho} \;\;(\text{compressible}) \tag{62} $$

$$ {}_s^e\boldsymbol{\sigma}^{(0)} = p(\theta)\boldsymbol{\delta} \;\;(\text{incompressible}) \tag{63} $$

The reduced form of entropy inequality in the Lagrangian description follows directly from (61).

$$ -{}_s^d\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \boldsymbol{m} : \left({}^{c\Theta}\dot{\boldsymbol{\varepsilon}} \right) + \frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta} \le 0 \tag{64} $$

In the following, we present derivation of constitutive theories for ${}_s^d\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ and $\boldsymbol{q}$ using representation theorem [60]-[71]. ${}_s^d\boldsymbol{\sigma}, \boldsymbol{S}, \boldsymbol{m}$ are symmetric tensors of rank two and their conjugate $\dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{\varepsilon}}^{(\alpha)}$ and ${}_s^{c\Theta}\dot{\boldsymbol{J}}$ are also symmetric tensors of rank two. $\boldsymbol{q}$ and $\boldsymbol{g}$ are tensors of rank one. Thus, there is no difficulty in deriving constitutive theories for all four constitutive tensors using the representation theorem. Furthermore, in the constitutive theories for ${}_s^d\boldsymbol{\sigma}, \boldsymbol{S}$ and $\boldsymbol{m}$, we consider elasticity, dissipation and rheology mechanisms. Dissipation and rheology mechanisms are ordered rate mechanisms, and hence yield dissipation and relaxation spectra in each constitutive theory.

4.3. Constitutive Theory for Cauchy Stress Tensor ${}_s^d\boldsymbol{\sigma}$

We consider the medium to be linear elastic. We begin with conjugate pair ${}_s^e\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ in the reduced form of the entropy inequality (64). This conjugate pair in conjunction with axiom of causality suggest that ${}_s^d\boldsymbol{\sigma}$ is the constitutive tensor and $\boldsymbol{\varepsilon}$ as its argument tensor. Thus, we can write (θ is included in the argument tensors due to non-isothermal physics)

$$ {}_s^d\boldsymbol{\sigma} = {}_s^d\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{65} $$

We know from the physics of viscous fluids that dissipation requires the strain rate, which is the same as the rate of strain in the Lagrangian description, thus $\dot{\boldsymbol{\varepsilon}}$ or $\boldsymbol{\varepsilon}_{(1)}$ should be an argument tensor of ${}_s^d\boldsymbol{\sigma}$. We generalize the dissipation mechanism by considering strain rates up to orders n, *i.e.*, by considering $\boldsymbol{\varepsilon}_{(i)}; i = 1, 2, \cdots, n$ as argument tensors of ${}_s^d\boldsymbol{\sigma}$. Rheology or memory requires the existence of memory modulus in the mathematical model, thus the constitutive theory for ${}_s^d\boldsymbol{\sigma}$ must at least be a first order differential equation in time, *i.e.*, ${}_s^d\boldsymbol{\sigma}$ and ${}_s^d\dot{\boldsymbol{\sigma}}$ (or ${}_s^d\boldsymbol{\sigma}^{(1)}$) must be considered in deriving the constitutive theory in which ${}_s^d\boldsymbol{\sigma}^{(1)}$ must be the constitutive tensor and ${}_s^d\boldsymbol{\sigma}$ as its argument tensor. We generalize this mechanism of rheology by considering rates of ${}_s^d\boldsymbol{\sigma}$ up to orders m, *i.e.*, we consider ${}_s^d\boldsymbol{\sigma}^{(j)}; j = 1, 2, \cdots, m$ in which ${}_s^d\boldsymbol{\sigma}^{(m)}$ must be the constitutive tensor and ${}_s^d\boldsymbol{\sigma}$ and ${}_s^d\boldsymbol{\sigma}^{(j)}; j = 1, 2, \cdots, m-1$ must be its argument tensors. Thus, finally, we have:

$$ {}_s^d\boldsymbol{\sigma}^{(m)} = {}_s^d\boldsymbol{\sigma}^{(m)}\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{(i)}, {}_s^d\boldsymbol{\sigma}, {}_s^d\boldsymbol{\sigma}^{(j)}, \theta \right); i = 1, 2, \cdots, n; j = 1, 2, \cdots, m-1 \tag{66} $$

${}_s^d\boldsymbol{\sigma}, {}_s^d\boldsymbol{\sigma}^{(j)}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{(i)}; i = 1, 2, \cdots, n; j = 1, 2, \cdots, m-1$ are symmetric tensors of rank two and θ is a tensor of rank zero. Thus, we can use representation theorem to derive constitutive theory for ${}_s^d\boldsymbol{\sigma}$.

Let ${}^\sigma\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^\sigma$ be the combined generators of the argument tensors of ${}_s^d\boldsymbol{\sigma}^{(m)}$ in (66) that are symmetric tensors of rank two and let ${}^\sigma\underset{\sim}{I}^j; j = 1,2,\cdots,M^\sigma$ be the combined invariants of the same argument tensors of ${}_s^d\boldsymbol{\sigma}^{(m)}$ in (66). Then, $\boldsymbol{I}, {}^\sigma\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^\sigma$ constitute the basis of the space of ${}_s^d\boldsymbol{\sigma}^{(m)}$, also referred to as integrity. Hence, we can express ${}_s^d\boldsymbol{\sigma}^{(m)}$ as a linear combination of the basis of its space in the current configuration.

$$
{}_s^d\boldsymbol{\sigma}^{(m)} = {}^\sigma\alpha^0\boldsymbol{I} + \sum_{i=1}^{N^\sigma} {}^\sigma\alpha^i\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right);\ {}^\sigma\underset{\sim}{\alpha}^i = {}^\sigma\underset{\sim}{\alpha}^i\left({}^\sigma\underset{\sim}{I}^j,\theta\right);\ i = 0,1,\cdots,N^\sigma; j = 1,2,\cdots,M^\sigma \quad (67)
$$

in which ${}^\sigma\alpha^i = {}^\sigma\alpha^i\left({}^\sigma\underset{\sim}{I}^j,\theta\right); j = 1,2,\cdots,M$ are coefficients in the linear combination (67). The material coefficients are determined by considering Taylor series expansion of ${}^\sigma\alpha^i; i = 0,1,\cdots,N$ in ${}^\sigma\underset{\sim}{I}^j$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^\sigma\underset{\sim}{I}^j; j = 1,2,\cdots,M$ and θ (for simplicity of resulting constitutive theory).

$$
{}^\sigma\underset{\sim}{\alpha}^i = {}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \left.\frac{\partial\, {}^\sigma\underset{\sim}{\alpha}^i}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j - {}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^\sigma\alpha^i}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right);\ i = 0,1,\cdots,{}^\sigma N \quad (68)
$$

Substituting ${}^\sigma\underset{\sim}{\alpha}^0$ and ${}^\sigma\underset{\sim}{\alpha}^i; i = 1,\cdots,{}^\sigma N$ from (68) into (67)

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{(m)} = &\left({}^\sigma\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \left.\frac{\partial\, {}^\sigma\underset{\sim}{\alpha}^0}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j - {}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^\sigma\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)\right)\boldsymbol{I} \\
&+ \sum_{i=1}^{N^\sigma}\left({}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \left.\frac{\partial\, {}^\sigma\underset{\sim}{\alpha}^i}{\partial\, {}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^\sigma\underset{\sim}{I}^j - {}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^\sigma\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)\right){}^\sigma\underset{\sim}{\boldsymbol{G}}^i
\end{aligned} \quad (69)
$$

Collecting coefficients of $\boldsymbol{I}, {}^\sigma\underset{\sim}{I}^j\boldsymbol{I}, {}^\sigma\underset{\sim}{\boldsymbol{G}}^i, {}^\sigma\underset{\sim}{I}^j\,{}^\sigma\underset{\sim}{\boldsymbol{G}}^i, \left(\theta - \theta|_{\underline{\Omega}}\right){}^\sigma\underset{\sim}{\boldsymbol{G}}^i$ and $\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I}$, we can write (69) as follows:

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma}^{(m)} = &\,\sigma_0\boldsymbol{I} + \sum_{i=1}^{N^\sigma} {}^\sigma\underset{\sim}{a}_j\left({}^\sigma\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{j=1}^{M^\sigma} {}^\sigma\underset{\sim}{b}_j\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{j=1}^{M^\sigma}\sum_{i=1}^{N^\sigma} {}^\sigma\underset{\sim}{c}_{ij}\left({}^\sigma\underset{\sim}{I}^j\right)\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right) \\
&- \sum_{i=1}^{N^\sigma} {}^\sigma\underset{\sim}{d}^i\left(\theta - \theta|_{\underline{\Omega}}\right)\left({}^\sigma\underset{\sim}{\boldsymbol{G}}^i\right) - (\alpha_{tm})_{\underline{\Omega}}\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned} \quad (70)
$$

The material coefficients ${}^\sigma\underset{\sim}{a}_j, {}^\sigma\underset{\sim}{b}_i, {}^\sigma\underset{\sim}{c}_{ij}, {}^\sigma\underset{\sim}{d}_i$ and ${}^\sigma\alpha_{tm}; i = 1,2,\cdots,N^\sigma; j = 1,2,\cdots,M^\sigma$ are defined in the following:

$$
\begin{aligned}
&\sigma_0 = \left({}^\sigma\underset{\sim}{\alpha}^0\Big|_{\underline{\Omega}} - \sum_{j=1}^{M^\sigma} \left.\frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^0\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}\right)\left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right);\quad \underset{\sim}{a}_j = \left.\frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^0\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \\
&\underset{\sim}{b}_i = {}^\sigma\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \left.\frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^i\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}\left(-{}^\sigma\underset{\sim}{I}^j\Big|_{\underline{\Omega}}\right);\quad \underset{\sim}{c}_{ij} = \left.\frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^i\right)}{\partial\left({}^\sigma\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}} \\
&{}^\sigma\underset{\sim}{d}_i = -\left.\frac{\partial\left({}^\sigma\underset{\sim}{\alpha}^i\right)}{\partial\theta}\right|_{\underline{\Omega}};\quad {}^\sigma\alpha_{tm} = -\left.\frac{\partial\,{}^\sigma\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned} \quad (71)
$$

The constitutive theory (70) with material coefficients (71) is based on integrity, complete basis of the space of constitutive tensor ${}_s^d\boldsymbol{\sigma}$. Desired simplified forms can be obtained from (70) by retaining specific generators and invariants of interest. This constitutive theory is ordered rate constitutive theory of orders n and m of strain and stress tensors. Material coefficients can be functions of ${}^{\sigma}\underset{\sim}{I}^j; j=1,2,\cdots,M$ and θ in a known configuration $\underline{\Omega}$.

Simplified form of (70) can be obtained by retaining desired generators and the invariants. Perhaps a simplified yet most general constitutive theory for ${}_s^d\boldsymbol{\sigma}$ is one in which ${}_s^d\boldsymbol{\sigma}$ is a linear function of the components of its argument tensors. Redefining material coefficients and rearranging terms in (70), we can write the following (${}_s^d\boldsymbol{\sigma}^{(0)} = {}_s^d\boldsymbol{\sigma}$):

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma} + \sum_{i=1}^{m}\left(\lambda_i^{\sigma}\right)\left({}_s^d\boldsymbol{\sigma}^{(i)}\right) = \sigma_0 \boldsymbol{I} + 2\mu^{\sigma}\boldsymbol{\varepsilon} + \lambda^{\sigma}\mathrm{tr}\boldsymbol{\varepsilon}\boldsymbol{I} + \sum_{i=1}^{n}\left(2\eta_i^{\sigma}\boldsymbol{\varepsilon}_{(i)} + \kappa_i^{\sigma}\left(\mathrm{tr}\left(\boldsymbol{\varepsilon}_{(i)}\right)\right)\boldsymbol{I}\right) \\
+\sum_{i=0}^{m-1}\beta_i^{\sigma}\,\mathrm{tr}\left({}_s^d\boldsymbol{\sigma}^{(i)}\right)\boldsymbol{I} - {}^{\sigma}\alpha_{tm}\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned}
\tag{72}
$$

in which σ_0 is initial stress field, μ^{σ} and λ^{σ} are Lames constants, η_i^{σ} and $\kappa_i^{\sigma}; i=1,2,\cdots,n$, is the spectrum of damping coefficients corresponding to strain rates $\boldsymbol{\varepsilon}_{(i)}; i=1,2,\cdots,n$, λ_i^{σ} is the spectrum of relaxation times corresponding to stress rates ${}_s^d\boldsymbol{\sigma}^{(j)}; j=1,\cdots,m$ and ${}^{\sigma}\alpha_{tm}$ is thermal modulus, β_i^{σ} are coefficients related to relaxation times, these are generally considered to be zero.

A further simplified model that is commonly used in polymer sciences is obtained for $n=1$ and $m=1$, *i.e.*, strain and stress rates of order one. In this case, (72) reduces to

$$
\begin{aligned}
{}_s^d\boldsymbol{\sigma} + \left(\lambda_1^{\sigma}\right){}_s^d\boldsymbol{\sigma}^{(1)} = \sigma_0\boldsymbol{I} + 2\mu^{\sigma}\left(\boldsymbol{\varepsilon}\right) + \lambda^{\sigma}\mathrm{tr}\left(\boldsymbol{\varepsilon}\right)\boldsymbol{I} + 2\eta_1^{\sigma}\boldsymbol{\varepsilon}_{(1)} + \kappa_1^{\sigma}\mathrm{tr}\left(\boldsymbol{\varepsilon}_{(1)}\right)\boldsymbol{I} \\
+\beta_0^{\sigma}\,\mathrm{tr}\left({}_s^d\boldsymbol{\sigma}^{(0)}\right)\boldsymbol{I} - {}^{\sigma}\alpha_{tm}\left(\theta - \theta|_{\underline{\Omega}}\right)\boldsymbol{I}
\end{aligned}
\tag{73}
$$

4.4. Constitutive Theory for Microconstituent Stress Tensor $\boldsymbol{S}$

We also consider microconstituent to have elasticity, dissipation and rheology mechanisms. Thus, following Section 4.3, we can choose the following for the constitutive tensor and its argument tensors.

$$
\boldsymbol{S}^{\left(m^s\right)} = \boldsymbol{S}\left(\boldsymbol{\varepsilon}^{(\alpha)}, \boldsymbol{\varepsilon}_{(i)}^{(\alpha)}, \boldsymbol{S}, \boldsymbol{S}^{(j)}, \theta\right); i=1,2,\cdots,n^s; j=1,2,\cdots,\left(m^s-1\right) \tag{74}
$$

where n^s and m^s are the highest order of rate of strain $\boldsymbol{\varepsilon}^{(\alpha)}$ and highest order of the rate of stress $\boldsymbol{S}$. Let ${}^s\underset{\sim}{\boldsymbol{G}}^i; i=1,2,\cdots,N^s$ be the combined generators of the argument tensors of $\boldsymbol{S}^{\left(m^s\right)}$ in (74) and let ${}^s\underset{\sim}{I}^j; j=1,2,\cdots,M^s$ be the combined invariants of the same argument tensors of $\boldsymbol{S}^{\left(m^s\right)}$ in (74), then $\boldsymbol{I}, {}^s\underset{\sim}{\boldsymbol{G}}^i; i=1,2,\cdots,N^s$ constitutes the basis of the space of constitutive tensor $\boldsymbol{S}^{\left(m^s\right)}$ and we can write the following for $\boldsymbol{S}^{\left(m^s\right)}$.

$$
\boldsymbol{S}^{\left(m^s\right)} = {}^s\alpha^0\boldsymbol{I} + \sum_{i=1}^{N^s}{}^s\alpha^i\left({}^s\underset{\sim}{\boldsymbol{G}}^i\right) \tag{75}
$$

in which

$$ {}^{s}\alpha^{i} = {}^{s}\alpha^{i}\left({}^{s}\underset{\sim}{I}^{j},\theta\right);\ j=1,2,\cdots,M^{s} \tag{76}$$

Following the procedure described in Section 4.3 (Taylor series expansion), we can derive the following constitutive theory for $\boldsymbol{S}^{\left(m^{s}\right)}$

$$\begin{aligned}\boldsymbol{S}^{\left(m^{s}\right)} &= S_0\boldsymbol{I} + \sum_{j=1}^{M^s} {}^{s}\underset{\sim}{a}_j\left({}^{s}\underset{\sim}{I}^{j}\right) + \sum_{i=1}^{N^s} {}^{s}\underset{\sim}{b}_i\left({}^{s}\underset{\sim}{\boldsymbol{G}}^{i}\right) + \sum_{i=1}^{N^s}\sum_{j=1}^{M^s} {}^{s}\underset{\sim}{c}_{ij}\left({}^{s}I^{j}\right)\left({}^{s}\underset{\sim}{G}^{i}\right) \\ &\quad - \sum_{i=1}^{N^s} {}^{s}\underset{\sim}{d}_i\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right){}^{s}\underset{\sim}{\boldsymbol{G}}^{i} - {}^{s}\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{77}$$

in which material coefficients are given by (71) after replacing ${}^{\sigma}\alpha^{i};i=0,1,\cdots,N^{\sigma}$ with ${}^{s}\alpha^{i};i=0,1,\cdots,N^{s}$ and replacing ${}^{\sigma}\underset{\sim}{a}_0, {}^{\sigma}\underset{\sim}{b}_i, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_i$ and ${}^{\sigma}\alpha_{tm};i=1,2,\cdots,N^{\sigma};j=1,2,\cdots,M^{\sigma}$ by ${}^{s}a_0, {}^{s}\underset{\sim}{b}_i, {}^{s}\underset{\sim}{c}_{ij}, {}^{s}\underset{\sim}{d}_i$ and ${}^{s}\alpha_{tm};i=1,2,\cdots,N^{s};j=1,2,\cdots,M^{s}$ and σ_0 by S_0. The material coefficients can be functions of ${}^{s}\underset{\sim}{I}^{j};j=1,2,\cdots,M^{s}$ and θ in a known configuration $\underset{\sim}{\Omega}$. This constitutive theory is based on integrity. A constitutive theory for $\boldsymbol{S}$ that is linear in the components of the argument tensors and is of orders n^s and m^s is given by (after redefining material coefficients and defining $\boldsymbol{S}^{(0)}=\boldsymbol{S}$)

$$\begin{aligned}\boldsymbol{S} + \sum_{i=1}^{m^s}\lambda_i^s\left(\boldsymbol{S}^{(i)}\right) &= S_0\boldsymbol{I} + 2\mu^s\left(\boldsymbol{\varepsilon}^{(\alpha)}\right) + \lambda^s\left(\operatorname{tr}\left(\boldsymbol{\varepsilon}^{(\alpha)}\right)\right)\boldsymbol{I} \\ &\quad + \sum_{i=1}^{n^s}\left(2\eta_i^s\boldsymbol{\varepsilon}_{(i)}^{(\alpha)} + \kappa_i^s\left(\operatorname{tr}\boldsymbol{\varepsilon}_{(i)}^{(\alpha)}\right)\boldsymbol{I}\right) \\ &\quad + \sum_{i=0}^{m^s-1}\beta_i^s\left(\operatorname{tr}\left(S^{(i)}\right)\right)\boldsymbol{I} - {}^{s}\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{78}$$

This constitutive theory is of orders n^s and m^s in rates of $\boldsymbol{\varepsilon}^{(\alpha)}$ and $\boldsymbol{S}$ respectively. When $n^s=1$ and $m^s=1$, we obtain the most simplified possible constitutive theory for $\boldsymbol{S}$.

$$\begin{aligned}\boldsymbol{S} + \lambda_1^s\left(\boldsymbol{S}^{(1)}\right) &= s_0\boldsymbol{I} + 2\mu^s\left(\boldsymbol{\varepsilon}^{(\alpha)}\right) + \lambda^s\left(\operatorname{tr}\left(\boldsymbol{\varepsilon}^{(\alpha)}\right)\right)\boldsymbol{I} + 2\eta_1^s\boldsymbol{\varepsilon}_{(1)}^{(\alpha)} \\ &\quad + \kappa_1^s\operatorname{tr}\left(\boldsymbol{\varepsilon}_{(1)}^{(\alpha)}\right)\boldsymbol{I} + \beta_0^s\left(\operatorname{tr}\left(\boldsymbol{S}^{(0)}\right)\right)\boldsymbol{I} - {}^{s}\alpha_{tm}\left(\theta-\theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \tag{79}$$

The constitutive theory (79) has a spectrum $\left(\eta_i^s,\kappa_i^s\right);i=1,2,\cdots,n^s$ of dissipation coefficients corresponding to strain rates $\boldsymbol{\varepsilon}_{(i)}^{(\alpha)};i=1,2,\cdots,n^s$ and $\lambda_j^s;j=1,2,\cdots,m^s$ is the spectrum of relaxation times corresponding to the rates $\boldsymbol{S}^{(i)};i=1,2,\cdots,m^s$ associated with the microconstituents.

4.5. Constitutive Theory for Moment Tensor $\boldsymbol{m}$

Rigid rotations and rotation rates of the microconstituents in an elastic and viscous medium with long chain polymer molecules of the medium result in: 1) elasticity due to the rotation gradient tensor, 2) dissipation due to viscous drag experienced by the microconstituents which is a function of the rates of the symmetric part of the rotation gradient tensor and 3) rheology due to the interaction of microconstituents with the long chain molecules in the viscous medium. Upon cessation of an external stimulus, the microconstituents exhibit a relaxation phenomenon in returning to their original position. If ${}_{s}^{c}{}^{\Theta}\boldsymbol{J}_{(1)}; i = 0,1,\cdots,n^m$ are the rates of the symmetric part of the rotation gradient tensor and $\boldsymbol{m}^{(j)}; j = 0,1,\cdots,m^m$, are the rates of moment tensor, then $\boldsymbol{m}^{\left(m^s\right)}$ is the constitutive tensor and its argument tensors are given by

$$\boldsymbol{m}^{\left(m^m\right)} = \boldsymbol{m}^{\left(m^m\right)}\left({}^{c\Theta}\boldsymbol{J}, {}^{c\Theta}\boldsymbol{J}_{(i)}, \boldsymbol{m}, \boldsymbol{m}^{(j)}, \theta\right); i = 1,2,\cdots,n^m; j = 1,2,\cdots,\left(m^m - 1\right) \quad (80)$$

Let ${}^{m}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^m$ be the combined generators of the argument tensors of $\boldsymbol{m}^{\left(m^m\right)}$ in (80) and let ${}^{m}\underset{\sim}{I}^j; j = 1,2,\cdots,M^m$ be the combined invariants of the same argument tensors, then $\boldsymbol{I}, {}^{m}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\cdots,N^m$ form the basis of the space (integrity) of the constitutive tensor $\boldsymbol{m}^{\left(m^m\right)}$ and we can write the following for $\boldsymbol{m}^{\left(m^m\right)}$:

$$\boldsymbol{m}^{\left(m^m\right)} = {}^{m}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{\alpha}^i \left({}^{m}\underset{\sim}{\boldsymbol{G}}^i\right) \quad (81)$$

in which coefficients

$$ {}^{m}\alpha^i = {}^{m}\alpha^i\left({}^{m}\underset{\sim}{I}^j, \theta\right); i = 0,1,\cdots,N^m; j = 1,2,\cdots,M^m \quad (82)$$

Following the procedure described in Section 4.3 (Taylor series expansion), we can derive the following constitutive theory for $\boldsymbol{m}$:

$$\begin{aligned}\boldsymbol{m}^{\left(m^m\right)} = {} & m_0 \boldsymbol{I} + \sum_{j=1}^{M^m} {}^{m}\underset{\sim}{a}_i \left({}^{m}\underset{\sim}{I}^j\right)\boldsymbol{I} + \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{b}_i \left({}^{m}\underset{\sim}{\boldsymbol{G}}^i\right) + \sum_{i=1}^{N^m}\sum_{j=1}^{M^m} {}^{m}\underset{\sim}{c}_{ij}\left({}^{m}\underset{\sim}{I}^j\right)\left({}^{m}\underset{\sim}{\boldsymbol{G}}^i\right) \\ & - \sum_{i=1}^{N^m} {}^{m}\underset{\sim}{d}_i \left(\theta - \theta|_{\underset{\sim}{\Omega}}\right) {}^{m}\underset{\sim}{\boldsymbol{G}}^i - {}^{m}\alpha_{tm}\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \quad (83)$$

in which material coefficients are given by after replacing ${}^{\sigma}\alpha^i; i = 0,1,\cdots,N^\sigma$; ${}^{\sigma}a_0, {}^{\sigma}\underset{\sim}{b}_i, {}^{\sigma}\underset{\sim}{c}_{ij}, {}^{\sigma}\underset{\sim}{d}_i; i = 1,2,\cdots,N^\sigma; j = 1,2,\cdots,M^\sigma$ by ${}^{m}\alpha^i; i = 0,1,\cdots,n^m$ and ${}^{m}\underset{\sim}{a}_j, {}^{m}\underset{\sim}{b}_i, {}^{m}\underset{\sim}{c}_{ij}, {}^{m}\underset{\sim}{d}_j; i = 1,2,\cdots,n^m; j = 1,2,\cdots,m^m$ and σ_0 by m_0. The material coefficients can be functions of ${}^{m}\underset{\sim}{I}^\theta; j = 1,2,\cdots,M^m$ and θ in a known configuration $\underset{\sim}{\Omega}$.

A constitutive theory that is linear in the components of the argument tensor of orders n^m and m^m is given by (using $\boldsymbol{m} = \boldsymbol{m}^{(0)}$):

$$\begin{aligned}\boldsymbol{m} + \sum_{i=1}^{m^m} \lambda_i^m \boldsymbol{m}^{(i)} = {} & m_0 \boldsymbol{I} + 2\left(\underset{\sim}{\mu}_i^m\right)\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right) + \left(\underset{\sim}{\lambda}^m\right)\left(\operatorname{tr}\left({}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}\right)\right)\boldsymbol{I} \\ & + \sum_{i=1}^{n^m}\left(\eta_i^m\left({}_{s}^{c}{}^{\Theta}\boldsymbol{J}_{(i)}\right) + \kappa_i^m \operatorname{tr}\left({}_{s}^{c}{}^{\Theta}\boldsymbol{J}_{(i)}\right)\boldsymbol{I}\right) \\ & + \sum_{i=0}^{m^m-1} \beta_i^m \operatorname{tr}\left(\boldsymbol{m}^{(i)}\right)\boldsymbol{I} - \left({}^{m}\alpha_{tm}\right)\left(\theta - \theta|_{\underset{\sim}{\Omega}}\right)\boldsymbol{I}\end{aligned} \quad (84)$$

When $n^m = 1$, $m^m = 1$, we have the simplest possible constitutive theory for $\boldsymbol{m}$:

$$\boldsymbol{m}+\lambda_1^m\left(\boldsymbol{m}^{(1)}\right)=m_0\boldsymbol{I}+2\mu^m\left({}_s^c\Theta\boldsymbol{J}\right)+\lambda^m\left(\operatorname{tr}\left({}_s^c\Theta\boldsymbol{J}\right)\right)\boldsymbol{I}+2\eta_1^m\left({}_s^c\Theta\boldsymbol{J}_{(1)}\right)$$
$$+\kappa_1^m\left(\operatorname{tr}\left({}_s^c\Theta\boldsymbol{J}_{(1)}\right)\right)\boldsymbol{I}+\beta_0^m\left(\operatorname{tr}(m)\right)\boldsymbol{I}-{}^m\alpha_{tm}\left(\theta-\theta|_{\Omega}\right)\boldsymbol{I} \tag{85}$$

The constitutive theory (84) has a spectrum of dissipation coefficients $\left(\eta_i^m,\kappa_i^m\right);i=1,2,\cdots,n^m$ corresponding to rates ${}_s^c\Theta\boldsymbol{J}_{(i)};i=1,2,\cdots,n^m$ and a spectrum of relaxation times $\lambda_i^m;i=1,2,\cdots,m^m$ corresponding to the rates $\boldsymbol{m}^{(i)};i=1,2,\cdots,m^m$ due to interaction of the microconstituents with the solid medium.

4.6. Constitutive Theory for q

In this derivation, we consider (based on conjugate pairs in the reduced entropy inequality)

$$\boldsymbol{q}=\boldsymbol{q}\left(\boldsymbol{g},\theta\right) \tag{86}$$

Tensors $\boldsymbol{q}$ and $\boldsymbol{g}$ are tensors of rank one and θ is a tensor of rank zero. The only combined generator of rank one of the argument tensor $\boldsymbol{g}$ and θ is $\boldsymbol{g}$, hence based on representation theorem, we can write:

$$\boldsymbol{q}=-{}^q\alpha\boldsymbol{g} \tag{87}$$

The coefficient ${}^q\alpha$ is a function of the combined invariants of $\boldsymbol{g},\theta$, *i.e.*, $\boldsymbol{g}\cdot\boldsymbol{g}$ and temperature θ. Let us define ${}^q\underset{\sim}{I}=\boldsymbol{g}\cdot\boldsymbol{g}$ to simplify the details of further derivation. We note that (87) holds in the current configuration in which the deformation is not known. Hence, in (87), ${}^q\alpha={}^q\alpha\left({}^q\underset{\sim}{I},\theta\right)$ is not yet deterministic and it is not a material coefficient. To determine material coefficients in (87), we expand ${}^q\alpha\left({}^q\underset{\sim}{I},\theta\right)$ in Taylor series about a known configuration $\underset{\sim}{\Omega}$ in ${}^q\underset{\sim}{I}$ and θ and retain only up to linear terms in ${}^q\underset{\sim}{I}$ and θ (for simplicity)

$${}^q\alpha={}^q\alpha\Big|_{\underset{\sim}{\Omega}}+\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left({}^q\underset{\sim}{I}-\left({}^q\underset{\sim}{I}\right)_{\underset{\sim}{\Omega}}\right)+\left.\frac{\partial{}^q\alpha}{\partial\theta}\right|_{\underset{\sim}{\Omega}}\left(\theta-\theta_{\underset{\sim}{\Omega}}\right) \tag{88}$$

Substituting (88) into (87)

$$\boldsymbol{q}=-\left({}^q\alpha\Big|_{\underset{\sim}{\Omega}}+\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left({}^q\underset{\sim}{I}-\left({}^q\underset{\sim}{I}\right)_{\underset{\sim}{\Omega}}\right)+\left.\frac{\partial{}^q\alpha}{\partial\theta}\right|_{\underset{\sim}{\Omega}}\left(\theta-\theta_{\underset{\sim}{\Omega}}\right)\right)\boldsymbol{g} \tag{89}$$

We note that ${}^q\alpha\Big|_{\underset{\sim}{\Omega}},\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}$ and $\left.\frac{\partial{}^q\alpha}{\partial\theta}\right|_{\underset{\sim}{\Omega}}$ are functions of $\left({}^q\underset{\sim}{I}\right)_{\underset{\sim}{\Omega}}$ and $\theta\big|_{\underset{\sim}{\Omega}}$, whereas ${}^q\alpha$ in (87) is a function of ${}^q\underset{\sim}{I}$ and θ in the current configuration. From (89), we can write the following, noting that ${}^q\underset{\sim}{I}=\boldsymbol{g}\cdot\boldsymbol{g}$

$$\boldsymbol{q}=-{}^q\alpha\Big|_{\underset{\sim}{\Omega}}\boldsymbol{g}-\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)\boldsymbol{g}-\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)_{\underset{\sim}{\Omega}}\boldsymbol{g}-\left.\frac{\partial{}^q\alpha}{\partial\theta}\right|_{\underset{\sim}{\Omega}}\left(\theta-\theta_{\underset{\sim}{\Omega}}\right)\boldsymbol{g} \tag{90}$$

or $$\boldsymbol{q}=-\left({}^q\alpha\Big|_{\underset{\sim}{\Omega}}\boldsymbol{g}+\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)\right)\boldsymbol{g}-\left.\frac{\partial{}^q\alpha}{\partial{}^q\underset{\sim}{I}}\right|_{\underset{\sim}{\Omega}}\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)_{\underset{\sim}{\Omega}}\boldsymbol{g}-\left.\frac{\partial{}^q\alpha}{\partial\theta}\right|_{\underset{\sim}{\Omega}}\left(\theta-\theta_{\underset{\sim}{\Omega}}\right)\boldsymbol{g} \tag{91}$$

Let

$$\kappa\left(\theta\big|_{\Omega},\left({}^{q}\underset{\sim}{I}\right)_{\Omega}\right)={}^{q}\alpha\big|_{\Omega}-\frac{\partial\,{}^{q}\alpha}{\partial\,{}^{q}\underset{\sim}{I}}\bigg|_{\Omega}\left(\{g\}^{\mathrm{T}}\{g\}\right)_{\Omega} \tag{92}$$

$$\kappa_1\left(\theta\big|_{\Omega},\left({}^{q}\underset{\sim}{I}\right)_{\Omega}\right)=\frac{\partial\,{}^{q}\alpha}{\partial\,{}^{q}\underset{\sim}{I}}\bigg|_{\Omega} \tag{93}$$

$$\kappa_2\left(\theta\big|_{\Omega},\left({}^{q}\underset{\sim}{I}\right)_{\Omega}\right)=\frac{\partial\,{}^{q}\alpha}{\partial\theta}\bigg|_{\Omega} \tag{94}$$

Then,

$$\boldsymbol{q}=-\kappa\boldsymbol{g}-\kappa_1\left(\boldsymbol{g}\cdot\boldsymbol{g}\right)\boldsymbol{g}-\kappa_2\left(\theta-\theta_{\Omega}\right)\boldsymbol{g} \tag{95}$$

This is the simplest possible constitutive theory based on conjugate pairs in the entropy inequality, representation theorem and (86). This constitutive theory uses integrity, the complete basis of the space of $\boldsymbol{q}$. The only assumption in this theory beyond (86) is the truncation of the Taylor series in (88) beyond linear terms in ${}^{q}\underset{\sim}{I}$ and θ. The constitutive theory for $\boldsymbol{q}$ in (95) is cubic in $\boldsymbol{q}$. It contains linear and cubic terms in $\boldsymbol{g}$, but does not contain a quadratic term in $\boldsymbol{g}$. Simplified linear theory is given by (95) by retaining only the first term on the right-hand side of (95) (Fourier heat conduction law).

5. Thermodynamic and Mathematical Consistency of the Micromorphic Theory Presented in This Paper

The laws of classical thermodynamics used in classical continuum mechanics are well founded and accepted laws. Microcontinuum theories contain new physics beyond classical continuum mechanics, and hence may require new considerations. For establishing conservation and balance laws for these theories, we must begin with classical thermodynamics, but can only make changes to them and incorporate new conservation and balance laws if the classical thermodynamics framework supports these. The resulting microcontinuum theory will be referred to as thermodynamically consistent with the laws of classical thermodynamics, *i.e.*, classical continuum mechanics. We list important features of the present work that establish the thermodynamic and mathematical consistency of the theory presented here.

1) The existence of a moment independent of forces and conjugate to rotations is a result of the resistance offered by the medium to the rigid rotations of the microconstituent. The balance of angular momenta, which is a statement of the balance of moments (of forces in classical continuum mechanics), permits inclusion of the moment tensor in the balance of angular momenta. Thus, this modification of the balance law of classical thermodynamics is supported by the classical thermodynamics.

2) In classical thermodynamics, a kinematically conjugate pair requires two balance laws. The kinematically conjugate pair of displacements and forces requires two balance laws: balance of forces and balance of moment of forces, *i.e.*, balance

of linear momenta and balance of angular momenta. Based on this, classical thermodynamics will permit two additional balance laws for each new kinematically conjugate pair. Thus, for the kinematically conjugate pair of rotations and moments in microcontinuum theories, we need balance of moments which already exists as the balance of angular momenta and can be modified to include the moment tensor as discussed in 1) and a balance of moment of moments, which is a new balance law needed in the 3M theories. The consequence of this balance law is that the Cauchy moment tensor is symmetric. In the absence of this, the dynamic equilibrium of moment of moments is violated, hence thermodynamic consistency is violated.

3) It has been shown by Surana *et al.* that if classical rotations are not used as rigid rotations of the microconstituents, the entropy inequality is violated. That is a microcontinuum theory based on ${}_{\alpha}\boldsymbol{\Theta}$ as unknown rigid rotations of the microconstituents or ${}_{c}\boldsymbol{\Theta} + {}_{\alpha}\boldsymbol{\Theta}$ as rigid rotations of the microconstituents result in a violation of the entropy inequality. These choices produce additional terms in the entropy inequality that cannot be accounted for, thus resulting in thermodynamic inconsistency.

4) Since rotations and moments are a new kinematically conjugate pair in 3M theories that does not exist in classical continuum mechanics, the integral-average definition of the moment cannot be derived using the microconstituent Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$, as this stress is due to classical continuum mechanics. Insisting on doing so will result in a theory that violates thermodynamic consistency.

5) In the micropolar microcontinuum theory, 1) - 4) are supported by classical thermodynamics and are sufficient to yield a microcontinuum theory that is thermodynamically consistent and has closure when the constitutive theories are included.

6) When the microconstituents are deformable, 1) - 4) are not sufficient (along with constitutive theories) to provide closure to the mathematical model. In the case of a micromorphic theory, six additional equations are needed, and in the case of a microdilation theory, only one additional equation is needed. We have shown that the balance of angular momenta, in fact, contains nine equations; six of these are eliminated due to presence of permutation tensor with the stress terms. We have shown that by premultiplying the balance of angular momenta with the inverse of the permutation tensor, we can recover the six additional equations needed for closure. This part of the derivation is related to the balance of angular momenta and hence does not violate thermodynamic consistency.

7) Thus, we note that use of 1) - 4) or 1) - 4) and 6) which are supported by classical thermodynamics, yields conservation and balance laws of three 3M microcontinuum theories, confirming that the conservation and balance laws in these theories are thermodynamically consistent.

8) In case of constitutive theories, we must use conjugate pairs in the entropy inequality and the axiom of causality to determine constitutive tensors and their

argument tensors that are supported by the theory of isotropic tensors (as done in the present work). A violation of this results in thermodynamic and mathematical inconsistency.

9) Constitutive theories must be derived strictly using the representation theorem (as done in the present work) to ensure mathematical consistency of the resulting constitutive theories. If the constitutive theories are derived using any other means such as potentials and energy functional, then we must show that the same theories can be derived using the representation theorem, otherwise the constitutive theories are mathematically inconsistent. Clearly the constitutive theories presented in this paper are mathematically and thermodynamically consistent.

10) The conservation and balance laws introduced by Eringen: a) conservation of microinertia and b) balance of moment of symmetric stresses with the gradients of the symmetric part of the moment tensor are not supported by the classical thermodynamics, *i.e.*, classical continuum mechanics, hence can only be viewed as phenomenological or ad-hoc. Inclusion of these in the laws of thermodynamics results in a thermodynamically inconsistent theory.

6. Linear Micromorphic Theory of Eringen

We summarize some aspects of Eringen's theories that lead to their thermodynamic and mathematical inconsistencies.

1) Use of ${}_{\alpha}\boldsymbol{\Theta}$ or ${}_{\alpha}\boldsymbol{\Theta} + {}_{c}\boldsymbol{\Theta}$ as rigid rotations of the microconstituents results in a violation of entropy inequality, and hence in the thermodynamic inconsistency of the resulting theory.

2) Including rigid rotations in the strain measures in Eringen's work results in tensors that cannot be used in the constitutive theories without violating the physics of deformation.

3) Eringen's work defines the integral-average moment tensor (nonclassical physics) using the microconstituent Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$. This is due to classical continuum mechanics. This is obviously wrong. The origin of the moment is due to the resistance offered to the rigid rotations of the microconstituents by the medium and not due to $\boldsymbol{\sigma}^{(\alpha)}$. Due to this wrong definition the balance of angular momenta is of concern.

4) The use of a weighted integral of the balance of micro-linear momenta using a weight function $\tilde{\phi}^{(\alpha)}(\bar{\underset{\sim}{x}}_m)$ has no thermodynamic foundations. Our work in this paper shows that this is neither needed nor used.

5) The use of nonsymmetric tensors of rank two as constitutive tensors and nonsymmetric tensors as their argument tensors is not supported by the theory of isotropic tensor. It results in constitutive theories that are mathematically inconsistent and are nonphysical.

6) Constitutive theories derived using potentials or energy functions (as in Eringen's work) are nonphysical, not valid and mathematically inconsistent if the same theories cannot be derived using the representation theorem.

7) Due to omission of the balance of moment of moments balance law, the dy-

namic equilibrium is not satisfied in the Eringen's mathematical model. Its consequence is non-symmetric moment tensor and the spurious constitutive theories are other negative and detrimental aspects.

8) The use of principle of equipresence provides nonphysical coupling between classical and nonclassical physics and results in nonphysical material coefficients.

9) Lack of various additive decompositions of the stress tensor leads to nonphysical and invalid constitutive tensors. For example, ${}_{a}\boldsymbol{\sigma}$ must be eliminated from $\boldsymbol{\sigma}$ as it is defined by the balance of angular momenta, and hence cannot be part of the constitutive theory. Further decomposition of ${}_{s}\boldsymbol{\sigma} = {}_{s}^{e}\boldsymbol{\sigma} + {}_{s}^{d}\boldsymbol{\sigma}$ is necessary to address volumetric and distortional physics correctly. None of these are used in Eringen's work, hence the constitutive theories are of concern.

10) In micromorphic theories additional six equations are needed for closure. Eringen proposes a new balance law to obtain these, balance of moments of symmetric stresses with the gradient of the symmetric part of the moment tensor. This law is not supported by classical thermodynamics, hence its use will yield a thermodynamically inconsistent theory.

11) Eringen also proposes the conservation of microinertia to obtain the three equations needed for closure. This conservation law is also not supported by classical thermodynamics, hence its use will lead to thermodynamically inconsistent theory.

We have presented plenty of evidence based on thermodynamics and well-established principles of mathematics showing that Eringen's 3M theories are thermodynamically and mathematically inconsistent, hence are not valid microcontinuum theories.

7. Summary and Conclusions

The conservation and balance laws of the linear micromorphic microcontinuum theory derived in reference [1], which are shown to be thermodynamically and mathematically consistent, are utilized in the present work to derive thermodynamically and mathematically consistent constitutive theories for a linear micromorphic polymeric solid matter. The most notable aspects of this work are summarized in the following:

1) Deformational and rigid rotation physics are always separated additively in order to facilitate the derivation of correct and physical constitutive theories.

2) Additive decomposition of the stress tensor $\boldsymbol{\sigma}$ is performed to separate volumetric and distortional aspects of the physics in $\boldsymbol{\sigma}$.

3) The choice of constitutive tensors and their argument tensors is always in accordance with the representation theorem, thus always ensuring mathematical consistency of the resulting constitutive theories.

4) Determination of the constitutive tensor and initial determination of its argument tensors are always made using the conjugate pairs in the entropy inequality.

5) Derivation of constitutive theories is strictly based on the representation the-

orem and integrity. This ensures mathematical consistency and yields constitutive theories based on a complete basis.

6) The constitutive theories presented in this paper for micromorphic polymeric solids consider mechanisms of elasticity, dissipation and rheology for the microconstituents, for the medium, and for the interaction of the microconstituents with the medium. All three dissipation and rheology mechanisms are ordered rate mechanisms employing rate of strain tensors as well as rate of constitutive tensors up to any desired orders. The resulting constitutive theories have spectra of dissipation coefficients as well as spectra of relaxation times for the microconstituents, for the medium, and for the interaction of the microconstituents with the medium.

7) The complete mathematical model consisting of the conservation and balance laws and the constitutive theories is thermodynamically and mathematically consistent.

Acknowledgements

The first author is grateful for his endowed professorship and the Department of Mechanical Engineering of the University of Kansas for providing financial support to the second author. The computational facilities provided by the Computational Mechanics Laboratory of the mechanical engineering departments are also acknowledged.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Surana, K.S. and Mathi, S.S.C. (2025) Thermodynamically and Mathematically Consistent Linear Micromorphic Microcontinuum Theory for Solid Continua. *Journal of Applied Mathematics and Physics*, **13**, 3616-3661. https://doi.org/10.4236/jamp.2025.1310202

[2] Voigt, W. (1887) Theoretische studien uber die elastirifikaterhaltnisse der krystalle. *Abhandlungen der Wissenschaften Gesellschaft*, **34**, 3-51.

[3] Grad, H. (1952) Statistical Mechanics, Thermodynamics, and Fluid Dynamics of Systems with an Arbitrary Number of Integrals. *Communications on Pure and Applied Mathematics*, **5**, 455-494. https://doi.org/10.1002/cpa.3160050405

[4] Gunther, W. (1988) Zur statik und kinematik des cosseratschen kontinuums. *Abhandlungen der Wissenschaften Gesellschaft*, **10**, Article 195.

[5] Cosserat, E. and Cosserat, P. (1909) Thorie des Corps Dformables. Hermann.

[6] Schaefer, H. (1967) Das Cosserat Kontinuum. *Journal of Applied Mathematics and Mechanics*, **47**, 485-498. https://doi.org/10.1002/zamm.19670470802

[7] Eringen, A.C. (1967) Linear Theory of Micropolar Viscoelasticity. *International Journal of Engineering Science*, **5**, 191-204. https://doi.org/10.1016/0020-7225(67)90004-3

[8] Eringen, A. (1969) Compatibility Conditions of the Theory of Micromorphic Elastic Solids. *Indiana University Mathematics Journal*, **19**, 473-481.

https://doi.org/10.1512/iumj.1970.19.19044

[9] Eringen, A.C. (1967) Theory of Micropolar Plates. *Zeitschrift für angewandte Mathematik und Physik*, **18**, 12-30. https://doi.org/10.1007/bf01593891

[10] Eringen, A.C. and Suhubi, E.S. (1964) Nonlinear Theory of Simple Micro-Elastic Solids—I. *International Journal of Engineering Science*, **2**, 189-203. https://doi.org/10.1016/0020-7225(64)90004-7

[11] Eringen, A.C. (1964) Simple Microfluids. *International Journal of Engineering Science*, **2**, 205-217. https://doi.org/10.1016/0020-7225(64)90005-9

[12] Eringen, A.C. (1964) Mechanics of Micromorphic Materials. In: Grtler, H., Ed., *Proceedings of the 11th International Congress of Applied Mechanics*, Springer, 131-138.

[13] Eringen, A.C. (1966) Theory of Micropolar Continua. *Proceedings of the* 9*th Midwestern Mechanics Conference*, Wisconsin, 16-18 August 1965, 23-40.

[14] Eringen, A.C. (1966) Linear Theory of Micropolar Elasticity. *Journal of Mathematics and Mechanics*, **15**, 909-923.

[15] Eringen, A. (1966) Theory of Micropolar Fluids. *Indiana University Mathematics Journal*, **16**, 1-18. https://doi.org/10.1512/iumj.1967.16.16001

[16] Eringen, A.C. (1968) Theory of Micropolar Elasticity. In: Idekowitz, H., Ed., *Fracture*, Academic Press, 621-729.

[17] Eringen, A.C. (1968) Mechanics of Micromorphic Continua. In: *Mechanics of Generalized Continua*, Springer, 18-35. https://doi.org/10.1007/978-3-662-30257-6_2

[18] Eringen, A.C. (1970) Foundations of Micropolar Thermoelasticity. International Center for Mechanics Studies, Course and Lectures No. 23.

[19] Bringen, A.C. (1970) Balance Laws of Micromorphic Mechanics. *International Journal of Engineering Science*, **8**, 819-828. https://doi.org/10.1016/0020-7225(70)90084-4

[20] Eringen, A.C. (1972) Theory of Micromorphic Materials with Memory. *International Journal of Engineering Science*, **10**, 623-641. https://doi.org/10.1016/0020-7225(72)90089-4

[21] Eringen, A.C. (1992) Balance Laws of Micromorphic Continua Revisited. *International Journal of Engineering Science*, **30**, 805-810. https://doi.org/10.1016/0020-7225(92)90109-t

[22] Eringen, A.C. and Claus Jr, W.D. (1970) A Micromorphic Approach to Dislocation Theory and Its Relation to Several Existing Theories. In: Shimron, J.A., DeWit, R. and Bullough, R., Eds., *Fundamental Aspects of Dislocation Theory*, National Bureau of Standards Publications, 1023-1040.

[23] Eringen, A.C. (1999) Microcontinuum Field Theories I. Foundations and Solids. Springer.

[24] Eringen, A. and Ryan, M. (2002) Microcontinuum Field Theories II: Fluent Media. *Applied Mechanics Reviews*, **55**, B15. https://doi.org/10.1115/1.1445333

[25] Vernerey, F., Liu, W.K. and Moran, B. (2007) Multi-Scale Micromorphic Theory for Hierarchical Materials. *Journal of the Mechanics and Physics of Solids*, **55**, 2603-2651. https://doi.org/10.1016/j.jmps.2007.04.008

[26] Chen, Y., Lee, J. and Xiong, L. (2009) A Generalized Continuum Theory and Its Relation to Micromorphic Theory. *Journal of Engineering Mechanics*, **135**, 149-155. https://doi.org/10.1061/(asce)0733-9399(2009)135:3(149)

[27] Regueiro, R.A. (2010) On Finite Strain Micromorphic Elastoplasticity. *International Journal of Solids and Structures*, **47**, 786-800.

https://doi.org/10.1016/j.ijsolstr.2009.11.006

[28] Wang, X. and Lee, J.D. (2010) Micromorphic Theory: A Gateway to Nano World. *International Journal of Smart and Nano Materials*, **1**, 115-135. https://doi.org/10.1080/19475411.2010.484207

[29] Lee, J.D. and Wang, X. (2011) Generalized Micromorphic Solids and Fluids. *International Journal of Engineering Science*, **49**, 1378-1387. https://doi.org/10.1016/j.ijengsci.2011.04.001

[30] Isbuga, V. and Regueiro, R.A. (2011) Three-Dimensional Finite Element Analysis of Finite Deformation Micromorphic Linear Isotropic Elasticity. *International Journal of Engineering Science*, **49**, 1326-1336. https://doi.org/10.1016/j.ijengsci.2011.04.006

[31] Reges, P.D.N., Pitangueira, R.L.S. and Silva, L.L. (2024) Modeling of Micromorphic Continuum Based on a Heterogeneous Microscale. *International Journal of Non-Linear Mechanics*, **167**, Article ID: 104881. https://doi.org/10.1016/j.ijnonlinmec.2024.104881

[32] McAvoy, R.C. (2024) Consistent Linearization of Micromorphic Continuum Theories. *Mathematics and Mechanics of Solids*, **30**, 1366-1392. https://doi.org/10.1177/10812865241280280

[33] Surana, K. and Mathi, S.S.C. (2025) Nonlinear Deformation/Strains for 3M Continua and Consistency of Linear Micropolar Theories. *Journal of Applied Mathematics and Physics*, **13**, 933-989. https://doi.org/10.4236/jamp.2025.133049

[34] Surana, K.S. and Carranza, C.H. (2023) Nonclassical Continuum Theories for Fluent Media Incorporating Rotation Rates and Their Thermodynamic Consistency. *Journal of Applied Mathematics and Mechanics*, **103**, e202200079. https://doi.org/10.1002/zamm.202200079

[35] Surana, K.S. and Kendall, J.K. (2020) Existence of Rotational Waves in Non-Classical Thermoelastic Solid Continua Incorporating Internal Rotations. *Continuum Mechanics and Thermodynamics*, **32**, 1659-1683. https://doi.org/10.1007/s00161-020-00872-6

[36] Surana, K.S. and Carranza, C.H. (2021) Dynamic Behavior of Thermoelastic Solid Continua Using Mathematical Model Derived Based on Non-Classical Continuum Mechanics with Internal Rotations. *Meccanica*, **56**, 1345-1375. https://doi.org/10.1007/s11012-020-01221-2

[37] Surana, K.S. and Kendall, J.K. (2022) Rotational Inertial Physics in Non-Classical Thermoviscous Fluent Continua Incorporating Internal Rotation Rates. *Applied Mathematics*, **13**, 453-487. https://doi.org/10.4236/am.2022.136030

[38] Surana, K.S. and Kendall, J.K. (2023) NCCT for Micropolar Solid and Fluid Media Based on Internal Rotations and Rotation Rates with Rotational Inertial Physics: Model Problem Studies. *Applied Mathematics*, **14**, 612-651. https://doi.org/10.4236/am.2023.149037

[39] Surana, K.S. and Long, S.W. (2020) Ordered Rate Constitutive Theories for Non-Classical Thermofluids Based on Convected Time Derivatives of the Strain and Higher Order Rotation Rate Tensors Using Entropy Inequality. *Entropy*, **22**, Article 443. https://doi.org/10.3390/e22040443

[40] Surana, K.S., Joy, A.D. and Reddy, J.N. (2019) A Finite Deformation, Finite Strain Nonclassical Internal Polar Continuum Theory for Solids. *Mechanics of Advanced Materials and Structures*, **26**, 381-393.

[41] Surana, K.S., Joy, A.D. and Reddy, J.N. (2017) Non-classical Continuum Theory for Solids Incorporating Internal Rotations and Rotations of Cosserat Theories. *Contin-*

uum Mechanics and Thermodynamics, **29**, 665-698.
https://doi.org/10.1007/s00161-017-0554-1

[42] Surana, K.S., Joy, A.D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Thermoviscoelastic Solids without Memory Incorporating Internal and Cosserat Rotations. *Acta Mechanica*, **229**, 3189-3213.
https://doi.org/10.1007/s00707-018-2163-x

[43] Surana, K.S., Long, S.W. and Reddy, J.N. (2016) Rate Constitutive Theories of Orders n and ^1n for Internal Polar Non-Classical Thermofluids without Memory. *Applied Mathematics*, **7**, 2033-2077. https://doi.org/10.4236/am.2016.716165

[44] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Fluids with Internal Rotation Rates. *Applied Mathematics*, **9**, 907-939. https://doi.org/10.4236/am.2018.98063

[45] Surana, K.S., Long, S.W. and Reddy, J.N. (2018) Necessity of Law of Balance/equilibrium of Moment of Moments in Non-Classical Continuum Theories for Fluent Continua. *Acta Mechanica*, **229**, 2801-2833. https://doi.org/10.1007/s00707-018-2143-1

[46] Surana, K.S., Mohammadi, F., Reddy, J.N. and Dalkilic, A.S. (2016) Ordered Rate Constitutive Theories for Non-Classical Internal Polar Thermoviscoelastic Solids without Memory. *International Journal of Mathematics, Science, and Engineering Applications*, **20**, 99-121.

[47] Surana, K.S., Mysore, D. and Reddy, J.N. (2018) Ordered Rate Constitutive Theories for Non-Classical Thermoviscoelastic Solids with Dissipation and Memory Incorporating Internal Rotations. *Polytechnica*, **1**, 19-35.
https://doi.org/10.1007/s41050-018-0004-2

[48] Surana, K.S., Mysore, D. and Reddy, J.N. (2019) Non-Classical Continuum Theories for Solid and Fluent Continua and Some Applications. *International Journal of Smart and Nano Materials*, **10**, 28-89. https://doi.org/10.1080/19475411.2018.1530700

[49] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Fluent Continua. *Journal of Thermal Engineering*, **1**, 460-475.

[50] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A More Complete Thermodynamic Framework for Solid Continua. *Journal of Thermal Engineering*, **1**, 446-459.

[51] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) A Polar Continuum Theory for Fluent Continua. *International Journal of Engineering Research and Interdisciplinary Applications*, **6**, 107-146.

[52] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Constitutive Theories for Internal Polar Thermoelastic Solid Continua. *Journal of Pure and Applied Mathematics Advances and Applications*, **14**, 89-150.

[53] Surana, K.S., Powell, M.J. and Reddy, J.N. (2015) Ordered Rate Constitutive Theories for Internal Polar Thermofluids. *International Journal of Mathematics, Science, and Engineering Applications*, **9**, 51-116.

[54] Surana, K.S., Reddy, J.N., Nunes, D. and Powell, M.J. (2015) A Polar Continuum Theory for Solid Continua. *International Journal of Engineering Research and Industrial Applications*, **8**, 77-106.

[55] Surana, K.S., Shanbhag, R. and Reddy, J.N. (2018) Necessity of Law of Balance of Moment of Moments in Non-Classical Continuum Theories for Solid Continua. *Meccanica*, **53**, 2939-2972. https://doi.org/10.1007/s11012-018-0851-1

[56] Surana, K.S. and Mathi, S.S.C. (2025) A Nonlinear Micropolar Continuum Theory for Thermoviscoelastic Solid Medium Based on Classical Rotations. *Applied Mathematics*, **16**, 235-261. https://doi.org/10.4236/am.2025.163012

[57] Surana, K.S. and Mathi, S.S.C. (2025) Finite Deformation, Finite Strain Nonlinear Micropolar NCCT for Thermoviscoelastic Solids with Rheology. *Applied Mathematics*, **16**, 143-168. https://doi.org/10.4236/am.2025.161006

[58] Surana, K.S. (2015) Advanced Mechanics of Continua. CRC/Taylor and Francis.

[59] Surana, K.S. (2022) Classical Continuum Mechanics. CRC/Taylor and Francis.

[60] Smith, G.F. (1965) On Isotropic Integrity Bases. *Archive for Rational Mechanics and Analysis*, **18**, 282-292. https://doi.org/10.1007/bf00251667

[61] Smith, G.F. (1970) On a Fundamental Error in Two Papers of C.-C. Wang "on Representations for Isotropic Functions, Parts I and II". *Archive for Rational Mechanics and Analysis*, **36**, 161-165. https://doi.org/10.1007/bf00272240

[62] Smith, G.F. (1971) On Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors and Vectors. *International Journal of Engineering Science*, **9**, 899-916. https://doi.org/10.1016/0020-7225(71)90023-1

[63] Spencer, A.J.M. (1971) Theory of Invariants. In: Eringen, A.C., Ed., *Treatise on Continuum Physics I*, Academic Press, 239-353.

[64] Spencer, A.J.M. and Rivlin, R.S. (1989) The Theory of Matrix Polynomials and Its Application to the Mechanics of Isotropic Continua. *Archive for Rational Mechanics and Analysis*, **2**, 309-336. https://doi.org/10.1007/bf00277933

[65] Spencer, A.J.M. and Rivlin, R.S. (1960) Further Results in the Theory of Matrix Polynomials. *Archive for Rational Mechanics and Analysis*, **4**, 214-230. https://doi.org/10.1007/bf00281388

[66] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **33**, 249-267. https://doi.org/10.1007/bf00281278

[67] Wang, C.C. (1969) On Representations for Isotropic Functions. *Archive for Rational Mechanics and Analysis*, **33**, 268-287. https://doi.org/10.1007/bf00281279

[68] Wang, C.C. (1970) A New Representation Theorem for Isotropic Functions, Part I and II. *Archive for Rational Mechanics and Analysis*, **36**, 166-223.

[69] Wang, C.-C. (1971) Corrigendum to My Recent Papers on "Representations for Isotropic Functions". *Archive for Rational Mechanics and Analysis*, **43**, 392-395. https://doi.org/10.1007/bf00252004

[70] Zheng, Q.-S. (1993) On the Representations for Isotropic Vector-Valued, Symmetric Tensor-Valued and Skew-Symmetric Tensor-Valued Functions. *International Journal of Engineering Science*, **31**, 1013-1024. https://doi.org/10.1016/0020-7225(93)90109-8

[71] Zheng, Q.S. (1993) On Transversely Isotropic, Orthotropic and Relatively Isotropic Functions of Symmetric Tensors, Skew-Symmetric Tensors, and Vectors. *International Journal of Engineering Science*, **31**, 1399-1453.