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The Oscillation of a Class of Fractional Impulsive Partial Differential Equations

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Abstract

In this paper, we study the forced oscillation properties of a class of fractional impulsive delay partial differential equations under the conformable fractional calculus definition with Neumann and Dirichlet boundary conditions. Using the properties of fractional calculus and integral mean value method, we obtain some new oscillation criteria for the fractional partial differential equations, which are the generalization of some classical results involving partial differential equations. One example is given to show the application of our main results.

Keywords

Fractional Partial Differential Equation, Oscillation, Impulse, Conformable Fractional Calculus

1. Introduction

Since fractional calculus is well-suited to characterizing materials and processes with memory and genetic properties, many important mathematical models are described by differential equations containing fractional derivatives, which are often neglected in classical models. Nowadays, fractional differential equations are increasingly used to describe problems in optics, thermal systems, rheology, fluid mechanics systems, signal processing, system identification, control, robotics and other applications [1]-[5].

There have been many studies on ordinary differential equations and fractional differential equations [6]-[12]. Recently, many scholars have studied the oscillation properties of fractional partial differential equations [13]-[20]. However, so far, the oscillation properties of fractional partial differential equations with several time delays are still rare. In 2017, Raheem A and Maqbul M [21] used differential inequality methods to study the oscillation of a class of fractional partial differential equations with impulse and forced terms under Robin and Dirichlet boundary conditions. In 2023, Chatzarakis and Logaarasi [22] studied the oscillation of impulsive fractional partial differential equations, and some sufficient conditions were established to guarantee the oscillation of the solutions. Both of them dealt with their problems with Riemann-Liouville fractional derivatives. In this paper, we study the oscillation of the impulsive partial differential equations with the conformable fractional derivatives, and the coefficients of Laplacian operator are nonlinear.

The conformable fractional derivative satisfies the chain and product rules, which simplifies the calculation of composite function derivatives [23]. A similar proof approach can be extended to the Riemann-Liouville derivative in some cases; whether it can be applied to Caputo derivative needs more exploration [24].

In this paper, we study the forced oscillation properties of the following nonlinear fractional impulsive delay partial differential equations under the conformable fractional calculus definition.

$$\begin{cases} T_\alpha\left(T_\alpha u(t,x)\right)+p(t)T_\alpha u(t,x) \\ \quad =a(t)h(u)\Delta u+\sum_{i=1}^{m}a_i(t)h_i\left(u(t-\tau_i,x)\right)\Delta u(t-\tau_i,x) \\ \quad -\sum_{j=1}^{n}q_j(t,x)f_j\left(u(t-\delta_j,x)\right)-g(t,x), \\ \quad t\neq t_k,(t,x)\in R_+\times\Omega, \\ T_\alpha u\left(t_k^+,x\right)-T_\alpha u\left(t_k^-,x\right)=\sigma(t_k,x)T_\alpha u(t_k,x), \\ \quad u\left(t_k^+,x\right)-u\left(t_k^-,x\right)=\theta(t_k,x)u(t_k,x), \\ \quad t=t_k,(t,x)\in R_+\times\Omega,k=1,2,\cdots. \end{cases} \tag{1.1}$$

We consider Neumann and Dirichlet boundary conditions

$$\frac{\partial u(t,x)}{\partial n}=0,(t,x)\in R_+\times\partial\Omega,t\neq t_k, \tag{1.2}$$

or

$$u(t,x)=0,(t,x)\in R_+\times\partial\Omega,t\neq t_k. \tag{1.3}$$

where $\alpha\in(0,1)$ is a constant, T_α is the conformable fractional derivative of order α with respect to t of a function $u(x,t)$, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$, $\bar{\Omega}=\Omega\cup\partial\Omega$; $R_+:=(0,+\infty)$, Δ is the Laplacian operator, n is the outer normal vector of the boundary $\partial\Omega$, τ_i,δ_j are nonnegative constants: $i=1,2,3,\cdots,m$ and $j=1,2,3,\cdots,n$; $p(t),a(t),a_i(t)\in PC[R_+,R_+]$, $\lim_{k\to+\infty}t_k=+\infty$, $0<t_1<\cdots<t_k<\cdots$ and the forced term $g(t,x)\in PC\left[R_+\times\bar{\Omega},R\right]$, PC denotes the class of piecewise continuous functions that have $t=t_k,k=1,2,\cdots$ as first-kind discontinuity points and are left-continuous at these points. The solutions $u(t,x)$ of problems (1.1) and (1.2) (or (1.1) and (1.3)) and their fractional derivatives $T_\alpha u(t,x)$ are piecewise continuous functions with $t=t_k$ as first-kind discontinuity points and are left-continuous at these points; that is,

$$u\left(t_k^-,x\right)=u\left(t_k,x\right);\ T_\alpha u\left(t_k^-,x\right)=T_\alpha u\left(t_k,x\right).$$

The following are the basic assumptions of this paper:

H1. $f_j:\mathbb{R}\to\mathbb{R}$ are continuous functions, and for $u\neq 0$, there exist positive constants k_j such that $\frac{f_j(u)}{u}\geq k_j>0$.

H2. $q_j(t,x)\in PC\left[\mathbb{R}_+\times\bar{\Omega},\mathbb{R}_+\right]$, and $q(t)=\min_{1\leq j\leq n}\min_{x\in\bar{\Omega}}q_j(t,x)$.

H3. $h(u),h_i(u)\in C(\mathbb{R},\mathbb{R})$; $uh'(u)\geq 0$ and $uh_i'(u)\geq 0$.

H4. $\sigma(t_k,x)\leq\alpha_k$, $\theta(t_k,x)\leq\theta_k\leq 0$, where α_k and θ_k are constants.

H5. $g(t,x)\in PC\left[\mathbb{R}_+\times\bar{\Omega}\right]$.

In this paper, for the sake of convenience, we introduce the following notations:

$$U(t)=\int_\Omega u(t,x)\mathrm{d}x,\ G(t)=\int_\Omega g(t,x)\mathrm{d}x,$$

$$\tilde{U}(\xi)=U(t),\ \tilde{G}(\xi)=G(t),\ \tilde{V}(\xi)=V(t),$$

$$\xi=\frac{1}{\alpha}t^\alpha,\ \xi_l=\frac{1}{\alpha}t_l^\alpha,\ p\left(\tilde{\xi}\right)=p(t),\ q\left(\tilde{\xi}\right)=q(t).$$

Definition 1.1. A nonzero solution $u(t,x)$ of problems (1.1) and (1.2) or ((1.1) and (1.3)) is nonoscillatory in the region G if there exists a constant $\tau\geq 0$ such that for $(t,x)\in[\tau,+\infty)\times\Omega$, either $u(t,x)<0$ or $u(t,x)>0$ always holds; otherwise, it is called oscillatory.

Definition 1.2. [23] Given a function $f:[0,\infty)\to R$. Then the conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon},$$

for all $t>0,\alpha\in(0,1)$. If f is α-differentiable in some $(0,a),a>0$, and $\lim_{t\to 0^+}T_\alpha f(t)$ exists, then we define $T_\alpha f(0)=\lim_{t\to 0^+}T_\alpha f(t)$.

Definition 1.3. [23] The conformable fractional integral operator of order α, $\alpha\in(0,1)$, of a function f is defined as

$$I_\alpha^a(f)(t)=I_1^a\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}\mathrm{d}x,$$

where the integral is defined in the sense of the improper Riemann integral.

The following presents some important related properties under the conformable fractional calculus definition [23]:

Let $\alpha\in(0,1]$ and f,g be α-differentiable at a point $t>0$. Then

1) $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$ for all $a,b\in\mathbb{R}$.

2) $T_\alpha\left(t^p\right)=pt^{p-\alpha}$ for all $p\in\mathbb{R}$.

3) $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$.

4) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.

5) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.

6) If f is differentiable, then $T_\alpha f(t)=t^{1-\alpha}f'(t)$.

Lemma 1.1. [12] Let λ be a constant. Then for the following problem

$$\begin{cases}\Delta w(x)+\lambda w(x)=0, & x\in\Omega,\\ w(x)=0, & x\in\partial\Omega.\end{cases}$$

Its minimum eigenvalue λ_0 is positive, and the corresponding eigenfunction $\phi_0(x)$ is also positive for $x\in\Omega$.

Lemma 1.2. [21] Suppose the following inequalities hold:

$$\omega'(t)\le g_1(t)\omega(t)+g_2(t),\ t\ne t_k, t\ge\mu,$$
$$\omega\left(t_k^+\right)\le(1+a_k)\omega(t_k),\ k=1,2,\cdots,$$

where $0<t_1<t_2<\cdots<t_k<\cdots$ and $\lim_{k\to\infty}t_k=+\infty$. $\omega\in PC^1[\mathbb{R}_+,\mathbb{R}]$, $g_1,g_2\in[\mathbb{R}_+,\mathbb{R}]$, and a_k are constants. Then,

$$\omega(t)\le\omega(t_0)\prod_{t_0<t_l<t}(1+\alpha_k)\exp\left(\int_{t_0}^t g_1(s)\mathrm{d}s\right)$$
$$+\int_{t_0}^t\prod_{s<t_l<t}(1+\alpha_k)\exp\left(\int_s^t g_1(\sigma)\mathrm{d}\sigma\right)g_2(s)\mathrm{d}s, t\ge\mu.$$

2. Main Theorems and Their Proofs

Theorem 2.1. Suppose that the assumption (H1)-(H5) hold. If the fractional impulsive differential inequalities

$$\begin{cases}T_\alpha\left(T_\alpha U(t)\right)+p(t)T_\alpha U(t)\le -G(t),\\ T_\alpha U\left(t_k^+\right)\le(1+\alpha_k)T_\alpha U(t_k),\\ U\left(t_k^+\right)\le(1+\theta_k)U(t_k),\ k=1,2,\cdots\end{cases}\tag{2.1}$$

have no eventually positive solutions and the fractional impulsive differential inequalities

$$\begin{cases}T_\alpha\left(T_\alpha U(t)\right)+p(t)T_\alpha U(t)\ge -G(t),\\ T_\alpha U\left(t_k^+\right)\ge(1+\alpha_k)T_\alpha U(t_k),\\ U\left(t_k^+\right)\ge(1+\theta_k)U(t_k),\ k=1,2,\cdots\end{cases}\tag{2.2}$$

have no eventually negative solutions, then every nonzero solution $u(t,x)$ of problems (1.1) and (1.2) is oscillatory in G.

Proof (by contradiction) Suppose that $u(t,x)$ is a non-oscillatory solution of problems (1.1) and (1.2). Without loss of generality, we assume that $u(t,x)$ is an eventually positive solution of problems (1.1) and (1.2), that is, there exists $\mu>0$ such that when $(t,x)\in[\mu,+\infty)\times\Omega$, $u(t,x)>0$, $u(t-\tau_i,x)>0$ and $u(t-\delta_j,x)>0$.

1) When $t\ne t_k$, integrating the first equation of (1.1) with respect to x over the bounded domain Ω on both sides, we get:

$$\begin{aligned}&T_\alpha\left(T_\alpha\int_\Omega u(t,x)\mathrm{d}x\right)+p(t)T_\alpha\int_\Omega u(t,x)\mathrm{d}x\\ &=a(t)\int_\Omega h\left(u(t,x)\right)\Delta u\mathrm{d}x+\int_\Omega\sum_{i=1}^m a_i(t)h_i\left(u(t-\tau_i,x)\right)\Delta u(t-\tau_i,x)\mathrm{d}x\\ &\quad-\sum_{j=1}^n\int_\Omega q_j(t,x)f_j\left(u(t-\delta_j,x)\right)\mathrm{d}x-\int_\Omega g(t,x)\mathrm{d}x, t\ne t_k,(t,x)\in\mathbb{R}_+\times\Omega.\end{aligned}\tag{2.3}$$

According to Green's formula, combining with the boundary condition (1.2) and the assumption (H3), we obtain:

$$\begin{aligned}\int_\Omega h(u)\Delta u(t,x)\mathrm{d}x &= \int_{\partial\Omega} h(u)\frac{\partial u(t,x)}{\partial n}\mathrm{d}x - \int_\Omega h'(u)\left|\mathrm{grad}u\right|^2 \mathrm{d}x \\ &= -\int_\Omega h'(u)\left|\mathrm{grad}u\right|^2 \mathrm{d}x \le 0, t \ge t_0.\end{aligned} \tag{2.4}$$

Similarly, we can obtain

$$\int_\Omega h_i\left(u\left(t-\tau_i\right),x\right)\Delta u\left(t-\tau_i,x\right)\mathrm{d}x \le 0. \tag{2.5}$$

According to the assumption (H1) and (H2), we obtain

$$\sum_{j=1}^{n}\int_\Omega q_j(t,x) f_j\left(u\left(t-\delta_j,x\right)\right)\mathrm{d}x \ge \sum_{j=1}^{n} k_j q(t)\int_\Omega u\left(t-\delta_j,x\right)\mathrm{d}x \ge 0, t \ge t_0. \tag{2.6}$$

By combining (2.3) - (2.6), we can obtain

$$T_\alpha\left(T_\alpha U(t)\right) + p(t) T_\alpha U(t) \le -G(t),\ t \ge t_0. \tag{2.7}$$

2) When $t = t_k$, integrating the second and third equations of (1.1) with respect to x over the bounded domain Ω on both sides and combining with the assumption (H4), we obtain

$$\begin{aligned}&T_\alpha \int_\Omega u\left(t_k^+,x\right)\mathrm{d}x = T_\alpha U\left(t_k^+,x\right) \le \left(1+\alpha_k\right) T_\alpha \int_\Omega u\left(t_k,x\right)\mathrm{d}x = \left(1+\alpha_k\right) T_\alpha U\left(t_k\right), \\ &U\left(t_k^+\right) = \int_\Omega u\left(t_k^+,x\right)\mathrm{d}x \le \left(1+\theta_k\right)\int_\Omega u\left(t_k,x\right)\mathrm{d}x = \left(1+\theta_k\right) U\left(t_k\right),\ k = 1,2,3,\cdots\end{aligned} \tag{2.8}$$

Therefore, from the impulsive differential inequalities (2.7)-(2.8), we know that the function $U(t) = \int_\Omega u(t,x)\mathrm{d}x$ is an eventually positive solution of the fractional impulsive differential inequality (2.1), which contradicts the assumed conditions.

On the other hand, if $u(t,x)$ is an eventually negative solution of problems (1.1) and (1.2) in G, that is, there exists $\mu > 0$ such that when $(t,x) \in [\mu,+\infty)\times\Omega$, $u(t,x) < 0$, $u(t-\tau_i,x) < 0$ and $u\left(t-\delta_j,x\right) < 0$.

3) When $t \ne t_k$, by using Green's formula for Equation (2.3) and combining with the boundary condition (1.2) and the assumption (H3), we have:

$$\begin{aligned}\int_\Omega h(u)\Delta u(t,x)\mathrm{d}x &= \int_{\partial\Omega} h(u)\frac{\partial u(t,x)}{\partial n}\mathrm{d}x - \int_\Omega h'(u)\left|\mathrm{grad}u\right|^2 \mathrm{d}x \\ &= -\int_\Omega h'(u)\left|\mathrm{grad}u\right|^2 \mathrm{d}x \ge 0, t \ge t_0.\end{aligned} \tag{2.9}$$

Similarly, we can obtain

$$\int_\Omega h_i\left(u\left(t-\tau_i,x\right)\right)\Delta u\left(t-\tau_i,x\right)\mathrm{d}x \ge 0, t \ge t_0. \tag{2.10}$$

According to the conditions (H1) and (H2), we get

$$\sum_{j=1}^{n}\int_\Omega q_j(t,x) f_j\left(u\left(t-\delta_j,x\right)\right)\mathrm{d}x \le \sum_{j=1}^{n} k_j q(t)\int_\Omega u\left(t-\delta_j,x\right)\mathrm{d}x \le 0, t \ge t_0. \tag{2.11}$$

By combining (2.3), (2.9) - (2.11), we can obtain

$$T_\alpha\left(T_\alpha U(t)\right) + p(t) T_\alpha U(t) \ge -G(t), t \ge t_0. \tag{2.12}$$

4) When $t = t_k$, integrating the second and third equations of (1.1) with respect

to x over the bounded domain Ω on both sides and combining with the assumption (H4), we obtain:

$$\begin{aligned} &T_\alpha \int_\Omega u\left(t_k^+, x\right)\mathrm{d}x = T_\alpha U\left(t_k^+\right) \ge \left(1+\alpha_k\right) T_\alpha \int_\Omega u\left(t_k, x\right)\mathrm{d}x = \left(1+\alpha_k\right) T_\alpha U\left(t_k\right), \\ &U\left(t_k^+\right) = \int_\Omega u\left(t_k^+, x\right)\mathrm{d}x \ge \left(1+\theta_k\right)\int_\Omega u\left(t_k, x\right)\mathrm{d}x = \left(1+\theta_k\right) U\left(t_k\right),\ k = 1,2,3,\cdots \end{aligned} \tag{2.13}$$

Therefore, from the impulsive differential inequalities (2.12)-(2.13), we know that the function $U(t) = \int_\Omega u(t,x)\mathrm{d}x$ is an eventually negative solution of the fractional impulsive differential inequality (2.2), which contradicts the assumed conditions. The proof is completed.

Theorem 2.2. Suppose that $h(u) = h_i(u) = 1$, and the conditions (H1) and (H2), (H4) and (H5) hold. If the fractional impulsive differential inequality (2.1) has no eventually positive solutions and inequality (2.2) has no eventually negative solutions, then every nonzero solution $u(t,x)$ of problems (1.1) and (1.3) is oscillatory in G.

Proof (by contradiction) Suppose that $u(t,x)$ is solution of problems (1.1) and (1.3) (where $h(u) = h_i(u) = 1$). Without loss of generality, we assume that $u(t,x)$ is an eventually positive solution of problems (1.1) and (1.3), that is, there exists $\mu > 0$ such that when $(t,x) \in [\mu, +\infty) \times \Omega$, $u(t,x) > 0$, $u(t-\tau_i, x) > 0$ and $u(t-\delta_j, x) > 0$.

1) When $t \ne t_k$, multiplying the first equation of (1.1) by $\phi_0(x)$ on both sides and integrating with respect to x over the bounded domain Ω, we obtain:

$$\begin{aligned} &T_\alpha\left(T_\alpha \int_\Omega u(t,x)\phi_0(x)\mathrm{d}x\right) + p(t) T_\alpha \int_\Omega u(t,x)\phi_0(x)\mathrm{d}x \\ &= a(t)\int_\Omega \phi_0(x)\Delta u \mathrm{d}x + \int_\Omega \sum_{i=1}^m a_i(t)\phi_0(x)\Delta u(t-\tau_i, x)\mathrm{d}x \\ &\quad -\sum_{j=1}^n \int_\Omega q_j(t,x)\phi_0(x) f_j\left(u\left(t-\delta_j, x\right)\right)\mathrm{d}x - \int_\Omega \phi_0(x) g(t,x)\mathrm{d}x,\ t \ge \mu. \end{aligned} \tag{2.14}$$

According to Green's formula, the boundary condition (1.3), and Lemma 1.1, we obtain:

$$\int_\Omega \phi_0(x)\Delta u(t,x)\mathrm{d}x = \int_\Omega \Delta\phi_0(x) u(t,x)\mathrm{d}x = -\lambda_0 \int_\Omega u(t,x)\phi_0(x)\mathrm{d}x < 0, t \ge \mu. \tag{2.15}$$

$$\int_\Omega \phi_0(x)\Delta u(t-\tau_i, x)\mathrm{d}x = -\lambda_0 \int_\Omega u(t-\tau_i, x)\phi_0(x)\mathrm{d}x < 0,\ t \ge \mu. \tag{2.16}$$

And according to the assumptions (H1) - (H2), we have

$$\begin{aligned} &\sum_{j=1}^n \int_\Omega q_j(t,x) f_j\left(u\left(t-\delta_j, x\right)\right)\phi_0(x)\mathrm{d}x \\ &\ge \sum_{j=1}^n k_j q(t)\int_\Omega u\left(t-\delta_j, x\right)\phi_0(x)\mathrm{d}x \ge 0, t \ge \mu. \end{aligned} \tag{2.17}$$

Furthermore, we have

$$-\int_\Omega g(t,x)\phi_0(x)\mathrm{d}x = -Q(t). \tag{2.18}$$

Combining (2.14)-(2.18), we get

$$T_\alpha\left(T_\alpha Y(t)\right) + p(t) T_\alpha Y(t) \le -Q(t),\ t \ge \mu, \tag{2.19}$$

where $Y(t)=\int_{\Omega}u(t,x)\phi_0(x)\mathrm{d}x$, $Q(t)=\int_{\Omega}g(t,x)\phi_0(x)\mathrm{d}x$.

2) When $t=t_k$, multiplying the second and third equations of (1.1) by $\phi_0(x)$ on both sides and integrating with respect to x over the bounded domain Ω, according to the condition (H4), similarly, we can obtain

$$\begin{aligned}&T_\alpha Y\left(t_k^+\right)\le\left(1+\alpha_k\right)T_\alpha Y\left(t_k\right),\\&Y\left(t_k^+\right)=\int_{\Omega}u\left(t_k^+,x\right)\phi_0(x)\mathrm{d}x\\&\qquad\le\left(1+\theta_k\right)\int_{\Omega}u\left(t_k,x\right)\phi_0(x)\mathrm{d}x=\left(1+\theta_k\right)Y\left(t_k\right),\\&k=1,2,3,\cdots\end{aligned}\tag{2.20}$$

Therefore, from the impulsive differential inequalities (2.19)-(2.20), we know that the function $Y(t)=\int_{\Omega}u(t,x)\phi_0(x)\mathrm{d}x$ is an eventually positive solution of the fractional impulsive differential inequality (2.1), which contradicts the assumed conditions.

On the other hand, if $u(t,x)$ is an eventually negative solution of problems (1.1) and (1.3) in G, similarly, we can know that the function $Y(t)=\int_{\Omega}u(t,x)\phi_0(x)\mathrm{d}x$ is an eventually negative solution of the fractional impulsive differential inequality (2.2), which contradicts the assumed conditions. The proof is completed.

Theorem 2.3. Suppose that the conditions (H1)-(H5) hold. If there exists $\mu_2\ge 0$ such that

$$\int_{\mu_2}^{\infty}\exp\left(-\int_{t_0}^{t}p(\sigma)\mathrm{d}\sigma\right)\mathrm{d}t=\infty,\tag{2.21}$$

and there exists $\mu_1\ge 0$ such that

$$\limsup_{\xi\to\infty}\frac{\int_{\mu_1}^{\xi}\prod_{s<\xi_l<\xi}(1+\alpha_k)\exp\left(-\int_s^{\xi}\tilde{p}(\sigma)\mathrm{d}\sigma\right)\tilde{G}(s)\mathrm{d}s}{\prod_{\mu_1<\xi_l<\xi}(1+\alpha_k)\exp\left(-\int_{\mu_1}^{\xi}\tilde{p}(s)\mathrm{d}s\right)}=\infty,\xi_l=\xi\left(t_l\right)\tag{2.22}$$

$k=1,2,3,\cdots$

$$\liminf_{\xi\to\infty}\frac{\int_{\mu_1}^{\xi}\prod_{s<\xi_l<\xi}(1+\alpha_k)\exp\left(-\int_s^{\xi}\tilde{p}(\sigma)\mathrm{d}\sigma\right)\tilde{G}(s)\mathrm{d}s}{\prod_{\mu_1<\xi_l<\xi}(1+\alpha_k)\exp\left(-\int_{\mu_1}^{\xi}\tilde{p}(s)\mathrm{d}s\right)}=-\infty,\xi_l=\xi\left(t_l\right)\tag{2.23}$$

$k=1,2,3,\cdots$

Then every solution of problems (1.1) and (1.2) is oscillatory in G.

Proof (by contradiction) To prove this theorem, we first prove that the fractional impulsive differential inequality (2.1) has no eventually positive solutions and the fractional impulsive differential inequality (2.2) has no eventually negative solutions. Without loss of generality, we assume that $U(t)$ is an eventually positive solution of the fractional impulsive differential inequality (2.1), then there exists $\mu_1\ge 0$ such that $U(t)\ge 0$, $U(t-\tau)\ge 0$, $t\ge\mu_1$. Let $V(t)=\exp\left(\int_{t_0}^{t}p(s)\mathrm{d}s\right)$, by property (2) in Definition 1.3, $T_\alpha\xi(t)=T_\alpha\left[\frac{1}{\alpha}t^\alpha\right]=1$. Hence, we have:

$$\begin{aligned}&T_\alpha U(t)=T_\alpha\tilde{U}(\xi)=\tilde{U}'(\xi)t^{\alpha-1}t^{1-\alpha}=\tilde{U}'(\xi),\\&T_\alpha\left(T_\alpha U(t)\right)=T_\alpha\tilde{U}'(\xi)=\tilde{U}''(\xi)t^{\alpha-1}t^{1-\alpha}=\tilde{U}''(\xi).\end{aligned}\tag{2.24}$$

From Equation (2.24), Equation (2.19) can be transformed into

$$\tilde{U}''(\xi)+\tilde{p}(\xi)\tilde{U}'(\xi)\le -\tilde{G}(\xi). \tag{2.25}$$

So we have

$$\left[\tilde{U}'(\xi)V(\xi)\right]'=\tilde{U}''(\xi)V(\xi)+\tilde{U}'(\xi)V(\xi)\tilde{p}(\xi)\le -\tilde{G}(\xi)V(\xi)<0. \tag{2.26}$$

Also, according to the third formula of (1.1), when $\xi\in[\xi_1,\infty)$, $\tilde{U}'(\xi)V(\xi)$ is strictly monotonically decreasing and non-sign-changing. Since $V(\xi)>0$ when $\xi\in[\xi_1,\infty)$, it can be known that $\tilde{U}'(\xi)$ is eventually not changing in sign. When $\xi\in[\xi_1,\infty)$, $\tilde{U}'(\xi)>0$, otherwise, $\tilde{U}'(\xi)<0$, that is, there exists $\xi_2\in[\xi_1,\infty)$ such that $\tilde{U}'(\xi)<0$. Since $\tilde{U}'(\xi)V(\xi)$ is strictly monotonically decreasing on $\xi\in[\xi_1,\infty)$, then when $\xi\in[\xi_2,\infty)$, $\tilde{U}'(\xi)V(\xi)\le\tilde{U}'(\xi_2)V(\xi_2)=c_1<0$. So the following inequality holds

$$\tilde{U}^{'}(\xi)\le\frac{c_1}{V(\xi)}=c_1\exp\left(-\int_{t_0}^{\xi}p(v)\,\mathrm{d}v\right)<0,\xi\in[\xi_2,\infty). \tag{2.27}$$

Dividing both sides of the above formula by c_1 and integrating, we get

$$\int_{\xi_2}^{\xi}\exp\left(-\int_{t_0}^{s}p(v)\,\mathrm{d}v\right)\mathrm{d}s\le\frac{\tilde{U}(\xi)-\tilde{U}(\xi_2)}{c_1}<\frac{-\tilde{U}(\xi_2)}{c_1},\xi\in[\xi_2,\infty). \tag{2.28}$$

Let $\xi\to\infty$, we can obtain $\int_{\xi_2}^{\xi}\exp\left(-\int_{t_0}^{s}p(v)\,\mathrm{d}v\right)\mathrm{d}s<\frac{-\tilde{U}(\xi_2)}{c_1}<\infty$, which contradicts Equation (2.21), so $\tilde{U}'(\xi)>0$.

Let $\omega(\xi)=\tilde{U}'(\xi)>0$. According to the first two formulas of the fractional impulsive differential inequality (2.1), we can get

$$\begin{aligned}&\omega'(\xi)\le-\tilde{p}(\xi)\omega(\xi)-\tilde{G}(\xi),\ \xi\ge\xi_0,\xi\ne\xi_k,\\&\omega\left(t_k^+\right)\le(1+\alpha_k)\omega(t_k),\ k=1,2,3,\cdots\end{aligned} \tag{2.29}$$

By Lemma (1.2), we can obtain

$$\begin{aligned}\omega(\xi)\le\omega(\xi_0)&\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi}\tilde{p}(s)\,\mathrm{d}s\right)\\&-\int_{\xi_0}^{\xi}\prod_{s<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{s}^{\xi}\tilde{p}(\sigma)\,\mathrm{d}\sigma\right)\tilde{G}(s)\,\mathrm{d}s,\end{aligned} \tag{2.30}$$

Therefore,

$$\begin{aligned}&\frac{\omega(\xi)}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi}\tilde{p}(s)\,\mathrm{d}s\right)}\\&\le\omega(\xi_0)-\frac{\int_{\xi_0}^{\xi}\prod_{s<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{s}^{\xi}\tilde{p}(\sigma)\,\mathrm{d}\sigma\right)\tilde{G}(s)\,\mathrm{d}s}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi}\tilde{p}(s)\,\mathrm{d}s\right)}.\end{aligned} \tag{2.31}$$

When $\xi\to\infty$, and by Formula (2.22), we can get

$$\liminf_{\xi\to\infty}\frac{\omega(\xi)}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi}\tilde{p}(s)\,\mathrm{d}s\right)}=-\infty, \tag{2.32}$$

It can be seen that this contradicts $\omega(\xi) > 0$. The proof is completed.

On the other hand, if the fractional impulsive differential inequality (2.2) has an eventually negative solution $U(t)$, then there exists $\tau_1 \ge 0$ such that $U(t) < 0$, $U(t-\delta_i) < 0$, $G(t) < 0$, $t \ge \tau_1$, $\tilde{\omega}(\xi) = \tilde{U}'(\xi)$. Similarly, we can get $\tilde{\omega}(\xi) = \tilde{U}'(\xi) < 0$. According to the inequality (2.2), we have

$$\begin{aligned} &\tilde{\omega}'(\xi) \ge -\tilde{p}(\xi)\tilde{\omega}(\xi) - \tilde{G}(\xi), \xi \ge \xi_0, \xi \ne \xi_k, \\ &\tilde{\omega}(t_k^+) \ge (1+\alpha_k)\omega(t_k), k = 1,2,3,\cdots \end{aligned} \tag{2.33}$$

Let $\tilde{\omega}(\xi) = -m(\xi)$, then we can get

$$\begin{aligned} &m'(\xi) \le -\tilde{p}(\xi)m(\xi) + \tilde{G}(\xi), \xi \ge \xi_0, \xi \ne \xi_k, \\ &m(t_k^+) \le (1+\alpha_k)m(t_k), k = 1,2,3,\cdots \end{aligned} \tag{2.34}$$

According to Lemma 1.2, we have

$$\begin{aligned} m(\xi) \le\, & m(\xi_0) \prod_{\xi_0<\xi_1<\xi} (1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right) \\ & + \int_{\xi_0}^{\xi} \prod_{s<\xi_1<\xi} (1+\alpha_k)\exp\left(-\int_{s}^{\xi} \tilde{p}(\sigma)\mathrm{d}\sigma\right)\tilde{G}(s)\mathrm{d}s, \end{aligned} \tag{2.35}$$

So, we have

$$\begin{aligned} & \frac{m(\xi)}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right)} \\ & \le m(\xi_0) + \frac{\int_{\xi_0}^{\xi} \prod_{s<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{s}^{\xi} \tilde{p}(\sigma)\mathrm{d}\sigma\right)\tilde{G}(s)\mathrm{d}s}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right)}, \end{aligned} \tag{2.36}$$

Then

$$\begin{aligned} & \frac{\tilde{\omega}(\xi)}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right)} \\ & \ge \tilde{\omega}(\xi_0) - \frac{\int_{\xi_0}^{\xi} \prod_{s<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{s}^{\xi} \tilde{p}(\sigma)\mathrm{d}\sigma\right)\tilde{G}(s)\mathrm{d}s}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right)}. \end{aligned} \tag{2.37}$$

When $\xi \to \infty$, we can get

$$\limsup_{\xi\to\infty} \frac{\tilde{\omega}(\xi)}{\prod_{\xi_0<\xi_1<\xi}(1+\alpha_k)\exp\left(-\int_{\xi_0}^{\xi} \tilde{p}(s)\mathrm{d}s\right)} = \infty, \tag{2.38}$$

which contradicts $\tilde{\omega}(\xi) < 0$, the proof is completed.

Theorem 2.4. Under the conditions of Theorem 2.2, if there exist $\mu_2 \ge 0, \mu_1 \ge 0$ such that (2.21)-(2.23) hold, then every solution of problems (1.1) and (1.3) is oscillatory in G.

3. Example

Example 3.1. Consider the following fractional impulsive partial differential equations with multiple delays:

$$\begin{cases} T_{\frac{1}{3}}\left(T_{\frac{1}{3}}u(t,x)\right)+\dfrac{1}{t}T_{\frac{1}{3}}u(t,x)=t\mathrm{e}^{-t}u^2(t,x)\Delta u(t,x) \\ \quad +\sum_{i=1}^{3}\dfrac{t^2}{i+1}u^2\left(t-\dfrac{i}{2},x\right)\Delta u\left(t-\dfrac{i}{2},x\right) \\ \quad -\sum_{j=1}^{3}\left(j+t^2+x^2\right)u\left(t-\dfrac{2j\pi}{3},x\right)\mathrm{e}^{\left[u\left(t-\frac{2j\pi}{3},x\right)\right]^2} \\ \quad -t\cos x,\ t\neq t_k,(t,x)\in R_+\times\Omega=G, \\ T_{\frac{1}{3}}u\left(t_k^+,x\right)-T_{\frac{1}{3}}u\left(t_k^-,x\right)=t_k^{-3}\cos x\cdot T_{\frac{1}{3}}u\left(t_k,x\right), \\ \quad u\left(t_k^+,x\right)-u\left(t_k^-,x\right)=-t_k^{-2}\sin x\cdot u\left(t_k,x\right),\ t=t_k,k=1,2,\cdots \end{cases} \tag{3.1}$$

The boundary conditions satisfy

$$\frac{\partial u(t,x)}{\partial n}=0,(t,x)\in R_+\times\partial\Omega,t\neq t_k, \tag{3.2}$$

where $\alpha=\dfrac{1}{3}$, $\Omega=\left(0,\dfrac{\pi}{2}\right)$, $m=n=3$, $p(t)=\dfrac{1}{t}$, $r(t)=\dfrac{1}{t}$, $a(t)=t\mathrm{e}^{-t}$, $a_i(t)=\dfrac{t^2}{i+1}$, $h_i(u(t,x))=u^2(t,x)$, $\tau_i=\dfrac{i}{2}$, $q_j(t,x)=x^2+t^2+j$, $f_j(u(t,x))=u(t,x)\mathrm{e}^{u^2}$, $\delta_j=\dfrac{2j\pi}{3}$, $g(t,x)=t\cos x$, $\sigma(t_k,x)=t_k^{-3}\cos x$, $\theta(t_k,x)=-t_k^{-2}\sin x$, $\alpha_k=t_k^{-3}$, $\theta_k=-t_k^{-2}$, $(t,x)\in R^+\times\left(0,\dfrac{\pi}{2}\right)$. It can be calculated that

$$\int_{\tau_2}^{\infty}\exp\left(-\int_{t_0}^{t}p(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s=\int_{\tau_2}^{\infty}\exp\left(-\int_{t_0}^{t}\frac{1}{\sigma}\mathrm{d}\sigma\right)\mathrm{d}s=\int_{\tau_2}^{\infty}\frac{t_0}{t}\mathrm{d}t=\infty, \tag{3.3}$$

$$\limsup_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_s^t p(\sigma)\mathrm{d}\sigma\right)G(s)\mathrm{d}s}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp\left(-\int_{\tau_1}^{t}p(s)\mathrm{d}s\right)}=\infty, \tag{3.4}$$

$$\liminf_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_s^t p(\sigma)\mathrm{d}\sigma\right)G(s)\mathrm{d}s}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp\left(-\int_{\tau_1}^{t}p(s)\mathrm{d}s\right)}=-\infty. \tag{3.5}$$

Therefore, all the conditions of Theorem 2.3 are satisfied. So all nonzero solutions of problems (3.1)-(3.2) are oscillatory.

The oscillation criteria derived in this paper may be applied to viscoelastic beam vibration models (fractional-order systems with memory effects) and thermal diffusion processes (delayed diffusion equations with impulsive disturbances). For instance, both in the vibration of a viscoelastic beam and thin film, the impulsive terms can represent instantaneous external force impacts, while the multiple time delays correspond to the hysteretic effects of stress transmission inside the material. The results of this paper may be used to determine whether the beam undergoes sustained vibration or tends to be stable.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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