

Thermodynamically and Mathematically Consistent Linear Micromorphic Microcontinuum Theory for Solid Continua

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Abstract

This paper presents derivation of micro and macro conservation and balance laws and the constitutive theories for the linear elastic micromorphic theory, in which elasticity is considered for microconstituents, the solid medium, and for the interaction of microconstituents with the solid medium. The conservation and balance laws are initiated for micro deformation, followed by consistent “*integral-average*” definitions valid at the macro level. These permit the derivation of the conservation and balance laws at the macro level. Significant aspects of this theory are: 1) microconstituent rigid rotation physics is the same in all 3M theories. The rigid rotations of the microconstituents are in fact classical rotations; hence, they do not introduce three unknown degrees of freedom at the material point and also can not be part of the strain measures. Thus, in this theory, a microconstituent has only six unknown degrees of freedom, six independent components of the symmetric part of the micro deformation gradient tensor, as opposed to Eringen’s theory, in which all nine components of the micro deformation gradient tensor are unknown degrees of freedom. 2) The balance of moment of moments balance law is shown to be essential in all 3M theories and hence is considered here, due to which the Cauchy moment tensor is symmetric. This avoids a spurious constitutive theory for the moment tensor. 3) In the case of nonsymmetric macro Cauchy stress tensor, the constitutive theory is needed only for the symmetric part, as the skew-symmetric part is defined by the balance of angular momenta. 4) The smoothing weighting function $\phi^{(\alpha)}$ for the microconstituent, as advocated by Eringen and used to multiply the balance of linear momenta of the micro deformation physics, has no thermodynamic, physical or mathematical basis; hence, it is not used in the present work. 5) In contrast with published works of Eringen and others, all constitutive tensors of rank two are always symmetric, hence always permitting the use of the representation theorem in deriving constitu-

tive theories, ensuring the mathematical consistency of the resulting theories. 6) Conservation of micro inertia, necessary in Eringen's theories to provide closure to the mathematical model, is neither needed nor used in the present work. The linear micromorphic theory derived here is compared with Eringen's theory to identify differences, discuss and evaluate these for their validity based on thermodynamic and mathematical principles to ultimately determine the thermodynamic and mathematical consistency of the published micromorphic theories.

Keywords

Micromorphic, Micro, Macro, Deformation/Strain Measures, Conservation and Balance Laws, Balance of Moment of Moments, Integral-Average, Representation Theorem, Constitutive Theories

1. Introduction

The polar nature of solids, in particular crystalline solids, was observed by Voigt in 1887 [1]. He presented equations of equilibrium for such solids, including moment equilibrium. In 1909, E. Cosserat and F. Cosserat presented a theory of elasticity by considering rotations about rigid directors using the principle of virtual work [2]. They derived the balance of momenta for the dynamic case. This work lacked conservation and balance laws and had no infrastructure for constitutive theories, hence it remained dormant till 1960. Grad in 1952 [3], Gunther in 1958 [4] and Schaefer in 1967 [5] revisited the Cosserat theory and established its connection to dislocation physics. Eringen in 1967 [6]-[8] presented the linear theory of micromorphic elastic solids and the theory of micropolar plates. In 1964, Eringen and Suhubi [9] presented a nonlinear theory for microelastic solids. Various works of Eringen [10]-[16] consider various aspects of micropolar continuum theories. The mechanics of micromorphic continua were introduced by Eringen in 1968 [17]. This was followed by subsequent works of Eringen: conditions for the theory of micromorphic solids in 1969 [18], balance laws for micromorphic mechanics in 1970 [19], micromorphic materials with memory in 1972 [20], balance laws for micromorphic continua revisited in 1992 [20], application of micromorphic theory to dislocation physics in 1970 [21].

A compilation of Eringen's work on 3M theories was published in the two books by Eringen [22] [23]. Since the publication of micromorphic theory and more generally 3M microcontinuum theories, many papers have appeared on micropolar theories, but not so many on micromorphic theory. Most published works on micropolar theory are largely following the balance laws and the constitutive theories derived by Eringen. We cite some more recent works in the following.

Vernerey, Liu and Moran in 2007 [24] presented a multiscale micromorphic theory for hierarchical materials. The authors employed the method of virtual power to derive the mathematical model for a particular scale, with its interaction with the

macroscale. Chen, Lee and Xiong in 2009 [25] considered the concept of deformable material point, thus not taking a crystal as structureless and hence not idealizing it as a mass point. The paper considers classical, micromorphic and generalized field theories and their applicability. Finite strain micromorphic elastoplasticity theory was presented by Regueiro in 2010 [26]. The basic foundations of the conservation and the balance laws and the constitutive theories in this work follow the approach presented by Eringen for micromorphic continua with extension to finite strain plasticity. Wang and Lee in 2010 presented micromorphic theory as a gateway to the nano world [27]. This work uses the basic micromorphic theory of Eringen but additionally introduces objective Eringen tensors in the kinematics and the balance laws are derived by requiring the energy equations to be form-invariant under generalized Galilean transformation. Lee and Wang in 2011 [28] presented a generalized micromorphic theory for solids and fluids based on Eringen's micromorphic theory with adjustments similar to those in Lee's work in reference [27]. Isbuga and Regueiro in 2011 [29] presented a three-dimensional finite element analysis of finite deformation micromorphic linear elasticity. The conservation and balance laws and the constitutive theories used in this work are the same as those due to Eringen. Reges, Petangueira and Silva in 2024 [30] presented a modeling of micromorphic continua based on a heterogeneous microscale. In ref. [31], McAvoy presented a consistent linearization technique for micromorphic continuum theories and presented its applicability to 3M theories. They advocated that this linearization technique yields tractable linear theories for nonlinear elastic microstructured materials.

From the literature review, it is clear that most published works on micromorphic microcontinuum theories follow the conservation and the balance laws and the constitutive theories derived and presented by Eringen (papers are cited here).

2. Scope of Work

In this paper, we undertake the derivation of the conservation and the balance laws and the constitutive theories for a linear elastic micromorphic continua in which: 1) the deformation/strain measures derived by Surana *et al.* [32] serve as the basic measures of deformation for the micromorphic theory. The microconstituents are deformable, hence there is a micro deformation gradient tensor associated with them. 2) All conservation and balance laws are initiated for the micro deformation of the microconstituent using the laws of thermodynamics of classical continuum mechanics, yielding micro conservation and balances. From micro conservation and balance laws, "integral-average" definitions are introduced that permit the derivation of macro conservation and balance laws and the constitutive theories using principles of thermodynamics and well-established concepts in applied mathematics. 3) In deriving conservation and balance laws and the constitutive theories for micromorphic continua, we maintain and adhere to the concepts of classical rotations, the Cauchy moment tensor, and the theory of isotropic tensors, etc. introduced and used by Surana *et al.* [32]-[54] in conjunction

with linear and nonlinear micropolar theories for solid and fluent continua. This is necessary because the physics of rigid rotations of microconstituents exists in all 3M theories as it arises from the skew-symmetric part of the micro deformation gradient tensor. Thus, we must have the exact same mathematical treatment of rigid rotation physics in 3M theories, requiring that we maintain the micropolar theory as a subset of micromorphic theory.

We first present conservation and balance laws for micro as well as macro deformation physics with clarity of valid “integral-average” definitions that are essential for deriving the conservation and the balance laws at the macro level. This is followed by constitutive theories for the macro Cauchy stress tensor, the microconstituent Cauchy stress tensor, the macro Cauchy moment tensor and the heat vector. Constitutive theories are initiated using conjugate pairs in the entropy inequality, establishing constitutive tensors and their argument tensors. Constitutive tensors and argument tensors are adjusted or augmented as required by the desired physics that may not have been considered while deriving the entropy inequality. In all three constitutive theories (microconstituent Cauchy stress tensor, macro Cauchy stress tensor and macro Cauchy moment tensor), we consider mechanisms of elasticity for microconstituent Cauchy stress tensor (\mathcal{S}) as well as the macro stress tensor σ (symmetric part) and the interaction of microconstituent in the elastic medium, Cauchy moment tensor m .

All constitutive theories are derived using the representation theorem [55]-[66], hence are always thermodynamically (not in violation of entropy inequality) and mathematically consistent. Constitutive theories and material coefficients are first derived for the constitutive theories based on integrity (complete basis) and then their simplified forms are presented, which are linear in the components of the argument tensors.

The micromorphic theory derived here is shown to be thermodynamically and mathematically consistent, hence it is a valid and physical linear micromorphic theory. The linear micromorphic theory derived here is compared with Eringen’s micromorphic theory. The differences in the two theories are identified, discussed and evaluated for their validity based on thermodynamic principles and well accepted mathematical concepts to establish their validity or lack of validity, hence the validity of the resulting micromorphic theory.

3. Preliminary Considerations and Definitions

3.1. Classical Rotations (\mathcal{C}), Deformation Gradient Tensor, Classical Rotation Gradient Tensor, and Micro Deformation

In classical continuum mechanics, the macro deformation gradient tensor $[J]$ is given in the Lagrangian description as

$$[J] = \left[\frac{\partial \{\bar{x}\}}{\partial \{x\}} \right] = [{}^d J] + [I]; \quad [{}^d J] = \left[\frac{\partial \{u\}}{\partial \{x\}} \right] \quad (1)$$

in which \bar{x} and x are the deformed and undeformed coordinates of a mate-

rial point in a fixed x -frame and \mathbf{u} are displacements of the material point also in x -frame. The additive decomposition of $[{}^d J]$ into symmetric $[{}_s J]$ and skew-symmetric $[{}_a J]$ tensor gives

$$[{}^d J] = [{}_s J] + [{}_a J]; [{}_s J] = \frac{1}{2}([{}^d J] + [{}^d J]^T); [{}_a J] = \frac{1}{2}([{}^d J] - [{}^d J]^T) \quad (2)$$

We also note that

$$\nabla \times \mathbf{u} = ({}_c \Theta_1) \mathbf{e}_1 + ({}_c \Theta_2) \mathbf{e}_2 + ({}_c \Theta_3) \mathbf{e}_3 \quad (3)$$

$${}_c \Theta_1 = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right); {}_c \Theta_2 = \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right); {}_c \Theta_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \quad (4)$$

where ${}_c \Theta$ are the classical rotations. The skew-symmetric matrix $[{}_a J]$ contains $\frac{{}_c \Theta}{2}$. ${}_c \Theta$ are the rotations about the axes of a triad located at the material point with axes parallel to the axes of the x -frame.

In classical continuum mechanics, ${}_c \Theta$ constitutes a free field. Thus, even though it is present in every deforming solid, the classical continuum theory is not influenced by its presence because it is a free field. In the presence of microconstituents causing obstruction to the free field ${}_c \Theta$, the rotation field due to ${}_c \Theta$ is no longer a free field and in fact describes the rotations of the microconstituents. A simple example illustrates this quite well. Consider 1D axial deformation of an unconstrained rod subjected to a force at the right end. The rigid body translations of the rod constitute a free field that has no effect on the deformation of the rod as all points of the rod are moving in the same direction by the same amount. If we constrain the left end of the rod from moving, then the displacement field is no longer a free field and is, in fact, the actual deformation field of the constrained rod with load on the right end. Thus, we see that the obstruction (constrained left end in this case) changes the free field to the actual deformation field of the constrained rod. Secondly, the free field has no influence on the actual deformation field of the constrained rod. Our situation of ${}_c \Theta$ as a free field and the microconstituents obstructing this free field is exactly similar to the axial rod. That is, the free field ${}_c \Theta$, in the presence of microconstituents, becomes a rotation field ${}_c \Theta$ describing the rotations of the microconstituents, meaning ${}_c \Theta$ is in fact the rigid rotations of the microconstituents. Thus, in micropolar theory, in which microconstituents only experience rigid rotations referred to as ${}_\alpha \Theta$ (defined by skew symmetric part of microconstituent deformation gradient tensor $[{}_a J^{(\alpha)}]$), ${}_c \Theta$ suffices to be the rigid rotations of the microconstituents as known degrees of freedom. Thus, ${}_c \Theta$ serves as rigid rotations of the microconstituents in all three 3M theories, hence not requiring ${}_\alpha \Theta$ as unknown rigid rotational degrees of freedom.

For classical rotations ${}_c \Theta$, we also note the following:

$$[{}^{c\Theta} J] = \left[\frac{\partial \{ {}_c \Theta \}}{\partial \{ x \}} \right] = [{}_s {}^{c\Theta} J] + [{}_a {}^{c\Theta} J] \quad (5)$$

$$[{}_s {}^{c\Theta} J] = \frac{1}{2}([{}^{c\Theta} J] + [{}^{c\Theta} J]^T); [{}_a {}^{c\Theta} J] = \frac{1}{2}([{}^{c\Theta} J] - [{}^{c\Theta} J]^T) \quad (6)$$

in which ${}^c J$ is the gradient tensor of classical rotations and ${}^s J$ and ${}^a J$ are its symmetric and skew-symmetric decompositions.

Next, we consider the concept of a deformable material point, rationalized through deformable directors contained in a material point. Consider a volume of matter V enclosed by surface ∂V in the reference configuration. Upon deformation, V and ∂V change to \bar{V} and $\partial\bar{V}$ at time $t > 0$. Let the volume $V + \partial V$ contain microconstituents uniformly dispersed in the volume. Let the volume V of a material particle P contain N microconstituents. Let $V^{(\alpha)}$ and $\partial V^{(\alpha)}$ be the volume and its closure for the α^{th} microconstituent with mass density $\rho^{(\alpha)}$.

The center of mass of V has the position coordinate \mathbf{x} in $V + \partial V$. Let $\mathbf{x}^{(\alpha)}$ be the location of the microconstituent α with respect to the center of mass of $dV + \partial(dV)$ and let $\mathbf{x}^{(\alpha)}$ be its position coordinate in the x -frame. Upon deformation, in the current configuration, \mathbf{x} changes to $\bar{\mathbf{x}}$, $\mathbf{x}^{(\alpha)}$ to $\bar{\mathbf{x}}^{(\alpha)}$ and $\mathbf{x}^{(\alpha)}$ to $\bar{\mathbf{x}}^{(\alpha)}$ (see **Figure 1**).

At this stage, we can possibly entertain two different methodologies in deriving the deformation/strain measures.

1) In the first case, we assume that each microconstituent located at a different position in \bar{V} has its own deformation physics, implying that there are N different deformation physics within the volume \bar{V} of the material point \bar{P} . We then assume that the material point \bar{P} only sees the homogenized response of N microconstituents. Since the homogenization yields surrogate behavior, homogenization must include boundary conditions and the load so that the homogenized model with surrogate material properties is representative of the true physics. It is obvious that homogenization is not practical for the volume \bar{V} of the material point.

2) In the second approach, we assume that the material point \bar{P} only sees the statistically averaged deformation physics of N microconstituents. This is a more practical viewpoint of considering complex physics within the volume \bar{V} . To simplify the consideration of varied deformation physics of microconstituents in this approach, we assume that there is a surrogate configuration of microconstituents in \bar{V} in which each of the N microconstituents has identical deformation physics. Thus, the average of the N microconstituent deformation physics in this case is the same as the deformation physics of one microconstituent, which is assumed to be the same as the statistically averaged deformation physics due to the original configuration of microconstituents in volume \bar{V} . Thus referring to **Figure 1**, $\mathbf{x}^{(\alpha)}$ is the director in the undeformed configuration and $\bar{\mathbf{x}}^{(\alpha)}$ is the director in the deformed configuration \bar{V} . The deformation of $\mathbf{x}^{(\alpha)}$ is assumed to represent the micro deformation of the microconstituents within the volume $(\bar{V} + \partial\bar{V})$ of the material point with the director $\mathbf{x}^{(\alpha)}$. In the following, we only present essential relations that are helpful in the derivation of the conservation and balance laws and the constitutive theories for linear micromorphic solids. With

this, we can proceed with the details of the derivations of deformation measures presented in ref. [32]. Referring to **Figure 1**, the following relations hold:

$$\mathbf{x}^{(\alpha)} = \mathbf{x} + \underline{\mathbf{x}}^{(\alpha)}; \quad \bar{\mathbf{x}}^{(\alpha)} = \bar{\mathbf{x}} + \bar{\underline{\mathbf{x}}}^{(\alpha)} \tag{7}$$

If we consider Lagrangian description, then $\bar{\mathbf{x}}^{(\alpha)}$ depends upon \mathbf{x} and $\underline{\mathbf{x}}^{(\alpha)}$ and we can write:

And we can write:

$$\bar{\underline{\mathbf{x}}}^{(\alpha)} = \bar{\underline{\mathbf{x}}}^{(\alpha)}(\mathbf{x}, \underline{\mathbf{x}}^{(\alpha)}, t) \tag{8}$$

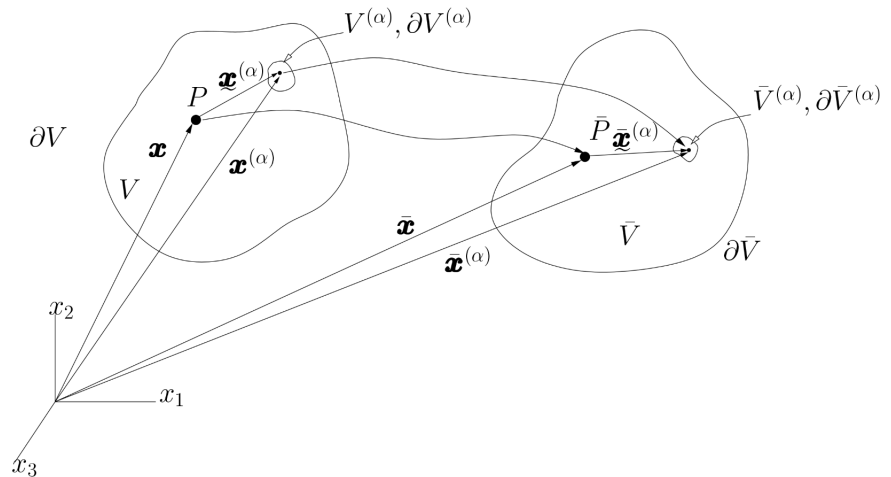


Figure 1. Undeformed and deformed configurations of material point volume.

We note that $\underline{\mathbf{x}}^{(\alpha)}$ and $\bar{\underline{\mathbf{x}}}^{(\alpha)}$ are undeformed and deformed coordinates. Hence,

$$\{\bar{\underline{\mathbf{x}}}^{(\alpha)}\} = [J^{(\alpha)}] \{\underline{\mathbf{x}}^{(\alpha)}\}; \quad [J^{(\alpha)}] = \begin{bmatrix} \frac{\partial \{\bar{\underline{\mathbf{x}}}^{(\alpha)}\}}{\partial \{\underline{\mathbf{x}}^{(\alpha)}\}} \end{bmatrix} \tag{9}$$

Substituting (9) into (7)

$$\{\bar{\mathbf{x}}^{(\alpha)}\} = \{\bar{\mathbf{x}}\} + [J^{(\alpha)}] \{\underline{\mathbf{x}}^{(\alpha)}\} \tag{10}$$

in which $[J^{(\alpha)}]$ is the micro deformation gradient tensor (similar to $[J]$ in macro physics). Additive decomposition of $[J^{(\alpha)}]$ gives

$$[J^{(\alpha)}] = [{}_s J^{(\alpha)}] + [{}_a J^{(\alpha)}] \tag{11}$$

Symmetric tensor $[{}_s J^{(\alpha)}]$ contains the deformation physics of the microconstituent and $[{}_a J^{(\alpha)}]$ contains $\frac{\alpha}{2} \Theta$, ${}_a \Theta$ being the rigid rotations of the microconstituents. Using (8), (9) and (10) the rest of the details regarding various micro deformation measures follow. For example, velocity \mathbf{v}^α , velocity gradient tensor $\mathbf{L}^{(\alpha)}$ and $\mathbf{J}^{(\alpha)}$ in the Lagrangian description are given by:

$$\mathbf{v}^\alpha = \frac{D(\bar{\mathbf{x}}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t))}{Dt} \quad (12)$$

$$[L^{(\alpha)}] = \left[\frac{\partial \mathbf{v}^\alpha}{\partial \mathbf{x}^{(\alpha)}} \right] \quad (13)$$

$$[j^{(\alpha)}] = [L^{(\alpha)}][J^{(\alpha)}] \quad (14)$$

and

$$\{\dot{\bar{\mathbf{x}}}^{(\alpha)}\} = \{v(\mathbf{x}, t)\} + [j^{(\alpha)}(\mathbf{x}^{(\alpha)})]\{\mathbf{x}^{(\alpha)}\} \quad (15)$$

In Eulerian description, we have the following:

$$[\bar{J}^{(\alpha)}] = \left[\frac{\partial \{\mathbf{x}^{(\alpha)}\}}{\partial \{\bar{\mathbf{x}}^{(\alpha)}\}} \right]; \text{ inverse microdeformation gradient tensor} \quad (16)$$

$$[\bar{J}^{(\alpha)}] = [J^{(\alpha)}]^{-1}; [J^{(\alpha)}] = [\bar{J}^{(\alpha)}]^{-1} \quad (17)$$

$$[J^{(\alpha)}][\bar{J}^{(\alpha)}] = [\bar{J}^{(\alpha)}][J^{(\alpha)}] = [I] \quad (18)$$

$$\{\dot{\bar{\mathbf{x}}}^{(\alpha)}\} = \{\bar{v}\} + [L^{(\alpha)}]\{\bar{\mathbf{x}}^{(\alpha)}\} \quad (19)$$

$$[L^{(\alpha)}] = \left[\frac{\partial \{\bar{v}^\alpha\}}{\partial \{\bar{\mathbf{x}}^{(\alpha)}\}} \right] \quad (20)$$

We also note the following useful relations.

$$[\dot{\bar{J}}^{(\alpha)}] = [\bar{L}^{(\alpha)}][J^{(\alpha)}] \quad (21)$$

$$[\dot{\bar{J}}^{(\alpha)}] = [\bar{J}^{(\alpha)}][\bar{L}^{(\alpha)}] \quad (22)$$

$$\frac{D(|J^{(\alpha)}|)}{Dt} = |j^{(\alpha)}| \text{tr}(\bar{L}^{(\alpha)}) \quad (23)$$

$$[\bar{L}^{(\alpha)}] = [\bar{D}^{(\alpha)}] + [\bar{W}^{(\alpha)}] \quad (24)$$

$$\frac{D}{DL} \{d\bar{A}^{(\alpha)}\} = [\text{tr}(\bar{D}^{(\alpha)})][I] - [\bar{L}^{(\alpha)}]^T \{d\bar{A}^{(\alpha)}\} \quad (25)$$

$$\frac{D(d\bar{V}^{(\alpha)})}{Dt} = d\dot{\bar{V}}^{(\alpha)} = (\text{tr}[\bar{L}^{(\alpha)}])d\bar{V}^{(\alpha)} \quad (26)$$

The relations (12) - (26) are helpful when deriving conservation and balance laws and the constitutive theories.

3.2. Microconstituent Stress Tensor S Due to Micro Cauchy Stress Tensor $\sigma^{(\alpha)}$

In the derivation of the conservation and balance laws, we use the following integral-average definition

$$\int_{\bar{v}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{mk} d\bar{V}$$

in which $\sigma^{(\alpha)}$ is the total stress tensor. Thus, \bar{S}_{mk} is the total microconstituent stress tensor. In this process, there is no concept of additive decomposition of $\bar{\sigma}^{(\alpha)}$ into equilibrium and deviatoric stress tensors, hence volumetric and distortional physics are not considered explicitly. Secondly, microconstituent density is eliminated through integral-average definitions. But $\rho^{(\alpha)}$ is needed if we are to consider a constitutive theory for the equilibrium stress for the microconstituents. Both of these considerations help us conclude that the stress tensor $\bar{\mathbf{S}}$ or \mathbf{S} is due to mechanical loading, hence it is a function of the work conjugate strain tensor and the elastic properties of the microconstituents. Henceforth, we do not consider any additive decomposition of \mathbf{S} , but instead consider the work conjugate strain tensor and temperature as its argument tensors of \mathbf{S} in deriving the constitutive theory for it.

4. Degrees of Freedom in Micro Deformation Physics

Determination of the degrees of freedom necessary to describe micro deformation physics is crucial. Since the deformation of microconstituents can be described by classical continuum mechanics, *i.e.*, by the micro displacement gradient tensor ${}^d\mathbf{J}^{(\alpha)}$. Additive decomposition of ${}^d\mathbf{J}^{(\alpha)}$ into symmetric (${}_s{}^d\mathbf{J}^{(\alpha)}$) and skew symmetric (${}_a{}^d\mathbf{J}^{(\alpha)}$) tensor separates rigid rotation of microconstituents (${}_a{}^d\mathbf{J}^{(\alpha)}$) and the deformation of the microconstituents (${}_s{}^d\mathbf{J}^{(\alpha)}$). ${}_s{}^d\mathbf{J}^{(\alpha)}$ is completely defined by the gradients of the microconstituent displacements. Thus, in principle $\mathbf{u}^{(\alpha)}$, three translational degrees of freedom are necessary to define ${}_s{}^d\mathbf{J}^{(\alpha)}$. But this requires the position coordinates of the microconstituents with respect to the center of mass of the material point, which we do not have. Thus, instead of three displacements of the microconstituents, we need to consider six independent components of ${}_s{}^d\mathbf{J}^{(\alpha)}$ as degrees of freedom for the micro constituents. Thus, in the linear micromorphic theory presented here, a microconstituents has nine degrees of freedom: three rigid rotations ${}_\alpha\Theta$, which are same as classical rotations ${}_\epsilon\Theta$ (hence do not add to unknown degrees of freedom), and six deformational degrees of freedom, which are six independent components of the symmetric part of the displacement gradient tensor ${}_s{}^d\mathbf{J}^{(\alpha)}$. We point out that this approach is totally different from that used by Eringen and those following Eringen's work. In Eringen's work, all nine components of $[J^\alpha]$ are treated as microconstituent degrees of freedom. It is clear that this consideration does not permit the separation of rigid rotations from deformations, which is essential in the development of the conservation and balance laws and the constitutive theories.

5. Conservation and Balance Laws for Linear Micromorphic Continua

In the following, we present the derivation of the conservation and balance laws:

conservation of mass, balance of linear momenta, balance of angular momenta, balance of moment of moments and the first and second laws of thermodynamics, in both Eulerian and Lagrangian description for linear micromorphic solid continua. The two descriptions can be derived from each other when displacements are kinematic variables in both. We always begin with the conservation or the balance laws derivation for micro deformation of the microconstituent and show that valid thermodynamic laws are possible to derive using classical continuum theory. This is followed by the introduction of “*integral-average*” definitions that hold at the macro level and are used to derive valid conservation and balance laws for macro physics. The conservation and balance laws of classical continuum mechanics are used in the micro deformation physics. Due to use of integral-average definitions at macro level, the conservation and balance laws of classical continuum mechanics are modified for macro physics. Introduction of a new kinematic conjugate pair, rotations and moments in addition to already existing displacements and forces requires an additional balance law, balance of moment of moments at the macro level [44] [54] [67].

5.1. Conservation of Mass

5.1.1. Conservation of Micro Mass

For the microconstituent in the reference and the deformed configurations, conservation of mass can be expressed as:

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (27)$$

If micro constituent mass is conserved, then

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} = 0 \quad (28)$$

Using transport theorem [68] [69], we can write the following for (28)

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\frac{D}{Dt} \bar{\rho}^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t) + \bar{\rho}^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t) \frac{\partial \bar{v}_i^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t)}{\partial \bar{x}_i^{(\alpha)}} \right) d\bar{V}^{(\alpha)} = 0 \quad (29)$$

Using localization theorem, we obtain the following from (29)

$$\frac{D}{Dt} \bar{\rho}^{(\alpha)} + \bar{\rho}^{(\alpha)} \frac{\partial \bar{v}_i^{(\alpha)}}{\partial \bar{x}_i^{(\alpha)}} = 0 \quad (30)$$

Equation (30) is the differential form of the conservation of mass in Eulerian description for the micro constituent based on classical continuum mechanics.

In Lagrangian description, using (27)

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{V^{(\alpha)}} \rho^{(\alpha)} |J^{(\alpha)}| dV^{(\alpha)} \quad (31)$$

Equation (31) implies that

$$\rho_0^{(\alpha)} = \rho^{(\alpha)} |J^{(\alpha)}| \quad (32)$$

Equation (32) is conservation of mass for microconstituent in Lagrangian description based on classical continuum mechanics.

5.1.2. Conservation of Macro Mass

Consider Eulerian description in (27) and integration over \bar{V} to obtain

$$\int_{\bar{V}} \left(\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \right) d\bar{V} \tag{33}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} d\bar{V} \tag{34}$$

Substituting (34) in (33) and setting its material derivative to zero (as mass is conserved for volume \bar{V})

$$\frac{D}{Dt} \int_{\bar{V}} \bar{\rho} d\bar{V} = 0 \tag{35}$$

Using transport theorem [68] [69], we obtain the following from (35)

$$\int_{\bar{V}} \left(\frac{D\bar{\rho}(\bar{x}, t)}{Dt} + \bar{\rho}(\bar{x}, t) \nabla \cdot \bar{\mathbf{v}}(\bar{x}, t) \right) d\bar{V} = 0 \tag{36}$$

using localization theorem

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho} \nabla \cdot \bar{\mathbf{v}} = 0 \tag{37}$$

Equation (37) is continuity equation at macro level in Eulerian description. Thus, conservation of mass holds at micro as well as macro level. In Lagrangian description, using (27) we can write the following:

$$\int_V \rho_0 dV = \int_{\bar{V}(t)} \bar{\rho} d\bar{V} = \int_V \rho |\mathbf{J}| dV \tag{38}$$

$$\text{Thus, } \rho_0(\mathbf{x}) = |\mathbf{J}| \rho(\mathbf{x}, t) \tag{39}$$

Equation (39) is the continuity equation at macro level in Lagrangian description.

5.2. Balance of Linear Momenta

5.2.1. Balance of Micro Linear Momenta

If $\bar{a}_k^{(\alpha)}$, ${}^b\bar{F}_k^{(\alpha)}$ and $\bar{\sigma}_{lk}^{(\alpha)}$ are microconstituent acceleration, body forces per unit mass and Cauchy stress tensor, then using balance of linear momenta of classical continuum mechanics, we can write the following:

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - {}^b\bar{F}_k^{(\alpha)} \bar{\rho} \right) d\bar{V} - \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \tag{40}$$

or

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - {}^b\bar{F}_k^{(\alpha)} \bar{\rho}^{(\alpha)} - \bar{\sigma}_{lk,l}^{(\alpha)} \right) d\bar{V} = 0 \tag{41}$$

Using localization theorem [69]

$$\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - {}^b \bar{F}_k^{(\alpha)} \bar{\rho}^{(\alpha)} - \bar{\sigma}_{lk,l}^{(\alpha)} = 0 \tag{42}$$

Equation (42) is balance of linear momenta for micro constituent based on classical continuum mechanics in Eulerian description. Micro balance of linear momenta in Lagrangian description can be directly written using (42).

$$\rho_0^{(\alpha)} a_k^{(\alpha)} - {}^b F_k^{(\alpha)} \rho_0^{(\alpha)} - \sigma_{lk,l}^{(\alpha)} = 0 \tag{43}$$

5.2.2. Balance of Macro Linear Momenta

Consider the following integral-average definitions:

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} \bar{a}_k d\bar{V} \tag{44}$$

$$\int_{\bar{V}^{(\alpha)}(t)} {}^b \bar{F}_k^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} {}^b \bar{F}_k^{(\alpha)} \bar{\rho} d\bar{V} \tag{45}$$

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\sigma}_{lk} \bar{n}_l d\bar{A} \tag{46}$$

Using (44) - (46) in (40) and integrating over \bar{V} and $\partial \bar{V}$

$$\int_{\bar{V}(t)} (\bar{\rho} \bar{a}_k - {}^b \bar{F}_k \bar{\rho}) d\bar{V} - \int_{\partial \bar{V}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l d\bar{A} = 0 \tag{47}$$

$$\text{or } \int_{\bar{V}(t)} (\bar{\rho} \bar{a}_k - {}^b \bar{F}_k \bar{\rho} - \bar{\sigma}_{lk,l}) d\bar{V} = 0 \tag{48}$$

Using localization theorem [69]

$$\bar{\rho} \bar{a}_k - {}^b \bar{F}_k \bar{\rho} - \bar{\sigma}_{lk,l} = 0 \tag{49}$$

5.3. Balance of Macro Angular Momenta

The simplified way to derive this is to consider the micro balance of linear momenta for $\bar{V}^{(\alpha)}$ and $\partial \bar{V}^{(\alpha)}$ and multiply it by $\epsilon_{mkn} \bar{x}_m^{(\alpha)}$, and integrate over $\bar{V}^{(\alpha)}$ and $\partial \bar{V}^{(\alpha)}$ and then integrate over \bar{V} and $\partial \bar{V}$ and including $\bar{M}_n^{(\alpha)}$ acting on $d\bar{A}^{(\alpha)}$. This can be written in three different forms, all three forms are conceptually identical, but there are some differences. We label these three forms as Case 1, Case 2 and Case 3.

Case 1

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} ({}^b \bar{F}_k^{(\alpha)}) \right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \tag{50}$$

Case 2

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} ({}^b \bar{F}_k^{(\alpha)}) \right) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \tag{51}$$

Case 3

We consider the following identity

$$\left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)}\right)_{,l} = \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} + \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} \tag{52}$$

$$\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} = \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)}\right)_{,l} - \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \tag{53}$$

We substitute from (53) in the second term of (51) to obtain

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\epsilon_{mkn} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)}\right)_{,l} - \epsilon_{nmk} \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)}\right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \tag{54}$$

Equation (54) is the third possible form that can be used to derive macro balance of angular momenta.

Integral terms that are common in Case 1, Case 2 and Case 3

In all three forms (50), (51) and (54), the first and the third terms are identically the same.

Thus, we consider the first and third terms appearing in (50), (51) and (54) first, and then provide individual details of the second term in (50), (51) and (54).

Consider the first term (say T1) in (50) or (51) or (54).

$$T1 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \tag{55}$$

Let

$$\bar{x}_m^{(\alpha)} = \bar{x}_m + \bar{x}_m^{(\alpha)} \tag{56}$$

Substituting from (56) in (55)

$$T1 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \tag{57}$$

$$T1 = \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \tag{58}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} \bar{a}_k d\bar{V} - \bar{\rho} \left({}^b \bar{F}_k\right) d\bar{V} \tag{59}$$

Using (59) in (57), we can write (59) as follows

$$T1 = \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho} \bar{a}_k - \bar{\rho} \left({}^b \bar{F}_k\right)\right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha) b} \bar{F}_k^{(\alpha)}\right) d\bar{V}^{(\alpha)} \tag{60}$$

In Lagrangian description

$$T1 = \int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b \bar{F}_k \right) \right) dV + \int_V \int_{\bar{V}^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \tag{61}$$

Consider the third term (say T3) in (50) or (51) or (54)

$$T3 = \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} \tag{62}$$

$$T3 = \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{63}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{m}_{ln} \bar{n}_l d\bar{A} \tag{64}$$

Using (64) in (63)

$$T3 = \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{ln} \bar{n}_l d\bar{A} = \int_{\bar{V}(t)} \bar{m}_{ln,l} d\bar{V} \tag{65}$$

This is the final form of T3 in Eulerian description. In Lagrangian description, (65) can be written as

$$T3 = \int_V m_{ln,l} dV \tag{66}$$

Integral terms that are not common in Case 1, Case 2 and Case 3

Case 1: Consider second term (say T2C1) and (58)

$$T2C1 = \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{67}$$

Substitute $\bar{x}_m^{(\alpha)}$ from (56) in (67)

$$\begin{aligned} T2C1 &= \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{x}_m^{(\alpha)} \right) \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}^{(\alpha)}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} d\bar{V}^{(\alpha)} \end{aligned} \tag{68}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\sigma}_{lk} \bar{n}_l d\bar{A} \tag{69}$$

Substituting (69) in (68)

$$\begin{aligned} T2C1 &= \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \bar{\sigma}_{lk} \bar{n}_l d\bar{A} + \int_{\bar{V}^{(\alpha)}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m \bar{\sigma}_{lk} \right)_{,l} d\bar{V} + \int_{\bar{V}^{(\alpha)}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_{m,k}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} + \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_{m,l} \bar{\sigma}_{lk} + x_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}^{(\alpha)}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk}^{(\alpha)} + \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \end{aligned} \tag{70}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{mk} d\bar{V} \tag{71}$$

Using (71) in (70)

$$\text{T2C1} = \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} + \bar{S}_{mk} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{72}$$

This is the final form of the second term in (58) (T2C1) for Case 1 in the Eulerian description. In the Lagrangian description, we can write T2C1 as follows:

$$\text{T2C1} = \int_{\bar{V}(t)} \epsilon_{mkn} \left(\sigma_{mk} + x_m \sigma_{lk,l} + S_{mk} \right) dV + \int_{V^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} dV^{(\alpha)} \tag{73}$$

Case 2: Consider second term (say T2C2) in (51)

$$\text{T2C2} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \tag{74}$$

Substitute (56) in (74)

$$\begin{aligned} \text{T2C2} &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{x}_m^{(\alpha)} \right) \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{75}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\sigma}_{lk} \bar{n}_l d\bar{A} \tag{76}$$

Substituting (76) in (75), we obtain the following.

$$\begin{aligned} \text{T2C2} &= \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m \bar{\sigma}_{lk} \bar{n}_l d\bar{A} + \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{x}_m \bar{\sigma}_{lk} \right)_{,l} d\bar{V} + \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \left(\bar{x}_{m,l} \bar{\sigma}_{lk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}(t)} \epsilon_{mkn} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}(t)} \epsilon_{mkn} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} + \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}(t)} \epsilon_{mkn} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{77}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{mk} d\bar{V} \tag{78}$$

Using (78) in (77)

$$T2C2 = \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} + \bar{S}_{mk} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (79)$$

This is the final form of T2C2 in the Eulerian description. T2C2 in the Lagrangian description can be written directly from (79)

$$T2C2 = \int_V \epsilon_{mkn} \left(\sigma_{mk} + x_m \sigma_{lk,l} + S_{mk} \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)} dV^{(\alpha)} \quad (80)$$

As expected, T2C2 in (79) for Case 2 is exactly same as T2C1 in (72) for Case 1 and (80) is same as (73).

Case 3: Consider second term (T2C2) in (54)

$$\begin{aligned} T2C3 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \quad (81)$$

Substitute for $\bar{x}_m^{(\alpha)}$ from (56) in (81)

$$\begin{aligned} T2C3 &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{x}_m^{(\alpha)} \right) \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &\quad - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \quad (82)$$

Define

$$\int_{\bar{V}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} = \bar{S}_{mk} d\bar{V} \quad (83)$$

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \bar{\sigma}_{lk} \bar{n}_l d\bar{A} \quad (84)$$

Substituting (83) and (84) in (82)

$$\begin{aligned} T2C3 &= \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m \bar{\sigma}_{lk} \bar{n}_l d\bar{A} + \int_{\partial \bar{V}(t)} \epsilon_{nmk} \bar{x}_m^{(\alpha)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{S}_{mk} d\bar{V} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{x}_m \bar{\sigma}_{lk} \right)_{,l} d\bar{V} + \int_{\bar{V}(t)} \epsilon_{mkn} \int_{\bar{V}^{(\alpha)}(t)} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{S}_{mk} d\bar{V} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{x}_{m,l} \bar{\sigma}_{lk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{S}_{mk} d\bar{V} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{S}_{mk} d\bar{V} \end{aligned} \quad (85)$$

or

$$T2C3 = \int_{\bar{V}(t)} \epsilon_{mkn} \left(\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} - \bar{S}_{mk} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (86)$$

This is the final form of T2C3 in the Eulerian description. T2C3 in the Lagrangian description can be written directly using (86), given below

$$T2C3 = \int_V \epsilon_{mkn} (\sigma_{mk} + x_m \sigma_{lk,l} - S_{mk}) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} dV^{(\alpha)} \tag{87}$$

Differential forms of balance of angular momenta for Case 1, Case 2 and Case 3

We note that T2C3, *i.e.*, (87), is exactly same as in T2C1 and T2C2, except here in (86), \mathbf{S} has a negative sign. This, of course, is due to introducing an identity to replace $\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)}$ term by the terms obtained due to identity. By using T2C1, T2C2 and T2C3 in Case 1, Case 2 and Case 3, and also using (61) and (65), we can derive the final expression for the balance of angular momenta for Case 1, Case 2 and Case 3, first in the Eulerian descriptions followed by Lagrangian description.

Balance of linear momenta: Case 1

Consider Eulerian description

$$\int_{\bar{V}(t)} \epsilon_{mkn} x_m (\bar{\rho} \bar{a}_k - \bar{\rho} ({}^b \bar{F}_k)) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} (\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} {}^b \bar{F}_k^{(\alpha)}) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \epsilon_{mkn} (\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} + \bar{S}_{mk}) d\bar{V} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{m}_{ln,l} d\bar{V} = 0 \tag{88}$$

Grouping terms in (88)

$$\int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m (\bar{\rho} \bar{a}_k - \bar{\rho} ({}^b \bar{F}_k) - \bar{\sigma}_{lk,l}) + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} (\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} {}^b \bar{F}_k^{(\alpha)} - \bar{\sigma}_{lk,l}^{(\alpha)}) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} (\epsilon_{mkn} (\bar{\sigma}_{mk} + \bar{S}_{mk}) + \bar{m}_{ln,l}) d\bar{V} = 0 \tag{89}$$

The first and the second terms in (89) are zero due to the macro and micro balance of linear momenta, thus (89) reduces to

$$\int_{\bar{V}(t)} \epsilon_{mkn} (\bar{\sigma}_{mk} + \bar{S}_{mk} + \bar{m}_{ln,l}) d\bar{V} = 0 \tag{90}$$

Using localization theorem (90) yields

$$\epsilon_{mkn} (\bar{\sigma}_{mk} + S_{mk}) + \bar{m}_{ln,l} = 0 \tag{91}$$

Equation (91) is the final form of the balance of macro angular momenta for Case 1 in the Eulerian description. Equation (91) in the Lagrangian description can be written as:

$$\epsilon_{mkn} (\sigma_{mk} + S_{mk}) + m_{ln,l} = 0 \tag{92}$$

Balance of linear momenta: Case 2

Consider the Eulerian description. Since the final form resulting for T2C2 for Case 2 (Equation (79)) is the same as the final form resulting for Case 1 for T2C1 (Equation (64)), the final form of the balance of macro angular momenta for this case is the same as that for Case 1, and we can write the following in the Eulerian description

$$\epsilon_{mkn} (\bar{\sigma}_{mk} + \bar{S}_{mk}) + \bar{m}_{ln,l} = 0 \tag{93}$$

In the Lagrangian description, (93) becomes

$$\epsilon_{mkn} (\sigma_{mk} + S_{mk}) + m_{ln,l} = 0 \quad (94)$$

Balance of linear momenta: Case 3

Consider the Eulerian description.

$$\int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m (\bar{\rho} \bar{a}_k - \bar{\rho}^b \bar{F}_k) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} (\bar{\rho} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)b} \bar{F}_k^{(\alpha)}) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \epsilon_{mkn} (\bar{\sigma}_{mk} + \bar{x}_m \bar{\sigma}_{lk,l} - \bar{S}_{mk}) d\bar{V} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \bar{m}_{ln,l} d\bar{V} = 0 \quad (95)$$

Collecting terms in (95)

$$\int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m (\bar{\rho} \bar{a}_k - \bar{\rho}^b \bar{F}_k - \bar{\sigma}_{lk,l}) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} (\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)b} \bar{F}_k^{(\alpha)} - \bar{\sigma}_{lk,l}^{(\alpha)}) d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} (\epsilon_{mkn} (\bar{\sigma}_{mk} - \bar{S}_{mk}) + \bar{m}_{ln,l}) d\bar{V} = 0 \quad (96)$$

The first term in (96) is zero due to balance of macro and micro linear momenta. Thus, (96) reduces to

$$\int_{\bar{V}(t)} (\epsilon_{mkn} (\bar{\sigma}_{mk} - \bar{S}_{mk}) + \bar{m}_{ln,l}) d\bar{V} = 0 \quad (97)$$

Using localization theorem (97) yields

$$\epsilon_{mkn} (\bar{\sigma}_{mk} - \bar{S}_{mk}) + \bar{m}_{ln,l} = 0 \quad (98)$$

This is the final form of the balance of macro angular momenta for Case 3 in the Eulerian description. In the Lagrangian description, (98) can be written as:

$$\epsilon_{mkn} (\sigma_{mk} - S_{mk}) + m_{ln,l} = 0 \quad (99)$$

We note that Case 1 and Case 2 use the actual balance of micro linear momenta in the derivation, whereas in Case 3, the micro gradient stress term in the balance of micro linear momenta is altered using an identity. The result of this change is a negative sign for \mathbf{S} term in the balance of angular momenta (Equation (98)). Equation (98) is what was derived by Eringen using a weighting function $\Phi^{(\alpha)}$ for the balance of microlinear momenta.

The derivation presented here for Case 3 shows that a weighting function is not needed. The end result of using the weight function is the same as what we have presented in this paper. The answer to the question of whether the correct form of the balance of angular momenta is (91) (or (93)) or (97) is important. Based on the derivation presented in Case 1 and Case 2 from the first principles, straight forward use of balance of angular momenta at this point we lean towards the forms (91) or (93). However, the use of identity resulting in change of sign for \mathbf{S} term may be meritorious of further consideration if supported by physics. Thus, at this stage we consider the following forms of balance of angular momenta in Eulerian and Lagrangian description that provides alternative to either forms

with plus or minus sign with \bar{S} or S tensors.

$$\epsilon_{mkn} (\bar{\sigma}_{mk} \pm \bar{S}_{mk}) + \bar{m}_{ln,l} = 0 \tag{100}$$

$$\epsilon_{mkn} (\sigma_{mk} \pm S_{mk}) + m_{ln,l} = 0 \tag{101}$$

We note that since $\bar{S}_{mk} = \bar{S}_{km}$ and $S_{mk} = S_{km}$, in (100) and (101), we have $\epsilon_{mkn} \bar{S}_{mk} = 0$ and $\epsilon_{mkn} S_{mk} = 0$, thus (100) and (101) reduce to the following.

$$\epsilon_{mkn} (\bar{\sigma}_{mk}) + \bar{m}_{ln,l} = 0 \tag{102}$$

$$\epsilon_{mkn} (\sigma_{mk}) + m_{ln,l} = 0 \tag{103}$$

Equations (102) and (103) are balance of angular momenta in Eulerian and Lagrangian descriptions. We keep (100) and (101) for further considerations.

Balance of angular momenta (102) or (103) is the same as balance of angular momenta for the micropolar microcontinuum theory, as it should be, because the rigid rotation physics of a micro constituent is identical in micropolar and micro-morphic microcontinuum theories. Equation (102) or (103) defines three balance of angular momenta equations about x_i -axes. Balance of angular momenta in the forms (100) and (101) is more informative. We use this in a later section.

5.4. First Law of Thermodynamics

Since the conservation and the balance laws of classical continuum mechanics hold for the micro deformation of the microconstituents, we can begin with the energy equation for the micro constituents over the volume $\bar{V}^{(\alpha)}$ and its surface $\partial\bar{V}^{(\alpha)}$ and integrate it over \bar{V} .

$$\int_{\bar{V}^{(t)}} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\bar{V}^{(t)}} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}^{(t)}} \int_{\bar{V}^{(\alpha)}(t)} \bar{q}_{k,k}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\partial\bar{V}^{(t)}} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} \left({}^r_c \bar{\Theta} \right)_k d\bar{A}^{(\alpha)} = 0 \tag{104}$$

in which $\bar{e}^{(\alpha)}$ is the specific internal energy, $\bar{q}^{(\alpha)}$ is the heat flux and ${}^r_c \bar{\Theta}$ are classical rotation rates (due to $\bar{\nabla}^{(\alpha)} \times \bar{v}^{(\alpha)}$). We consider each term in (104) (say t1).

Consider first term in (104)

$$\begin{aligned} \text{t1} &= \int_{\bar{V}^{(t)}} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{\bar{V}^{(t)}} \int_{\bar{V}^{(\alpha)}(t)} \rho_0^{(\alpha)} \dot{e}^{(\alpha)} dV^{(\alpha)} \\ &= \int_{\bar{V}^{(t)}} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \end{aligned} \tag{105}$$

Let

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 e dV \tag{106}$$

Using (106) in (105)

$$\text{t1} = \int_V \frac{D}{Dt} (\rho_0 e) dV = \int_V \rho_0 \dot{e} dV = \int_{\bar{V}^{(t)}} \bar{\rho} \dot{\bar{e}} d\bar{V} \tag{107}$$

Consider second term in (104) (say t2)

$$\begin{aligned}
 t2 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\left(\bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \right)_{,k} - \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\
 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} d\bar{V}^{(\alpha)}
 \end{aligned} \tag{108}$$

We note that

$$\bar{v}_l^{(\alpha)} = \bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \tag{109}$$

$$\text{and } \bar{\sigma}_{kl,k}^{(\alpha)} = \bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \tag{110}$$

Substituting (109) and (110) in (108)

$$\begin{aligned}
 t2 &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \right) \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\
 &\quad - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\
 &= \int_{\partial \bar{V}(t)} \bar{v}_l \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\
 &\quad - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} d\bar{V}^{(\alpha)} \\
 &\quad - \int_{\bar{V}(t)} \bar{v}_l \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}
 \end{aligned} \tag{111}$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\sigma}_{kl} \bar{n}_k d\bar{A} \tag{112}$$

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \left(\bar{\rho} \bar{a}_l - \bar{\rho} \left({}^b \bar{F}_l \right) \right) d\bar{V} \tag{113}$$

Substituting (112) and (113) in (111)

$$\begin{aligned}
 t2 &= \int_{\bar{V}(t)} \bar{v}_l \bar{\sigma}_{kl} \bar{n}_k d\bar{A} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \bar{v}_l \left(\bar{\rho} \bar{a}_l - \bar{\rho} \left({}^b \bar{F}_l \right) \right) d\bar{V} \\
 &\quad - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} d\bar{V}^{(\alpha)}
 \end{aligned} \tag{114}$$

We note that

$$\int_{\partial \bar{V}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \bar{n}_k d\bar{A} = \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl} \bar{v}_l \right)_{,k} d\bar{V} = \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl} \bar{v}_{l,k} + \bar{v}_l \bar{\sigma}_{kl,k} \right) d\bar{V} \tag{115}$$

and

$$\begin{aligned}
 &\int_{\bar{V}(t)} \bar{L}_{lm}^{(\alpha)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\
 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \left(\bar{\sigma}_{kl}^{(\alpha)} \bar{x}_m^{(\alpha)} \right)_{,k} d\bar{V}^{(\alpha)} \\
 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{x}_m^{(\alpha)} + \bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{kl}^{(\alpha)} \bar{x}_{m,k}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\
 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{ml}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} d\bar{V}^{(\alpha)}
 \end{aligned} \tag{116}$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \bar{\sigma}_{ml}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} \tag{117}$$

Using (117) in (116)

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{\alpha}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{\alpha}_m^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} \tag{118}$$

Substituting (115) and (118) in (114)

$$\begin{aligned} t2 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl} \bar{v}_{l,k} + \bar{v}_l \bar{\sigma}_{kl,k} \right) d\bar{V} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{\alpha}_m^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} \\ &\quad - \int_{\bar{V}(t)} \bar{v}_l \left(\bar{\rho} - \bar{\rho} \left({}^b \bar{F}_k \right) \right) d\bar{V} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{\alpha}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \tag{119}$$

Collecting coefficients in (119)

$$\begin{aligned} t2 &= \int_{\bar{V}(t)} \bar{\sigma}_{kl} \bar{v}_{l,k} d\bar{V} - \int_{\bar{V}(t)} \bar{v}_l \left(\bar{\rho} \bar{a}_l - \bar{\rho} \left({}^b \bar{F}_l \right) - \bar{\sigma}_{kl,k} \right) d\bar{V} \\ &\quad + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \bar{\alpha}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) - \bar{\sigma}_{kl,k}^{(\alpha)} \right) d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} \end{aligned} \tag{120}$$

The second and the third terms in (120) are zero due to the balance of macro linear momentum and the balance of micro linear momentum, and we can write (120) as follows

$$t2 = \int_{\bar{V}(t)} \left(\bar{\sigma}_{kl} \bar{v}_{l,k} + \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} \right) d\bar{V} \tag{121}$$

Consider the third term in (104) (say t3)

$$t3 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{q}_{k,k}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{\partial\bar{V}^{(\alpha)}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \tag{122}$$

Let

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{q}_k \bar{n}_k d\bar{A} \tag{123}$$

Using (123) in (122)

$$t3 = \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{q}_k \bar{n}_k d\bar{A} = \int_{\bar{V}(t)} \bar{q}_{k,k} d\bar{V} \tag{124}$$

Consider the fourth term in (104) (say t4)

$$t4 = \int_{\partial\bar{V}^{(\alpha)}(t)} \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} r_c \bar{\Theta}_k d\bar{A}^{(\alpha)} = \int_{\partial\bar{V}^{(\alpha)}(t)} r_c \bar{\Theta}_k \int_{\partial\bar{V}^{(\alpha)}(t)} \bar{m}_{lk} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \tag{125}$$

Define

$$\int_{\partial\bar{V}^{(\alpha)}(t)} \bar{m}_{lk} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{m}_{lk} \bar{n}_l d\bar{A} \tag{126}$$

Substituting (126) in (125), we can write (125) as follows

$$\begin{aligned}
 t4 &= \int_{\partial \bar{V}(t)} {}^r_c \bar{\Theta}_k \bar{m}_{lk} \bar{n}_l d\bar{A} \\
 &= \int_{\partial \bar{V}(t)} {}^r_c \bar{\Theta} \cdot (\bar{\mathbf{m}})^T d\bar{A} \\
 &= \int_{\bar{V}(t)} \bar{\nabla} \cdot ({}^r_c \bar{\Theta} \cdot (\bar{\mathbf{m}})^T) d\bar{V}
 \end{aligned}
 \tag{127}$$

We can show that [49] [53]

$$\bar{\nabla} \cdot ({}^r_c \bar{\Theta} \cdot \bar{\mathbf{m}})^T = {}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : {}^r_c \bar{\Theta} \bar{\mathbf{J}}
 \tag{128}$$

Using (128) in (127)

$$t4 = \int_{\bar{V}(t)} \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : {}^r_c \bar{\Theta} \bar{\mathbf{J}} \right) d\bar{V}
 \tag{129}$$

in which ${}^r_c \bar{\Theta} \bar{\mathbf{J}}$ are the gradients of the rotation rates.

Substituting (107), (121), (124) and (129) into (104), we can write the following for (104).

$$\begin{aligned}
 &\int_{\bar{V}(t)} \bar{\rho} \dot{\bar{e}} d\bar{V} - \int_{\bar{V}(t)} \bar{r}_{ki} \bar{v}_{l,k}^{(\alpha)} d\bar{V} - \int_{\bar{V}(t)} \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} d\bar{V} + \int_{\bar{V}(t)} \bar{q}_{k,k} d\bar{V} \\
 &- \int_{\bar{V}(t)} \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : ({}^r_c \bar{\Theta} \bar{\mathbf{J}}) \right) d\bar{V} = 0
 \end{aligned}
 \tag{130}$$

or

$$\int_{\bar{V}(t)} \left(\bar{\rho} \dot{\bar{e}} - \bar{\sigma}_{kl} \bar{v}_{l,k} - \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} + \bar{q}_{k,k} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : ({}^r_c \bar{\Theta} \bar{\mathbf{J}}) \right) \right) d\bar{V} = 0
 \tag{131}$$

using localization theorem

$$\bar{\rho} \dot{\bar{e}} - \bar{\sigma}_{kl} \bar{v}_{l,k} - \bar{S}_{ml} \bar{L}_{lm}^{(\alpha)} + \bar{q}_{k,k} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : ({}^r_c \bar{\Theta} \bar{\mathbf{J}}) \right) = 0
 \tag{132}$$

This is the final form of the macro energy equation in the Eulerian description. The energy equation in the Lagrangian description can be written directly from (132) and is given by

$$\rho_0 \dot{e} - \sigma_{kl} v_{l,k} - S_{ml} L_{lm}^{(\alpha)} + q_{k,k} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \mathbf{m}) + \mathbf{m} : {}^c \bar{\Theta} \bar{\mathbf{J}} \right) = 0
 \tag{133}$$

We note that

$$v_{lk} = \dot{J}_{l,k}
 \tag{134}$$

$$\text{and } L_{lm}^{(\alpha)} = \dot{J}_{lm}^{(\alpha)}
 \tag{135}$$

5.5. Second Law of Thermodynamics: Macro

Let $\bar{\eta}^{(\alpha)}$ be entropy density in microconstituent volume $\bar{V}^{(\alpha)}$, $\bar{h}^{(\alpha)}$ be the entropy flux imparted to volume $\bar{V}^{(\alpha)}$ by surrounding medium and $\bar{s}^{(\alpha)}$ be the source of entropy in $\bar{V}^{(\alpha)}$ due to noncontacting sources (or bodies), then the rate of increase of entropy in volume $\bar{V}^{(\alpha)}$ of the microconstituent from all contacting

and noncontacting sources is given by

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} dV^{(\alpha)} \geq \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (136)$$

Integrating (136) over \bar{V} and $\partial \bar{V}$

$$\int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (137)$$

Using

$$\bar{h}^{(\alpha)} = -\bar{\Psi}_k^{(\alpha)} \bar{n}_k^{(\alpha)} \quad (138)$$

$$\bar{\Psi}_k^{(\alpha)} = \frac{\bar{q}_k^{(\alpha)}}{\theta} \quad (139)$$

$$\bar{h}^{(\alpha)} = -\frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\theta} \quad (140)$$

$$\text{and } \bar{s}^{(\alpha)} = \frac{\bar{r}^{(\alpha)}}{\theta} \quad (141)$$

Using (140) and (141) in (137)

$$\int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial \bar{V}(t)} - \int_{\partial \bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\theta} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\theta} d\bar{V}^{(\alpha)} \quad (142)$$

Define

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\eta} \dot{\bar{\rho}} d\bar{V} \quad (143)$$

$$\int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\theta} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{q}_k \bar{n}_k}{\theta} d\bar{A} \quad (144)$$

$$\int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\theta} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{r} \bar{\rho}}{\theta} d\bar{V} \quad (145)$$

Using (143), (144) and (145) in (142)

$$\int_{\bar{V}(t)} \bar{\rho} \dot{\bar{\eta}} d\bar{V} \geq - \int_{\partial \bar{V}(t)} \frac{\bar{q}_k}{\theta} \bar{n}_k d\bar{A} + \int_{\bar{V}(t)} \frac{\bar{r} \bar{\rho}}{\theta} d\bar{V} \quad (146)$$

or

$$\int_{\bar{V}(t)} \bar{\rho} \dot{\bar{\eta}} d\bar{V} \geq - \int_{\partial \bar{V}(t)} \left(\frac{\bar{q}_k}{\theta} \right)_{,k} d\bar{V} + \int_{\bar{V}(t)} \frac{\bar{r} \bar{\rho}}{\theta} d\bar{V} \quad (147)$$

$$\text{or } \int_{\bar{V}(t)} \left(\bar{\rho} \dot{\bar{\eta}} + \frac{\bar{q}_{k,k}}{\theta} - \frac{\bar{q}_k}{\theta^2} \bar{\theta}_{,k} - \frac{\bar{r} \bar{\rho}}{\theta} \right) d\bar{V} \geq 0 \quad (148)$$

Using localization theorem

$$\bar{\rho} \dot{\bar{\eta}} + \frac{\bar{q}_{k,k}}{\theta} - \frac{\bar{q}_k \bar{\theta}_{,k}}{\theta^2} - \frac{\bar{r} \bar{\rho}}{\theta} \geq 0 \quad (149)$$

Multiply throughout by $\bar{\theta}$

$$\bar{\rho}\bar{\theta}\bar{\eta} + \bar{q}_{k,k} - \frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}} - \frac{\bar{r}\bar{\rho}}{\bar{\theta}} \geq 0 \quad (150)$$

Let

$$\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta} \quad (151)$$

$$\dot{\bar{\Phi}} = \dot{\bar{e}} - \dot{\bar{\eta}}\bar{\theta} - \bar{\eta}\dot{\bar{\theta}} \quad (152)$$

$$\therefore \bar{\rho}\bar{\theta}\dot{\bar{\eta}} = \bar{\rho}\dot{\bar{e}} - \bar{\rho}\dot{\bar{\Phi}} - \bar{\rho}\bar{\eta}\dot{\bar{\theta}} \quad (153)$$

Substituting from (153) into (150)

$$-\bar{\rho}(\bar{\Phi} + \bar{\eta}\dot{\bar{\theta}}) + \bar{\rho}\dot{\bar{e}} + \bar{q}_{k,k} - \frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}} - \bar{r}\bar{\rho} \geq 0 \quad (154)$$

Substituting $\bar{\rho}\dot{\bar{e}}$ from the energy Equation (133) in (154) after inserting $\bar{r}\bar{\rho}$ in the energy equation

$$\begin{aligned} & -\bar{\rho}(\dot{\bar{\Phi}} + \bar{\eta}\dot{\bar{\theta}}) + \bar{\sigma} : \bar{L} + \bar{S} : \bar{L}^{(\alpha)} - \bar{\nabla} \cdot \bar{q} + \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{m}) + \bar{m} : {}^r_c \bar{J} \right) \\ & + \bar{r}\bar{\rho} + \bar{\nabla} \cdot \bar{q} - \frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}} - \bar{r}\bar{\rho} \geq 0 \end{aligned} \quad (155)$$

$\bar{\nabla} \cdot \bar{q}$ term and $\bar{r}\bar{\rho}$ terms cancel and we can write the following after changing the sign.

$$\bar{\rho}(\dot{\bar{\Phi}} + \bar{\eta}\dot{\bar{\theta}}) - \bar{\sigma} : \bar{L} - \bar{S} : \bar{L}^{(\alpha)} + \frac{\bar{q}_k\bar{\theta}_{,k}}{\bar{\theta}} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{m}) + \bar{m} : {}^r_c \bar{J} \right) \leq 0 \quad (156)$$

This is the final form of the macro entropy inequality in the Eulerian description. In the Lagrangian description, we can write the following directly from (156).

$$\rho_0(\dot{\Phi} + \eta\dot{\theta}) - \sigma : J - S : J^{(\alpha)} - \frac{q_k\theta_{,k}}{\theta} - \left({}^r_c \Theta \cdot (\nabla \cdot m) + m : {}^r_c J \right) \leq 0 \quad (157)$$

5.6. Balance of Moment of Moments: Macro

Since in all 3M microcontinuum theories the classical rotations ${}^r_c \Theta$ and conjugate moment \bar{m} form an additional kinematically conjugate pair, in addition to displacements and forces, it follows that, based on Yang *et al.* [67] and Surana *et al.* [44] [54], the balance of moment of moments balance law is essential in all 3M theories. Using the macro definition of Cauchy stress tensor and Cauchy moment tensor, and following [44] [54], we can derive the following for this balance law:

$$\epsilon_{ijk}\bar{m}_{ij} = 0 \text{ or } \epsilon_{ijk}m_{ij} = 0 \quad (158)$$

That is Cauchy moment tensor is symmetric in all 3M microcontinuum theories.

6. Summary of Macro Conservation and Balance Laws in Eulerian Description

Conservation of mass, balance of linear momenta, balance of angular momenta, first and second law of thermodynamics and balance of moment of moments are

given in the following:

$$\dot{\bar{\rho}} + \bar{\rho}(\bar{\nabla} \cdot \bar{\mathbf{v}}) = 0 \tag{159}$$

$$\bar{\rho} \bar{a}_k - \bar{\rho}^b \bar{F}_k - \bar{\sigma}_{lk,l} = 0 \tag{160}$$

$$\epsilon_{mkn} (\bar{\sigma}_{mk} \pm \bar{S}_{mk}) + \bar{m}_{ln,l} = 0 \tag{161}$$

$$\bar{\rho} \dot{\bar{e}} - \bar{\sigma} : \bar{\mathbf{L}} - \bar{S} : \bar{\mathbf{L}}^{(\alpha)} + \bar{\nabla} \cdot \bar{\mathbf{q}} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : {}^r_c \bar{\mathbf{J}} \right) = 0 \tag{162}$$

$$\bar{\rho} \left(\dot{\bar{\Phi}} + \bar{\eta} \dot{\bar{\theta}} \right) - \bar{\sigma} : \bar{\mathbf{L}} - \bar{S} : \bar{\mathbf{L}}^{(\alpha)} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} - \left({}^r_c \bar{\Theta} \cdot (\bar{\nabla} \cdot \bar{\mathbf{m}}) + \bar{\mathbf{m}} : {}^r_c \bar{\mathbf{J}} \right) \leq 0 \tag{163}$$

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \tag{164}$$

This mathematical model in the Eulerian description consists of eight equations: conservation of mass (1), balance of linear momenta (3), balance of angular momenta (3) and energy Equation (1) in thirty five dependent variables:

$\bar{\rho}(1), \bar{\mathbf{v}}(3), \bar{\sigma}(9), \bar{\mathbf{m}}(6), \bar{S}(6), \bar{\theta}(1), \bar{\mathbf{q}}(3)$ and ${}^d_s \bar{\mathbf{J}}^{(\alpha)}(6)$. Thus, we need twenty-seven more equations for closure of the mathematical model. Constitutive theories yield twenty-one equations: $\bar{\sigma}(6), \bar{\mathbf{m}}(6), \bar{\mathbf{q}}(3), \bar{S}(6)$. Thus, we need an additional six equations for closure. These are discussed in Section 6.2.

6.1. Summary of Macro Conservation and Balance Laws in Lagrangian Description

Conservation of mass, balance of linear momenta, balance of angular momenta, first and second laws of thermodynamics and balance of moment of moments are given in the following:

$$\rho_0(\mathbf{x}) = |\mathbf{J}| \rho(\mathbf{x}, t) \tag{165}$$

$$\rho_0 a_k - \rho_0^b F_k - \sigma_{lk,l} = 0 \tag{166}$$

$$\epsilon_{mkn} (\sigma_{mk} \pm S_{mk}) + m_{ln,l} = 0 \tag{167}$$

$$\rho_0 \dot{e} - \sigma : \dot{\mathbf{J}} - \mathbf{S} : \dot{\mathbf{J}}^{(\alpha)} - \nabla \cdot \mathbf{q} - \left({}^r_c \Theta \cdot (\nabla \cdot \mathbf{m}) + \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} \right) = 0 \tag{168}$$

$$\rho_0 \left(\dot{\phi} + \eta \dot{\theta} \right) - \sigma : \dot{\mathbf{J}} - \mathbf{S} : \dot{\mathbf{J}}^{(\alpha)} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \left({}^r_c \Theta \cdot (\nabla \cdot \mathbf{m}) + \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} \right) \leq 0 \tag{169}$$

$$\epsilon_{ijk} m_{ij} = 0 \tag{170}$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3) and energy Equation (1) in thirty four dependent variables:

$\mathbf{u}(3), \sigma(9), \mathbf{S}(6), \mathbf{m}(6), \mathbf{q}(3), \theta(1), {}^d_s \mathbf{J}^{(\alpha)}(6)$. Thus, an additional twenty-seven equations are needed for closure. Constitutive theories provide twenty-one equations: $\sigma(6), \mathbf{S}(6), \mathbf{m}(6), \mathbf{q}(3)$. Thus, an additional six equations are needed for closure. These are discussed in Section 6.2.

6.2. Additional Six Equations in the Mathematical Model

From the conservation and the balance laws, we note that the microconstituent

stress \mathcal{S} only appears in the energy equation and the entropy inequality. This, of course, implies that if we were to solve a boundary value problem for isothermal physics, in which case the energy equation is not part of the mathematical model, then the microconstituent stress \mathcal{S} is completely absent from the mathematical model. This certainly is not physical, as the microconstituent deformation contributes to macro physics for stationary processes as well as evolutionary processes. Thus, we must have another relationship that considers symmetric stress tensor \mathcal{S} and the symmetric part of σ .

There are many differences between our work and Eringen's work on nonclassical theories.

1) Moment tensor (nonclassical mechanics) is defined using $\sigma^{(\alpha)}$ (due to classical mechanics, thus this definition is in error).

2) Due to not using balance of moment of moments balance law, the moment tensor is nonsymmetric.

3) Moment tensor in balance of angular momenta contains permutation tensor. This is obviously in error as the permutation tensor only appears in force terms due to their cross product with distance. This is obviously not needed in case of moment tensor as it is already a moment.

4) In Eringen's work in the derivation of balance of angular momenta, the skew symmetric components of σ are balanced by the gradients of the skew symmetric part of the moment tensor (as the moment tensor has permutation tensor in Eringen's derivations). Eringen [6]-[23] [70] and those following his work suggest that in the derivations of the balance of angular momenta, the permutation tensor must be dropped to obtain another balance law, moments of \mathcal{S} and σ (only symmetric part) that must balance with gradients of the symmetric part of the moment tensor to obtain additional equations.

5) In references [6]-[23] [70], it is stated that the three equations in (4) and the six equations in (5) are suitable for determining nine components of $\mathcal{J}^{(\alpha)}$.

6) It has been pointed, discussed and demonstrated that in 3M theories, balance of moment of moments balance law is essential [54]. Due to this balance law, the Cauchy moment tensor is symmetric. Thus, in the balance of angular momenta ${}_a\sigma$ are balanced by the gradients of symmetric moment tensor. This is the correct balance of angular momenta.

7) We must recognize that the permutation tensor in balance of angular moment only appears with force terms due to their cross product with distance vector, we just cannot discard it (as suggested by Eringen) as it is due to the physics of moment of forces. It is obvious that what is suggested in 4) has no basis, hence will not lead to any meaningful relations.

8) Thus, in Eringen's work on balance of angular momenta as well as six additional equations, both are in error. Our view, approach and outcome to obtain the six additional equations is completely different than Eringen.

Derivation of Additional Six Equations

From the derivation of balance of angular momenta leading to (101) (in Lagrangian

description), we note that σ has nine independent components, three in ${}_a\sigma$ and six in ${}_s\sigma$ and S has six independent components. However, presence of permutation tensor on the left side of Equation (101) forces us to discard six symmetric components of σ as well as S . This is an important observation that suggests that some how ϵ_{mkn} from the left side of (101) must be eliminated. This of course can be done by premultiplying Equation (101) or (167) with $(\epsilon_{mkn})^{-1}$, the inverse of ϵ_{mkn} . Symbolically, we can write

$$(\epsilon_{mkn})^{-1} \epsilon_{mkn} (\sigma_{mk} \pm S_{mk}) + (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{171}$$

$$\text{or } \sigma_{mk} \pm S_{mk} + (\epsilon_{mkn})^{-1} m_{ln,l} = 0 \tag{172}$$

But inverse of ϵ_{mkn} (for values of 1, 2, 3 for m, k, n) is ϵ_{mkn} , thus we can write (172) as

$$\sigma_{mk} \pm S_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{173}$$

$$\text{or } {}_a\sigma_{mk} + {}_s\sigma_{mk} \pm S_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{174}$$

Since

$${}_a\sigma_{mk} + \epsilon_{mkn} m_{ln,l} = 0 \tag{175}$$

is balance of angular momenta, (174) reduces to the following.

$${}_s\sigma_{mk} \pm S_{mk} = 0 \tag{176}$$

At this point, choice of negative sign is physical as it would suggest that symmetric part of σ and S balance each other, this obviously has to be true at an interface between the microconstituent and the medium, recalling that ${}_a\sigma$ are balanced by the gradients of m . Thus, we rewrite (176) with only negative sign.

$${}_s\sigma_{mk} - S_{mk} = 0 \tag{177}$$

Equations (177) are additional six equations that provide closure of the mathematical model.

Remarks

1) First, we note that (102) and (103) (balance of angular momenta) only contains nonclassical physics, both ${}_a\sigma$ and m are due to nonclassical physics, whereas (177) contains stresses due to classical mechanics. This is necessary for maintaining consistency of physics in the derivations.

2) When the microconstituents and the medium are of the same material, then naturally (177) must hold. When the microconstituents and the medium are of different material, (177) must also hold at the interface, continuity of stress due to classical physics, while ${}_a\sigma$ is taken care by the gradients of moment tensor, both ${}_a\sigma$ and moment tensor are nonclassical physics.

6.3. Constitutive Theories for Linear Elastic Micromorphic Solid Continua

We assume that the medium and microconstituents are linear elastic. Thus, in the derivation of the constitutive theories for σ, S and m , we consider only linear

elasticity in this paper.

6.3.1. Determination of Constitutive Tensors and Their Argument Tensors

In deriving constitutive theories, we always begin with the rate of work or otherwise conjugate pairs in the entropy inequality, for determination of the constitutive tensors based on the causality axiom of constitutive theory and their possible argument tensors. The choice of constitutive tensors can be altered or changed if the physics requires it, and the argument tensors of the constitutive tensors can be augmented with additional tensors if the physics requiring this has not been considered while deriving the entropy inequality. We follow the details and the guidelines presented in references [68] [69]. Once the constitutive tensors and their argument tensors are established, we follow the theory of isotropic tensors or representation theorem for deriving the constitutive theories and the standard procedure of Taylor series expansion of the coefficients in the linear combination of the basis of the space of constitutive tensor for determining the material coefficients [68] [69].

Consider the entropy inequality (169).

$$\rho_0 (\dot{\phi} + \eta \dot{\theta}) - \boldsymbol{\sigma} : \dot{\mathbf{J}} - \mathbf{S} : \dot{\mathbf{J}}^{(\alpha)} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - ({}_c \dot{\Theta} \cdot (\nabla \cdot \mathbf{m}) + \mathbf{m} : {}^c \Theta \dot{\mathbf{J}}) \leq 0 \quad (178)$$

The macro stress tensor $\boldsymbol{\sigma}$ is nonsymmetric, and hence cannot be a constitutive tensor due to representation theorem [55]-[66]. Thus, we need additive decomposition of $\boldsymbol{\sigma}$ into symmetric tensor ${}_s \boldsymbol{\sigma}$ and skew symmetric tensor ${}_a \boldsymbol{\sigma}$. There cannot be constitutive theory for ${}_a \boldsymbol{\sigma}$, as it is defined by the balance of angular momenta. Thus, ${}_s \boldsymbol{\sigma}$ is the constitutive tensor, and not $\boldsymbol{\sigma}$ or ${}_a \boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = {}_s \boldsymbol{\sigma} + {}_a \boldsymbol{\sigma} \quad (179)$$

Secondly

$$\dot{\mathbf{J}} = {}^d \dot{\mathbf{J}} = {}^d_s \dot{\mathbf{J}} + {}^d_a \dot{\mathbf{J}} = \dot{\boldsymbol{\epsilon}} + {}^d_a \dot{\mathbf{J}} \quad (180)$$

in which ${}^d \mathbf{J}$ is the displacement gradient tensor, and ${}^d_s \mathbf{J}$ and ${}^d_a \mathbf{J}$ are symmetric and skew-symmetric tensors obtained by additive decomposition of ${}^d \mathbf{J}$.

$${}^d \mathbf{J} = {}^d_s \mathbf{J} + {}^d_a \mathbf{J} \quad (181)$$

Likewise, additive decomposition of $({}^c \Theta \dot{\mathbf{J}})$ and $\mathbf{J}^{(\alpha)}$ into symmetric and skew-symmetric tensors gives:

$${}^c \Theta \dot{\mathbf{J}} = {}^c \Theta_s \dot{\mathbf{J}} + {}^c \Theta_a \dot{\mathbf{J}} \quad (182)$$

Also

$$\dot{\mathbf{j}}^{(\alpha)} = {}^d \dot{\mathbf{j}}^{(\alpha)} \quad (183)$$

$$\text{and } {}^d \mathbf{J}^{(\alpha)} = {}^d_s \mathbf{J}^{(\alpha)} + {}^d_a \mathbf{J}^{(\alpha)} \quad (184)$$

in which ${}^d \mathbf{J}^{(\alpha)}$ is micro displacement gradient tensor and ${}^d_s \mathbf{J}^{(\alpha)}$ and ${}^d_a \mathbf{J}^{(\alpha)}$ are symmetric and skew-symmetric tensors due to additive decomposition of ${}^d \mathbf{J}^{(\alpha)}$. Furthermore,

$${}^d \mathbf{J}^{(\alpha)} = {}^d \mathbf{J}^{(\alpha)} + {}^d \mathbf{J}^{(\alpha)} = \boldsymbol{\varepsilon}^{(\alpha)} + {}^d \mathbf{J}^{(\alpha)} \tag{185}$$

Also

$${}^c \Theta \mathbf{J} = {}^c \Theta \mathbf{J} + {}^c \Theta \mathbf{J} \tag{186}$$

Substituting (179) - (186) as needed in the entropy inequality (169) and noting that

$${}_s \boldsymbol{\sigma} : {}^d \dot{\mathbf{J}} = 0; \quad {}_a \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = 0; \quad \mathbf{S} : {}^d \dot{\mathbf{J}}^{(\alpha)} = 0; \quad \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} = 0 \tag{187}$$

We can write (169) as follows:

$$\rho_0 (\dot{\phi} + \eta \dot{\theta}) - {}_s \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_a \boldsymbol{\sigma} : {}^d \dot{\mathbf{J}} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} + {}^c \Theta \cdot (\nabla \cdot \mathbf{m}) \leq 0 \tag{188}$$

From balance of angular momenta

$$\nabla \cdot \mathbf{m} = -\epsilon : \boldsymbol{\sigma} \tag{189}$$

Substituting (189) in (188)

$$\rho_0 (\dot{\phi} + \eta \dot{\theta}) - {}_s \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}_a \boldsymbol{\sigma} : ({}^d \dot{\mathbf{J}}) - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} - {}^c \Theta \cdot (\epsilon : \boldsymbol{\sigma}) - \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \tag{190}$$

A simple calculation shows that

$${}_a \boldsymbol{\sigma} : {}^c \Theta \dot{\mathbf{J}} = {}^c \Theta \cdot (\epsilon : \boldsymbol{\sigma}) \tag{191}$$

Using (191) in (190), (191) reduces to

$$\rho_0 (\dot{\phi} + \eta \dot{\theta}) - {}_s \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} - \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \tag{192}$$

Further additive decomposition of ${}_s \boldsymbol{\sigma}$ into equilibrium and deviatoric stresses, ${}^e \boldsymbol{\sigma}$ and ${}^d \boldsymbol{\sigma}$, is used to derive the constitutive theory for volumetric and distortional deformation physics, which are mutually exclusive

$${}_s \boldsymbol{\sigma} = {}^e \boldsymbol{\sigma} + {}^d \boldsymbol{\sigma} \tag{193}$$

Substituting (193) in (192)

$$\rho_0 (\dot{\phi} + \eta \dot{\theta}) - {}^e \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - {}^d \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m} : {}^c \Theta \dot{\mathbf{J}} - \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \tag{194}$$

The rate of work conjugate pairs and the last term in (194) suggest, in conjunction with the axiom of causality [69], that ${}^e \boldsymbol{\sigma}, {}^d \boldsymbol{\sigma}, \mathbf{S}, \mathbf{m}$, and \mathbf{q} are valid choices of constitutive tensors. The initial choice of argument tensors is as follows (θ is included as an argument tensor in all constitutive tensors because of non-isothermal physics):

$${}^e \boldsymbol{\sigma} = {}^e \boldsymbol{\sigma}(\rho, \theta) \tag{195}$$

$${}^d \boldsymbol{\sigma} = {}^d \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{196}$$

$$\mathbf{S} = \mathbf{S}(\boldsymbol{\varepsilon}^{(\alpha)}, \theta) \tag{197}$$

$$\mathbf{m} = \mathbf{m}({}^c \Theta \mathbf{J}, \theta) \tag{198}$$

$$\mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \tag{199}$$

Even though we do not need a constitutive theory for Φ , its argument tensors

are essential to establish as it is used to simplify entropy inequality (194) as well as to derive constitutive theory for ${}^e_s\sigma$. The presence of η in (194) must be addressed as well. Clearly, ρ and θ must be argument tensors of both Φ and η . Other argument tensors of Φ and θ are chosen based on principle of equipresence. However, the principle of equipresence is not used in (195) - (199), since the conjugate pairs appearing in the entropy inequality (194) directly dictate their selection.

$$\Phi = \Phi\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}^c_s\mathbf{J}, \mathbf{q}, \theta\right) \tag{200}$$

$$\eta = \eta\left(\rho, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}, {}^c_s\mathbf{J}, \mathbf{q}, \theta\right) \tag{201}$$

The energy Equation (168) can be simplified in the same manner as the entropy inequality to obtain the following (similar to (194))

$$\rho \dot{e} - \nabla \cdot \mathbf{q} - {}_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \mathbf{S} : \boldsymbol{\varepsilon}^{(\alpha)} - \mathbf{m} \cdot {}^c_s\mathbf{J} = 0 \tag{202}$$

6.3.2. Constitutive Theory for Equilibrium Cauchy Stress Tensor ${}^e_s\sigma$

In Lagrangian description, density $\rho(\mathbf{x}, t)$ is determined directly from the conservation of mass $\rho(\mathbf{x}, t) = \frac{\rho_0}{|\mathbf{J}|}$, once the deformation gradient tensor \mathbf{J} is known. Therefore, density $\rho(\mathbf{x}, t)$ cannot be an argument tensor of the constitutive tensors [69]. However, compressibility and incompressibility physics is related to density and temperature. Consequently, the constitutive theory for ${}^e_s\sigma$ cannot be derived from the entropy inequality (194) in the Lagrangian description. Instead, we must begin with the entropy inequality similar to (194) in the Eulerian description to derive constitutive theory for ${}^e_s\sigma$ first.

$$\bar{\rho}\left(\dot{\bar{\Phi}} + \eta \dot{\bar{\theta}}\right) - {}_s\bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - {}^d_s\bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} - \bar{\mathbf{m}} : \left({}^r_s\bar{\mathbf{J}}\right) + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} \leq 0 \tag{203}$$

In this case, $\bar{\rho}$ is unknown and therefore treated a dependent variable in the mathematical model. Following same procedure as for Lagrangian description, the constitutive tensors and their argument tensors (including $\bar{\Phi}$ and $\bar{\eta}$) are given by:

$${}^e_s\bar{\boldsymbol{\sigma}} = {}^e_s\bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\theta}) \tag{204}$$

$${}^d_s\bar{\boldsymbol{\sigma}} = {}^d_s\bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\mathbf{D}}, \bar{\theta}) \tag{205}$$

$$\bar{\mathbf{S}} = \bar{\mathbf{S}}(\bar{\mathbf{D}}^{(\alpha)}, \bar{\theta}) \tag{206}$$

$$\bar{\mathbf{m}} = \bar{\mathbf{m}}\left(\bar{\rho}, {}^r_s\bar{\mathbf{J}}, \bar{\theta}\right) \tag{207}$$

$$\bar{\mathbf{q}} = \bar{\mathbf{q}}(\bar{\rho}, \bar{\mathbf{g}}, \bar{\theta}) \tag{208}$$

$$\bar{\Phi} = \bar{\Phi}\left(\bar{\rho}, \bar{\mathbf{D}}, \bar{\mathbf{D}}^{(\alpha)}, {}^r_s\bar{\mathbf{J}}, \bar{\mathbf{g}}, \bar{\theta}\right) \tag{209}$$

$$\bar{\eta} = \bar{\eta}\left(\bar{\rho}, \bar{\mathbf{D}}, \bar{\mathbf{D}}^{(\alpha)}, {}^r_s\bar{\mathbf{J}}, \bar{\mathbf{g}}, \bar{\theta}\right) \tag{210}$$

Using (209), we can write

$$\dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \dot{\bar{\rho}} + \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}} : \dot{\bar{\mathbf{D}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}^{(\alpha)}} : \dot{\bar{\mathbf{D}}^{(\alpha)}} + \frac{\partial \bar{\Phi}}{\partial \left({}_s^{\bar{\epsilon}} \bar{\mathbf{J}} \right)} : \left({}_s^{\bar{\epsilon}} \dot{\bar{\mathbf{J}}} \right) + \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{g}}} \cdot \dot{\bar{\mathbf{g}}} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \quad (211)$$

From conservation of mass in the Eulerian description

$$\dot{\bar{\rho}} = -\bar{\rho}(\bar{\nabla} \cdot \bar{\mathbf{v}}) = -\bar{\rho} \bar{D}_{kk} = -\bar{\rho} \bar{D}_{kk} \delta_{lk} = \bar{\rho} \bar{\mathbf{D}} : \boldsymbol{\delta} \quad (212)$$

Substituting from (212) for $\dot{\bar{\rho}}$ in (211) and then substituting (211) in (203), we obtain the following after regrouping the terms

$$\begin{aligned} & \left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_s^{\bar{e}} \bar{\boldsymbol{\sigma}} \right) : \dot{\bar{\mathbf{D}}} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}} : \dot{\bar{\mathbf{D}}} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}^{(\alpha)}} : \dot{\bar{\mathbf{D}}^{(\alpha)}} + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}_s^{\bar{\epsilon}} \bar{\mathbf{J}} \right)} : {}_s^{\bar{\epsilon}} \dot{\bar{\mathbf{J}}} \\ & + \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{g}}} \cdot \dot{\bar{\mathbf{g}}} + \bar{\rho} \left(\bar{\eta} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \right) \dot{\bar{\theta}} - {}_s^{\bar{d}} \bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} - \bar{\mathbf{m}} : {}_s^{\bar{\epsilon}} \bar{\mathbf{J}} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} \leq 0 \end{aligned} \quad (213)$$

The entropy inequality (213) holds for arbitrary but admissible choices of $\bar{\mathbf{D}}, \bar{\mathbf{D}}^{(\alpha)}, {}_s^{\bar{\epsilon}} \bar{\mathbf{J}}, \dot{\bar{\mathbf{g}}}$ and $\dot{\bar{\theta}}$ if the following conditions hold:

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}(\bar{\mathbf{D}}) \quad (214)$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{D}}^{(\alpha)}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}(\bar{\mathbf{D}}^{(\alpha)}) \quad (215)$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \left({}_s^{\bar{\epsilon}} \bar{\mathbf{J}} \right)} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi} \left({}_s^{\bar{\epsilon}} \bar{\mathbf{J}} \right) \quad (216)$$

$$\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\mathbf{g}}} = 0 \Rightarrow \bar{\Phi} \neq \bar{\Phi}(\bar{\mathbf{g}}) \quad (217)$$

$$\bar{\rho} \left(\bar{\eta} + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \right) = 0 \Rightarrow \bar{\eta} = -\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \quad (218)$$

Equations (214) - (218) imply that $\bar{\Phi}$ is not a function of $\bar{\mathbf{D}}, \bar{\mathbf{D}}^{(\alpha)}, {}_s^{\bar{\epsilon}} \bar{\mathbf{J}}$ and $\bar{\mathbf{g}}$. Equation (218) implies that $\bar{\eta}$ is deterministic from $\bar{\Phi}$; hence, $\bar{\eta}$ is not a constitutive or dependent variable. Using (214) - (218), the constitutive tensors and their argument tensors in (204) - (208) remain the same, but the argument tensors of $\bar{\Phi}$ and $\bar{\eta}$ can be modified:

$$\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \quad (219)$$

$$\bar{\eta} = \bar{\eta}(\bar{\rho}, \bar{\Phi}) \quad (220)$$

and the entropy inequality (213) reduces to

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} - {}_s^{\bar{e}} \bar{\boldsymbol{\sigma}} \right) : \bar{\mathbf{D}} - {}_s^{\bar{d}} \bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} - \bar{\mathbf{m}}^{(0)} : {}_s^{\bar{\epsilon}} \bar{\mathbf{J}} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\bar{\theta}} \leq 0 \quad (221)$$

Constitutive theory for ${}_s^{\bar{e}} \bar{\boldsymbol{\sigma}}$ for compressible matter can be obtained by setting coefficient of $\bar{\mathbf{D}}$ in the first term of (221) to zero.

$${}_s^{\bar{e}} \bar{\boldsymbol{\sigma}}(\bar{\rho}, \bar{\theta}) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \boldsymbol{\delta} = \bar{p}(\bar{\rho}, \bar{\theta}) \boldsymbol{\delta} \quad (222)$$

$$\bar{p}(\bar{\rho}, \bar{\theta}) = -\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{223}$$

in which $\bar{p}(\bar{\rho}, \bar{\theta})$ is thermodynamic pressure, the equation of state for compressible matter. When the deforming matter is incompressible, there is no change in volume. Thus, for a fixed mass, the density is constant, *i.e.*, $\bar{\rho}(\mathbf{x}, t) = \rho(\mathbf{x}, t) = \rho_0$. For this case, the conservation of mass gives:

$$\dot{\bar{\rho}} = -\bar{\rho}(\nabla \cdot \bar{\mathbf{v}}) = 0 \tag{224}$$

and

$$\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} = \frac{\partial \bar{\Phi}(\rho_0, \theta)}{\partial \rho} = 0 \tag{225}$$

Hence, for incompressible solid, the constitutive theory for ${}^e_s \boldsymbol{\sigma}$ cannot be derived using (222) and (223). First, using (225), the entropy inequality (221) reduces to

$$-{}^e_s \boldsymbol{\sigma} : \bar{\mathbf{D}} - {}^d_s \bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} - \bar{\mathbf{m}}^{(0)} : {}^r_s \bar{\boldsymbol{\sigma}} \bar{\mathbf{J}} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\theta} \leq 0 \tag{226}$$

In order to derive constitutive theory for ${}^e_s \bar{\boldsymbol{\sigma}}$ for incompressible solid matter, we must introduce incompressibility condition in (226). From the continuity equation, the velocity field for incompressible matter is divergence free, *i.e.*,

$$\bar{\nabla} \cdot \bar{\mathbf{v}} = \bar{D}_{kk} = \bar{D}_{kk} \delta_{lk} = \boldsymbol{\delta} : \bar{\mathbf{D}} = 0. \tag{227}$$

If (227) holds, then the following holds too:

$$\bar{p}(\bar{\theta}) \boldsymbol{\delta} : \bar{\mathbf{D}} = 0 \tag{228}$$

in which, $\bar{p}(\bar{\theta})$ is a Lagrange multiplier. Adding (228) to (226) and regrouping terms

$$(\bar{p}(\bar{\theta}) \boldsymbol{\delta} - {}^e_s \bar{\boldsymbol{\sigma}}) : \bar{\mathbf{D}} - {}^d_s \bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\theta} - \bar{\mathbf{m}}^{(0)} : {}^r_s \bar{\boldsymbol{\sigma}} \bar{\mathbf{J}} \leq 0 \tag{229}$$

Entropy inequality (229) holds for arbitrary but admissible $\bar{\mathbf{D}}$, if the coefficient of $\bar{\mathbf{D}}$ in the first term in (229) is set to zero, giving:

$${}^e_s \bar{\boldsymbol{\sigma}} = \bar{p}(\bar{\theta}) \boldsymbol{\delta} \tag{230}$$

The reduced form of entropy inequality is given by:

$$-{}^d_s \bar{\boldsymbol{\sigma}} : \bar{\mathbf{D}} - \bar{\mathbf{S}} : \bar{\mathbf{D}}^{(\alpha)} - \bar{\mathbf{m}}^{(0)} : {}^r_s \bar{\boldsymbol{\sigma}} \bar{\mathbf{J}} + \frac{\bar{\mathbf{q}} \cdot \bar{\mathbf{g}}}{\theta} \leq 0 \tag{231}$$

In the Lagrangian description, the constitutive theory for ${}^e_s \boldsymbol{\sigma}$ can be obtained directly from (222), (223) and (230).

$${}^e_s \boldsymbol{\sigma} = p(\rho, \theta) \boldsymbol{\delta}; \quad p(\rho, \theta) = -\rho^2 \frac{\partial \Phi}{\partial \rho} \quad (\text{compressible}) \tag{232}$$

$${}^e_s \boldsymbol{\sigma} = p(\theta) \boldsymbol{\delta} \quad (\text{incompressible}) \tag{233}$$

The reduced form of entropy inequality in the Lagrangian description follows directly from (231).

$$-{}^d_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \mathcal{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{(0)} : \left({}^c_s \mathbf{J} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \tag{234}$$

In the following, we present the derivation of constitutive theories for ${}^d_s\boldsymbol{\sigma}, \mathcal{S}, \mathbf{m}$ and \mathbf{q} using representation theorem [55]-[66]. The tensors ${}^d_s\boldsymbol{\sigma}, \mathcal{S}, \mathbf{m}$ are symmetric tensors of rank two and their work conjugate $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{(\alpha)}$, and ${}^c_s \mathbf{J}$ are also symmetric tensors of rank two. \mathbf{q} and \mathbf{g} are tensors of rank one. Thus, there is no difficulty in deriving constitutive theories for all four constitutive tensors using the representation theorem.

6.3.3. Constitutive Theory for ${}^d_s\boldsymbol{\sigma}$ Cauchy Stress Tensor

We consider the medium to be linear elastic. We begin with conjugate pair ${}^e_s\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$ in the reduced form of the entropy inequality (234). This conjugate pair, in conjunction with axiom of causality, suggests that ${}^d_s\boldsymbol{\sigma}$ is the constitutive tensor and $\boldsymbol{\varepsilon}$ as its argument tensor. Thus, we can write (with θ included in the argument tensors due to non-isothermal physics)

$${}^d_s\boldsymbol{\sigma} = {}^d_s\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{235}$$

Equation (235) suffices for thermoelastic deformation physics.

Let ${}^\sigma\mathcal{G}^i; i = 1, 2, \dots, N^\sigma$ be the combined generators of the argument tensors of ${}^d_s\boldsymbol{\sigma}$ in (235), which are symmetric tensors of rank two. Then, \mathbf{I} together with ${}^\sigma\mathcal{G}^i; i = 1, 2, \dots, N^\sigma$ constitute the basis of the space of tensor ${}^d_s\boldsymbol{\sigma}$, referred to as integrity basis. Now, ${}^d_s\boldsymbol{\sigma}$ can be expressed as a linear combination of the basis in the current configuration.

$${}^d_s\boldsymbol{\sigma} = {}^\sigma\alpha^0 \mathbf{I} + \sum_{i=1}^{N^\sigma} {}^\sigma\alpha^i ({}^\sigma\mathcal{G}^i); \quad {}^\sigma\alpha^i = {}^\sigma\alpha^i({}^\sigma I^j, \theta); \quad i = 0, 1, \dots, N^\sigma; \quad j = 1, 2, \dots, M^\sigma \tag{236}$$

in which ${}^\sigma I^j; j = 1, 2, \dots, M^\sigma$ are combined invariants of the argument tensors of ${}^d_s\boldsymbol{\sigma}$ in (235). The material coefficients in (236) are determined by expanding ${}^\sigma\alpha^i; i = 0, 1, \dots, N^\sigma$ in the invariants ${}^\sigma I^j; j = 1, 2, \dots, M^\sigma$ and the temperature θ about a known configuration $\underline{\Omega}$ and only retaining up to linear terms in ${}^\sigma I^j; j = 1, 2, \dots, M^\sigma$ and temperature θ .

$${}^\sigma\alpha^i = {}^\sigma\alpha^i|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial {}^\sigma\alpha^i}{\partial {}^\sigma I^j} \Big|_{\underline{\Omega}} \left({}^\sigma I^j - {}^\sigma I^j|_{\underline{\Omega}} \right) + \frac{\partial {}^\sigma\alpha^i}{\partial \theta} \Big|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right); \quad i = 0, 1, \dots, N^\sigma \tag{237}$$

We substitute ${}^\sigma\alpha^i; i = 0, 1, \dots, N^\sigma$ from (237) into (236)

$$\begin{aligned} {}^d_s\boldsymbol{\sigma} = & \left({}^\sigma\alpha^0|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial {}^\sigma\alpha^0}{\partial {}^\sigma I^j} \Big|_{\underline{\Omega}} \left({}^\sigma I^j - {}^\sigma I^j|_{\underline{\Omega}} \right) + \frac{\partial {}^\sigma\alpha^0}{\partial \theta} \Big|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) \right) \mathbf{I} \\ & + \sum_{i=1}^{N^\sigma} \left({}^\sigma\alpha^i|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial {}^\sigma\alpha^i}{\partial {}^\sigma I^j} \Big|_{\underline{\Omega}} \left({}^\sigma I^j - {}^\sigma I^j|_{\underline{\Omega}} \right) + \frac{\partial {}^\sigma\alpha^i}{\partial \theta} \Big|_{\underline{\Omega}} \left(\theta - \theta|_{\underline{\Omega}} \right) \right) {}^\sigma\mathcal{G}^i \end{aligned} \tag{238}$$

Collecting coefficients of $\mathbf{I}, {}^\sigma I^j \mathbf{I}, {}^\sigma\mathcal{G}^i, {}^\sigma I^j {}^\sigma\mathcal{G}^i, (\theta - \theta|_{\underline{\Omega}}) {}^\sigma\mathcal{G}^i$ and $(\theta - \theta|_{\underline{\Omega}}) \mathbf{I}$, we can write (238) as follows:

$$\begin{aligned}
 {}^d_s \boldsymbol{\sigma} = & \sigma_0 \mathbf{I} + \sum_{i=1}^{N^\sigma} {}^\sigma a_j ({}^\sigma \underline{I}^j) \mathbf{I} + \sum_{j=1}^{\sigma M} {}^\sigma b_j ({}^\sigma \mathbf{G}^i) + \sum_{j=1}^{\sigma M} \sum_{i=1}^{N^\sigma} {}^\sigma c_{ij} ({}^\sigma \underline{I}^j) ({}^\sigma \mathbf{G}^i) \\
 & - \sum_{i=1}^{N^\sigma} {}^\sigma d_i (\theta - \theta|_\Omega) ({}^\sigma \mathbf{G}^i) - (\alpha_m)_\Omega (\theta - \theta|_\Omega) \mathbf{I}
 \end{aligned} \tag{239}$$

The material coefficients ${}^\sigma a_j, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i$ and ${}^\sigma \alpha_m; i = 1, 2, \dots, N^\sigma; j = 1, 2, \dots, M^\sigma$ are defined in the following:

$$\begin{aligned}
 \sigma_0 = & \left({}^\sigma \alpha^0|_\Omega - \sum_{j=1}^{\sigma M} \frac{\partial ({}^\sigma \alpha^0)}{\partial ({}^\sigma \underline{I}^j)} \Big|_\Omega \right) ({}^\sigma \underline{I}^j|_\Omega); \quad a_j = \frac{\partial ({}^\sigma \alpha^0)}{\partial ({}^\sigma \underline{I}^j)} \Big|_\Omega \\
 b_i = & {}^\sigma \alpha^i|_\Omega + \sum_{j=1}^{\sigma M} \frac{\partial ({}^\sigma \alpha^i)}{\partial ({}^\sigma \underline{I}^j)} \Big|_\Omega ({}^\sigma \underline{I}^j|_\Omega); \quad c_{ij} = \frac{\partial ({}^\sigma \alpha^i)}{\partial ({}^\sigma \underline{I}^j)} \Big|_\Omega \\
 {}^\sigma d_i = & - \frac{\partial ({}^\sigma \alpha^i)}{\partial \theta} \Big|_\Omega; \quad {}^\sigma \alpha_m = - \frac{\partial ({}^\sigma \alpha^0)}{\partial \theta} \Big|_\Omega
 \end{aligned} \tag{240}$$

The constitutive theory (239), together with material coefficients (240), is based on integrity, *i.e.*, the complete basis of the space of constitutive tensor ${}^d_s \boldsymbol{\sigma}$. Simplified forms of (239) can be obtained from (239) by retaining specific generators and invariants. A simplified yet sufficiently general constitutive theory for ${}^d_s \boldsymbol{\sigma}$ is one in which ${}^d_s \boldsymbol{\sigma}$ is expressed as a linear function of the components of its argument tensors. By redefining the material coefficients and rearranging terms in (239), we can write the following:

$${}^d_s \boldsymbol{\sigma} = \sigma_0 \mathbf{I} + 2\mu \boldsymbol{\varepsilon} + \lambda \text{tr} \boldsymbol{\varepsilon} \mathbf{I} - {}^\sigma \alpha_m (\theta - \theta|_\Omega) \mathbf{I} \tag{241}$$

6.3.4. Constitutive Theory for Stress Tensor \mathcal{S}

Consider (197), *i.e.*,

$$\mathcal{S} = \mathcal{S}(\boldsymbol{\varepsilon}^{(\alpha)}, \theta) \tag{242}$$

Let ${}^s \underline{G}^i; i = 1, 2, \dots, N^s$ be the combined generators of the argument tensors of \mathcal{S} in (242), which are symmetric tensors of rank two and let ${}^s \underline{I}^j; j = 1, 2, \dots, M^s$ be the combined invariants of the same argument tensors of \mathcal{S} in (242). Then, $\mathbf{I}, {}^s \underline{G}^i; i = 1, 2, \dots, N^s$ constitute the basis of the space of constitutive tensor \mathcal{S} , and we can write the following expression for \mathcal{S} .

$$\mathcal{S} = {}^s \alpha_0 \mathbf{I} + \sum_{i=1}^{N^s} {}^s \alpha_i ({}^s \underline{G}^i) \tag{243}$$

in which

$${}^s \alpha_i = {}^s \alpha_i ({}^s \underline{I}^j, \theta); j = 1, 2, \dots, M^s; i = 0, 1, \dots, N^s \tag{244}$$

Following the procedure described in Section 6.3.3 (Taylor series expansion and collecting coefficients) we can derive the following constitutive theory for \mathcal{S}

$$\begin{aligned}
 \mathcal{S} = & S_0 \mathbf{I} + \sum_{j=1}^{M^s} {}^s a_j ({}^s \underline{I}^j) + \sum_{i=1}^{N^s} {}^s b_i ({}^s \underline{G}^i) + \sum_{i=1}^{N^s} \sum_{j=1}^{M^s} {}^s c_{ij} ({}^s \underline{I}^j) ({}^s \underline{G}^i) \\
 & - \sum_{i=1}^{N^s} {}^s d_i (\theta - \theta|_\Omega) ({}^s \underline{G}^i) - {}^s \alpha_m (\theta - \theta|_\Omega) \mathbf{I}
 \end{aligned} \tag{245}$$

in which material coefficients are given by (240) after replacing ${}^\sigma \alpha^i; i = 0, 1, \dots, N^\sigma$ with ${}^s \alpha_i; i = 0, 1, \dots, N^s$ and replacing ${}^\sigma a_0, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i$ and ${}^\sigma \alpha_{im}; i = 1, 2, \dots, N^\sigma; j = 1, 2, \dots, M^\sigma$ by ${}^s a_0, {}^s b_i, {}^s c_{ij}, {}^s d_i$ and ${}^s \alpha_{im}; i = 1, 2, \dots, N^s; j = 1, 2, \dots, M^s$ and σ_0 by S_0 . The material coefficients can be functions of ${}^s I^j; j = 1, 2, \dots, M^s$ and θ in a known configuration $\underline{\Omega}$. This constitutive theory is based on integrity. A constitutive theory that is linear in the components of the argument tensors is given by (after redefining material coefficients)

$$\mathbf{S} = S_0 \mathbf{I} + 2\mu^s \left(\boldsymbol{\varepsilon}^{(\alpha)} \right) + \lambda^s \left(\text{tr} \left(\boldsymbol{\varepsilon}^{(\alpha)} \right) \right) \mathbf{I} - {}^s \alpha_{im} \left(\theta - \theta|_{\underline{\Omega}} \right) \mathbf{I} \tag{246}$$

6.3.5. Constitutive Theory for Moment Tensor \mathbf{m}

Rigid rotations and rotation rates of the microconstituents in the medium require consideration of elasticity due to rotation gradient tensor. \mathbf{m} is the constitutive tensor and its argument tensors are given by:

$$\mathbf{m} = \mathbf{m} \left({}^c \boldsymbol{\varepsilon}_{[0]}, \theta \right) \tag{247}$$

Let ${}^m \mathbf{G}^i; i = 1, 2, \dots, N^m$ and ${}^m I^j; j = 1, 2, \dots, M^m$ be the combined generators and combined invariants of the argument tensors of \mathbf{m} in (247) in which ${}^m \mathbf{G}^i; i = 1, 2, \dots, N^m$ are symmetric tensors of rank two. Then based on the representation theorem, $\mathbf{I}, {}^m \mathbf{G}^i; i = 1, 2, \dots, N^m$ constitute basis of the space of constitutive tensor \mathbf{m} (integrity). Thus, we can represent \mathbf{m} as a linear combination of the basis in the current configuration.

$$\mathbf{m} = {}^m \alpha^0 \mathbf{I} + \sum_{i=1}^{N^m} {}^m \alpha^i \left({}^m \mathbf{G}^i \right) \tag{248}$$

in which coefficients

$${}^m \alpha^i = {}^m \alpha^i \left({}^m I^j, \theta \right); i = 0, 1, \dots, N^m; j = 1, 2, \dots, M^m \tag{249}$$

Following the procedure described in Section 6.3.3 (Taylor series expansion) we can derive the following constitutive theory for \mathbf{m} :

$$\begin{aligned} \mathbf{m} = m^0|_{\underline{\Omega}} \mathbf{I} + \sum_{j=1}^{M^m} {}^m a_i {}^m I^j \mathbf{I} + \sum_{i=1}^{N^m} {}^m b_i {}^m \mathbf{G}^i + \sum_{i=1}^{N^m} \sum_{j=1}^{M^m} {}^m c_{ij} {}^m I^j {}^m \mathbf{G}^i \\ - \sum_{i=1}^{N^m} {}^m d_i \left(\theta - \theta|_{\underline{\Omega}} \right) {}^m \mathbf{G}^i - {}^m \alpha_{im} \left(\theta - \theta|_{\underline{\Omega}} \right) \mathbf{I} \end{aligned} \tag{250}$$

in which material coefficients are given by after replacing ${}^\sigma \alpha^i; i = 0, 1, \dots, N^\sigma$ and $\sigma_0, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i; i = 1, 2, \dots, N^\sigma; j = 1, 2, \dots, M^\sigma$ with ${}^m \alpha^i; i = 0, 1, \dots, N^m$ and ${}^m a_i, {}^m b_i, {}^m c_{ij}, {}^m d_i, {}^m \alpha_{im}; i = 1, 2, \dots, N^m; j = 1, 2, \dots, M^m$. The material coefficients can be functions of ${}^m I^j; j = 1, 2, \dots, M^m$ and θ in a known configuration $\underline{\Omega}$.

A constitutive theory that is linear in the components of the argument tensors is given by:

$$\mathbf{m} = m_0 \mathbf{I} + 2 \left(\mu^m \right) \left({}^c \boldsymbol{\varepsilon} \right) + \left(\lambda^m \right) \left(\text{tr} \left({}^c \boldsymbol{\varepsilon} \right) \right) \mathbf{I} - \left({}^m \alpha_{im} \right) \left(\theta - \theta|_{\underline{\Omega}} \right) \mathbf{I} \tag{251}$$

6.4. Constitutive Theory for \mathbf{q}

Consider

$$\mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \quad (252)$$

Following references [68] [69], we can derive the following constitutive theory for \mathbf{q} using representation theorem.

$$\mathbf{q} = -\kappa \mathbf{g} - \kappa_1 (\mathbf{g} \cdot \mathbf{g}) \mathbf{g} - \kappa_2 (\theta - \theta|_{\Omega}) \mathbf{g} \quad (253)$$

where κ, κ_1 and κ_2 are material coefficients. These can be functions of $(\mathbf{g} \cdot \mathbf{g})|_{\Omega}$ and $\theta|_{\Omega}$. In (253), $\mathbf{g} \cdot \mathbf{g}$ is invariant of argument tensor \mathbf{g} . Simplified form of (253), the Fourier heat conduction law is given by

$$\mathbf{q} = -\kappa \mathbf{g} \quad (254)$$

7. Thermodynamic and Mathematical Consistency of the Micromorphic Theory Presented in the Paper

This requires that we establish that while deriving conservation and the balance laws and the constitutive theories the principles of thermodynamics and well established mathematical concepts have not been violated. The following list of what has been used in deriving the linear micromorphic theory establishes its thermodynamic and mathematical consistency.

1) Conservation and balance laws of classical continuum mechanics are applied and hold for microconstituent deformation.

2) Integral-average definitions derived from the microconstituents' conservation and balance laws permit the use of conservation and balance laws of classical continuum mechanics at micro level with appropriate modifications due to new microconstituent deformation physics.

3) Separation of micro constituent strains and rigid rotations by additive decomposition of the microconstituent deformation gradient tensor is essential for a thermodynamically correct and consistent treatment of the deformation physics and the rigid rotation physics. The constitutive theory for the microconstituent stress tensor obviously requires the use of strain measure (and not the rigid rotations or the micro deformation gradient tensor) while the rigid rotations are part of the constitutive theory for the moment tensor.

4) The rigid rotations of the microconstituents (present in all three 3M theories) and the conjugate moment constitute a new kinematically conjugate pairs in 3M theories in addition to displacements and forces already present (classical continuum mechanics). Each conjugate pair requires two balance laws: balance of forces and balance of moment of forces due to displacement and force kinematic pair and balance of moments and balance of moment of moments due to the new kinematically conjugate pair of rotations and moments. The balance of moment of moments balance law has been shown to be essential [44] [54] [67] in 3M theories. In its absence, incorrect conjugate pairs appear in the entropy inequality that yield nonphysical and invalid constitutive theories, resulting in violation of thermody-

dynamic consistency of the resulting theory.

5) Inequality restrictions imposed on the conjugate pairs in the entropy inequality are satisfied by the constitutive theories derived in the paper.

6) Choice of constitutive tensors and their argument tensors are decided from the conjugate pairs in the entropy inequality in conjunction with axiom of causality.

7) All constitutive tensors of rank two are symmetric tensors with symmetric tensors of rank two, tensors of rank zero and one as their argument tensors. This is supported by representation theorem, hence the resulting constitutive theories are mathematically consistent.

8) Deformation/strain measures derived in ref. [32] by Surana *et al.* are found to be valid for linear micromorphic theory, and hence are used in the present work.

9) All constitutive theories are derived using the complete basis (integrity) of the spaces of constitutive tensors first, followed by derivation of material coefficients. This is followed by simplified constitutive theories supported by the one derived using integrity.

10) Appropriate additive decomposition of the constitutive tensors is done to ensure that the physics of deformation is correctly described by the constitutive theories. Additive decomposition of $\sigma = {}_s\sigma + {}_a\sigma$, ${}_s\sigma = {}^e{}_s\sigma + {}^d{}_s\sigma$ are examples that help in determination of correct constitutive tensors for specific physics.

11) In summary all derivations presented in the paper for conservation and balance laws and specifically the constitutive theories are supported both thermodynamically and mathematically.

8. Linear Micromorphic Theory of Eringen

The linear micromorphic microcontinuum theory presented by Eringen [17]-[22] [70] is used almost exclusively in all published works related to micromorphic theories. We list important features and details of the linear micromorphic theory of Eringen and compare them with the corresponding details of the micromorphic theory presented in this paper. This is followed by an evaluation of their admissibility based on thermodynamic principles and the well established mathematical concepts.

1) In Eringen's linear micromorphic theory a microconstituent has nine degrees of freedom defined by the nine components of the microconstituent deformation gradient tensor $\mathbf{J}^{(\alpha)}$. Additive decomposition of $\mathbf{J}^{(\alpha)}$ shows that its skew-symmetric component contains rigid rotations of microconstituents. Thus, components of $\mathbf{J}^{(\alpha)}$ contain deformation as well as rigid rotations hence $\mathbf{J}^{(\alpha)}$ cannot be used as a strain measure.

In the present work, we consider additive decomposition of ${}^d\mathbf{J}^{(\alpha)}$ into ${}^d{}_s\mathbf{J}^{(\alpha)}$ and ${}^d{}_a\mathbf{J}^{(\alpha)}$ tensors. The three rigid rotations in ${}^d{}_a\mathbf{J}^{(\alpha)}$ are in fact classical rotations ${}_c\Theta$ in the present work, hence these are not unknown degrees of freedom. Six components of ${}^d{}_s\mathbf{J}^{(\alpha)}$ constitute unknown degrees of freedom for the micro-

constituents. In this approach rigid rotations of the microconstituents are ϵ_{Θ} that remain the same in all three 3M theories and their treatment in the development of the theories also remains the same.

In the works of Surana *et al.* on micropolar theories [24]-[55] and in the micromorphic theory presented here, rigid rotations ϵ_{Θ} result in the Cauchy moment tensor. In Eringen's work, the moment tensor definition is in error as it cannot be due to classical continuum physics of the microconstituent Cauchy stress tensor.

2) Micro and macro conservation of mass presented in this paper is exactly the same as in references [17]-[22] [70] presented by Eringen.

3) The balance of linear momenta, balance of angular momenta and the balance of first moment of momenta (in Eringen's works [17]-[22] [70]) are derived by considering weighted integral of the balance of microlinear momenta using a weight function $\bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$.

$$\int_{\bar{V}^{(\alpha)}} \left(\int_{\bar{V}^{(\alpha)}(t)} \bar{\phi}^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)b} \bar{F}_k^{(\alpha)} - \bar{\sigma}_{lk,l}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \right) = 0 \quad (255)$$

in which $\bar{\phi}^{(\alpha)} = \bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$.

Using

$$\bar{\phi}^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} = \left(\bar{\phi}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} - \bar{\phi}_{,l}^{(\alpha)} \bar{\sigma}_{lk} \quad (256)$$

Equation (255) can be written as:

$$\int_{\bar{V}^{(\alpha)}} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)b} \bar{F}_k^{(\alpha)} - \left(\bar{\phi}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} + \bar{\phi}_{,l}^{(\alpha)} \bar{\sigma}_{lk} \right) d\bar{V}^{(\alpha)} = 0 \quad (257)$$

or

$$\int_{\bar{V}^{(\alpha)}} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)b} \bar{F}_k^{(\alpha)} - \bar{\phi}_{,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}^{(\alpha)}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\phi}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} n_l^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \quad (258)$$

a) Macrobalance of linear momenta is derived using $\bar{\phi}^{(\alpha)} = 1$ in which case $\bar{\phi}_{,l}^{(\alpha)} = 0$ and Equation (258) reduces back to (255) with $\bar{\phi}^{(\alpha)} = 1$ but with the substitution (256). This form is the same as the standard approach for balance of macro linear momenta, hence yields the same balance of macro linear momenta equations as we have in this paper.

b) In Eringen's work balance of angular momenta is derived using

$$\bar{\phi}^{(\alpha)} = \epsilon_{mkn} \bar{x}_m^{(\alpha)} \quad \text{in (258).}$$

c) Balance of first moment of momentum is derived using $\bar{\phi}^{(\alpha)} = \bar{x}_m^{(\alpha)}$.

There are many issues, inconsistencies and possible errors in this approach. We discuss these in the following:

i) There is no basis for using $\bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$ as a weight function in (255).

ii) Introduction of (256) due to $\bar{\phi}^{(\alpha)}$ in (255) changes the original physics in (255) which is the correct physics of the balance of microlinear momenta. Hence, the derivation that follows is of concern.

iii) Introduction of a third rank moment tensor through

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\phi}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} n_l^{(\alpha)} d\bar{A}^{(\alpha)} = \bar{m}_{lk,m} \bar{n}_l d\bar{A} \quad (259)$$

is not valid due to the fact that $\bar{\sigma}^{(\alpha)}$ is a symmetric Cauchy stress tensor for the microconstituents (classical continuum mechanics), hence cannot possibly yield something that is purely related to nonclassical microcontinuum physics, *i.e.*, $\bar{m}_{lk,m}$ tensor.

iv) The fundamental mistake is that balance of angular momenta is the sum of the rate of change of moment of linear momenta, the moment of all other forces and the moments. Thus, the permutation tensor can only appear with the rate of change of moment of momenta and moment of forces and not with the Cauchy moment tensor as it is already a moment tensor. Due to this error, definition (259) is in error and the balance of angular momenta and balance of first moment of momentum are in error as well. Since, balance of angular momenta is used in the energy equation and the entropy inequality, these are in error also.

v) Due to i) - iv), the balance laws have incorrect definitions of “integral-average” moment tensor and as a consequence their derivations have errors and are in violation of thermodynamic consistency.

4) Constitutive tensors are nonsymmetric tensors of rank two and their argument tensors are also nonsymmetric tensors of rank two. This is not supported by the theory of isotropic tensors (representation theorem) [55]-[66]. For nonsymmetric tensors, the basis of the space of the constitutive tensors cannot be established. Thus, all constitutive theories for nonsymmetric constitutive tensors are in violation of mathematical consistency, as these cannot be supported by well established mathematical concepts of the representation theorem. For the most part, the derivation of constitutive theories presented by Eringen for nonsymmetric constitutive tensors using potentials or using the polynomial approach has no mathematical foundation either, hence it is ad hoc or phenomenological in our view.

5) The principle of equipresence used almost in all works of Eringen including micromorphic theories introduces nonphysical coupling between classical and nonclassical physics, hence results in many nonphysical material coefficients that either need to be justified or whose elimination needs to be proved.

6) Various additive decompositions shown to be essential in our work in deriving physical and valid constitutive theories are entirely missing in Eringen’s work.

7) The constitutive theory for the nonsymmetric stress tensor, of course, violates the representation theorem, but additionally it contains ${}_a\sigma$ for which there cannot be a constitutive theory as it is completely defined by the balance of angular momenta in terms of gradients of the Cauchy moment tensor.

8) Eringen’s work does not use the balance of moment of moments balance law; as a consequence Cauchy moment tensor is nonsymmetric, further leading to a nonphysical constitutive theory for nonsymmetric moment tensor and resulting

in thermodynamic inconsistency of the resulting micromorphic theory.

9) It is instructive to check the closure in our linear micromorphic theory and the linear micromorphic theory of Eringen. In the linear micromorphic theory presented in this paper, we have: 34 dependent variables

$\mathbf{u}(3), \boldsymbol{\sigma}(9), \mathbf{m}(6), \mathbf{S}(6), {}_s^d \mathbf{J}^{(\alpha)}(6), \mathbf{q}(3), \theta(1)$ and 34 equations: balance of linear momenta(3), balance of angular momenta(3), balance of moment of symmetric stresses (6), energy Equation (1), and constitutive theories for $\boldsymbol{\sigma}(6), \mathbf{m}(6), \mathbf{S}(6), \mathbf{q}(3)$. Hence, the mathematical model has closure.

In Eringen's linear micromorphic theory, there are 43 dependent variables: $\mathbf{u}(3), \boldsymbol{\sigma}(9), \mathbf{m}(9), \mathbf{S}(6), \mathbf{J}^{(\alpha)}(9), \mathbf{q}(3), \theta(1), {}_\alpha \boldsymbol{\Theta}(3)$ and 40 equations: balance of linear momenta (3), balance of angular momenta (3), balance of moment of momentum (6), energy Equation (1), and constitutive theories for: $\boldsymbol{\sigma}(9), \mathbf{m}(9), \mathbf{S}(6), \mathbf{q}(3)$, thus additional three equations are needed for closure. Eringen advocates these to be due to conservation of micro inertia.

We remark that in Eringen's work $\boldsymbol{\sigma}$ and \mathbf{m} have nine independent components each requiring nine constitutive equations. In our theory, \mathbf{m} and $\boldsymbol{\sigma}$ have six and nine independent components but only six constitutive equations for each are required as ${}_a \boldsymbol{\sigma}$ (three components) are balanced by the balance of angular momenta hence cannot be part of the constitutive theory. In Eringen's theory $\mathbf{J}^{(\alpha)}$ has nine independent components all of which are considered degrees of freedom in Eringen's theory. In our case ${}_s^d \mathbf{J}^{(\alpha)}$ has only six independent components and ${}_\alpha \boldsymbol{\Theta}$ are in fact ${}_c \boldsymbol{\Theta}$, hence are not unknown degrees of freedom. Thus, a micro-constituent only has six unknown degrees of freedom. These significant differences are necessary to note so that we can see why our micromorphic theory is quite different from that of Eringen.

10) The conservation of micro inertia law is proposed by Eringen to obtain the additional three equations needed for closure of Eringen's mathematical model. The laws of thermodynamic have no such conservation law. In our work presented in the paper, which is strictly based on the laws of thermodynamics and well-established mathematical principles, the need for an additional conservation or balance law does not arise. These equations proposed by Eringen are not part of the thermodynamic framework and laws. We keep in mind that we are not discarding the conservation of microinertia. This conservation law is not part of the laws of thermodynamics, and hence its need never arises in our micromorphic theory that is strictly based on the thermodynamic framework.

11) All issues and concerns regarding Eringen's work have been discussed and illustrated using the well-established laws of thermodynamics and well-established principles of applied mathematics; hence, our remarks and comments in the paper regarding Eringen's work are not to be misconstrued as speculative.

In summary, we have presented ample evidence that the linear micromorphic theory of Eringen is both thermodynamically and mathematically inconsistent as a microcontinuum theory, hence cannot be considered a valid micromorphic microcontinuum theory.

9. Summary and Conclusions

A linear micromorphic continuum theory has been presented in which the mechanism of elasticity is considered for the microconstituents, for the solid medium and for the interaction of the microconstituents with the solid medium. In the following, we summarize the work presented in the paper and draw some conclusions.

1) In the present micromorphic theory, a microconstituent also has nine degrees of freedom, as in the case of Eringen's theory, but the degrees of freedom are completely different. In our work, rotations of the microconstituents (described by ${}_{\epsilon}\Theta$, hence known) and the six independent components of the symmetric part of the micro displacement gradient tensor (unknown) constitute nine degrees of freedom, out of which ${}_{\epsilon}\Theta$ are known; thus, we have only six unknown degrees of freedom for microconstituents. In Eringen's work, all nine components of the micro deformation gradient tensor are considered unknown degrees of freedom. In these degrees of freedom, rigid rotations and deformations are not separated, leading to incorrect considerations in the derivation of the theory compared to the theory we have presented in this paper.

2) In the theory presented here, care is taken to ensure that the rigid body rotation physics of the microconstituent, which is common to all three 3M theories, is incorporated in an identical manner in all three 3M theories.

3) Our work recognizes that rotations ${}_{\epsilon}\Theta$ and the Cauchy moment tensor form a new kinematically conjugate pair in all three 3M theories, and hence will require two balance laws just as the displacement-force kinematic pair does in classical continuum mechanics. This necessitates a new balance law in all 3M theories [38] [54] [67]: the balance of moment of moments. This balance law is never used in Eringen's work, the consequence of this is spurious constitutive theories.

4) Varying rotations ${}_{\epsilon}\Theta$ in the deforming solid medium, when resisted, create moments. Our derivation shows that the Cauchy moment tensor and the symmetric part of the gradients of ${}_{\epsilon}\Theta$ are work conjugate. This physics is purely due to nonclassical mechanics, hence it has no interaction or any connection to classical continuum theory. Based on this, the "integral-average" definition of moment tensor (259) by Eringen is incorrect as it is based on $\bar{\sigma}^{(\alpha)}$ which is purely due to classical continuum mechanics.

5) Our derivation in this paper shows that the use of weight function $\bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$ in Eringen's work in the derivation of the macro balance of linear momenta, balance of angular momenta and moment of moment has no thermodynamic basis. Our work shows that the use of $\bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$ as advocated by Eringen is not justified based on the physics considered in the balance laws and leads to balance laws different from those obtained without using it.

6) All constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors, hence permitting the use of representation theorem in deriving constitutive theories that are naturally mathemat-

ically consistent. This is in contrast with published works in which the constitutive tensors of rank two are mostly nonsymmetric tensors with nonsymmetric argument tensors. Such constitutive theories derived using assumed potentials are nonphysical and cannot be justified based on representation theorem.

7) Conservation of micro inertia, advocated by Eringen to be necessary in 3M theories, is neither needed in the present work nor used. These additional equations are required primarily due to ${}_{\alpha}\mathbb{C}$ being unknown degrees of freedom, whereas in our work, ${}_{\alpha}\mathbb{C}$ are in fact ${}_{c}\mathbb{C}$, hence known. Other significant differences are that in Eringen's work, σ and m are nonsymmetric and nine constitutive equations are considered for σ , as well as m . In our work, $\sigma = {}_s\sigma + {}_a\sigma$ decomposition is used and there are only six constitutive equations needed for ${}_s\sigma$. m is symmetric due to the balance of moment of moments balance law, hence only six constitutive equations are needed for m as well. It is shown in Section 8 why Eringen's micromorphic theory does not have closure without the conservation of micro inertia conservation law.

8) The thermodynamic and mathematical consistency of the linear micromorphic theory presented in this paper has been established in Section 7. The lack of thermodynamic and mathematical consistency of Eringen's linear micromorphic theory has been discussed and illustrated in Section 8.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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