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General Analytical Solutions to Fractional Differential Equations

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Abstract

This research introduces a new global technique for addressing fractional differential equations (FDEs), referred to as the N-Transform (NT). This method serves as a comprehensive integral transform, encompassing both Laplace and Sumudu transforms. A fixed point theorem is used to prove the existence and uniqueness of solutions. Finding general solutions to specific fractional differential equations and proving their uniqueness through the application of a fixed point theorem are the main goals of this study. The fractional derivatives are expressed in the Caputo framework. The findings indicate that this method is highly effective, straightforward, and applicable to differential equations across various fields of science.

Keywords

Caputo Sense, N-Transform, Fractional Differential Equations, A Fixed Point Theorem, General Analytical Solutions

1. Introduction

The idea of fractional calculus has become notably popular and acknowledged in recent years, due to its demonstrated uses in multiple areas of science and engineering. It presents various efficient techniques for tackling fractional differential equations (FDEs) and other associated challenges that involve special functions from mathematical physics, alongside their characteristics and generalizations in one or multiple dimensions. A significant connection exists between fractional calculus and the behavior of intricate real-world issues. Owing to their non-local characteristics, fractional operators provide a more thorough and organized depiction of numeric. In many mathematical models, fractional-order differential equations provide efficient control. Because classical mathematical models are particular cases of fractional-order mathematical models, the results from the fractional mathematical model are therefore more accurate and widely applicable. FDEs, or fractional differential equations, shed light on these physical phenomena. Physical processes are represented using fractional differential equations in a variety of ways, including: [1] [2] the fractional exponential function method, [3] the modified exponential function method, [4] the fractional He-Laplace method, [5] [6] the sub-equation method. In recent years, researchers have worked to provide more useful definitions for derivatives' connotations. The goal of requiring these activities is to prevent norm derivatives from losing their ability to portray characters. These actions resulted in the development of several concepts for derivatives of fractional phenomena, which are logical extensions of normal derivatives of integer order. These derivative operators include the Caputo derivative and others. Recent theoretical, analytical, numerical, and computational contributions to this essentially multidisciplinary subject are the main focus of this study. Fractional calculus's connections to other branches of mathematics and physics may open up new lines of inquiry, discoveries, and uses. This study presents analytical fresh investigations of fractional differential using a revolutionary approach known as NT. Previous methods such as LT and ST suggest that NT offers new generic fractional solutions for fractional differential equations. In some cases, NT is used to receive the classical forms of fractional differential equations. They aid in our understanding and interpretation of fractional differential equation and system phenomena. Here, we'll review a few fundamental definitions of fractional derivatives:

Definition 1.1: [7] [8] The fractional derivative of $\Theta(\mathrm{I})$ for $\Phi-1<\Phi<\Phi$, $\Phi\in\mathrm{N}$, $\Phi>0$, $\mathrm{A},\Phi,\mathrm{I}\in R$ in the Caputo sense is defined by:

$$D_I^{\Phi}\Theta(\mathrm{I})=J^{\mathrm{P}-\Phi}D^{\Phi}\Theta(\mathrm{I})=\frac{1}{\Gamma(\mathrm{P}-\Phi)}\int_{\mathrm{A}}^{\mathrm{I}}(t-\tau)^{\mathrm{P}-\Phi-1}\Theta^{(\mathrm{P})}(\mathrm{I})\mathrm{dI},\ (\Phi>0). \tag{1}$$

Definition 1.2: [9] The Mittag-Leffler function $E_{\Phi}(\mathrm{H})$ with $\Phi>0$ is defined via the series representation, valid in the whole complex plane is:

$$E_{\Phi}(\mathrm{H})=\sum_{\iota=0}^{\infty}\frac{\mathrm{H}^{\iota}}{\Gamma(\Phi\iota+1)}. \tag{2}$$

A generalization of the Mittag-Leffler function is

$$E_{\Phi,r}(\mathrm{H})=\sum_{\iota=0}^{\infty}\frac{\mathrm{H}^{\iota}}{\Gamma(\Phi\iota+r)} \tag{3}$$

Definition 1.3: [10] For the function with $\Theta(\mathrm{I})$ in $(-\infty,\infty)$ the general integral transform is defined by:

$$\mathrm{N}\left[\Theta(\mathrm{I})\right](\wp,\lambda)=\int_{-\infty}^{\infty}J(\wp,\mathrm{I})\Theta(\wp\mathrm{I})\mathrm{dI}, \tag{4}$$

where $J(\wp,\mathrm{I})$ is the kernel of the transform and the variables λ and $\wp$ are the natural transform variables. When $J(\wp,\mathrm{I})=\mathrm{e}^{-st}$ Equation (4) gives Laplace transform. When $J(\wp,\mathrm{I})=\mathrm{e}^{-t}$ Equation (4) gives Sumudu transform.

Over the set of functions

$$X = \left\{ \Theta(I) : \exists Y, T_1, T_2 > 0, \left|\Theta(I)\right| < Ye^{|I|/T_j}, \text{ if } I \in (-1)^j \times [0,\infty) \right\} \tag{5}$$

The N-transform of $\Theta(I)$ is defined by:

$$N\left[\Theta(I)\right] = \Re(\wp, \lambda) = \int_0^\infty \Theta(\lambda I) e^{-\wp I} dI, \lambda > 0, \wp > 0 \tag{6}$$

where $\Re(\wp, \lambda)$ is the N-transform of the time function $\Theta(I)$ and the variables λ and $\wp$ are the natural transform variables. The inversion formula of the N-transform is

$$N^{-1}\left[\Re(\wp, \lambda)\right] = \Theta(I) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{\wp I}{\lambda}} \Re(\wp, \lambda) d\wp \tag{7}$$

Definition 1.4: [10], If $N\left[\Theta(I)\right]$ is the N-transform of the function $\left[\Theta(I)\right]$, then the N-transform of fractional derivative of order Φ is defined as:

$$N\left[\Theta^{(\Phi)}(I)\right] = \frac{\wp^{\Phi}}{\lambda^{\Phi}} \Re(\wp, \lambda) - \sum_{r=0}^{m-1} \frac{\wp^{\Phi-(r+1)}}{\lambda^{\Phi-r}} \Theta^{(r)}(0) \tag{8}$$

More details of N-transform found in [11] [12]. In terms of fractional differential operators, the Caputo fractional derivative appears to be the most widely used type. Its definition centers on the kernel of the power law function convolution with a function's local derivative. For the Caputo fractional differential equation, the derivative of a constant function is zero, hence the initial conditions can be properly expressed and handled identically to those of the ordinary derivative. In this study, we choose the Caputo fractional derivative because of these factors. Problems involving non-local characteristics and phenomena that take interactions into consideration benefit from the use of the Caputo fractional derivative. The equation can be thought of as having memory in this way. A finer feature of the Caputo fractional derivative is the memory effect. It is the most effective instrument for characterizing memories. Thus, we can say that the fractional Caputo derivative's physical meaning is an indicator of memory. Since the Caputo differential operator shares a number of traits with usual ones, it is frequently employed in fractional calculus applications to represent a variety of physical events. Lastly, to mimic real-world issues, the Caputo derivative ought to be employed as the fractional operator. This is the main motivation behind the use of the Caputo fractional-order derivative operator in different physical implementations. Using the N-transform, the study's primary goal is to solve fractional differential equations and provide general solutions. In addition to establishing the existence and uniqueness of solutions to fractional differential equations (FDEs) using fixed-point theory as follows:

First, consider a general fractional differential equation

A general Caputo fractional differential equation of $0 < \Phi < 1$ order can be written as:

$$D^{\Phi}\Theta(I) = \Im(I, \Theta(I)), I \in [0,1], \Theta(0) = \Theta_0, \tag{9}$$

where D^{Φ} is the Caputo fractional derivative and $\Im(I, \Theta(I))$ is a given function.

Using fractional integration, we rewrite Equation (8) as aVoltera integral equation:

$$\Theta(\mathrm{I}) = \Theta_0 + \frac{1}{\Gamma(\Phi)} \int_0^{\mathrm{I}} (\mathrm{I} - \wp)^{\Phi - 1} \Im(\wp, \Theta(\wp)) \, \mathrm{d}\wp \tag{10}$$

Secondly, we apply fixed-point theory to prove the existence and uniqueness. A fixed-point theory relies on Banach's fixed-point theorem. It states that a contraction mapping on a complete metric space has a unique fixed point.

Step 1: Define an Operator

Define an operator $\int$ on the space of continuous functions:

$$\left(\int \Theta\right)(\mathrm{I}) = \Theta_0 + \frac{1}{\Gamma(\Phi)} \int_0^{\mathrm{I}} (\mathrm{I} - \wp)^{\Phi - 1} \Im(\wp, \Theta(\wp)) \, \mathrm{d}\wp \tag{11}$$

Step 2: Show $\int$ is a Contraction

For uniqueness, we assume $\Im(\mathrm{I}, \Theta)$ satisfies a Lipschitz condition:

$$\left| \Im(\mathrm{I}, \Theta_1) - \Im(\mathrm{I}, \Theta_2) \right| \le \ell \left| \Theta_1 - \Theta_2 \right| \tag{12}$$

Then, for two functions Θ_1, Θ_2

$$\left| \left(\int \Theta_1\right)(\mathrm{I}) - \left(\int \Theta_2\right)(\mathrm{I}) \right| \le \frac{\ell}{\Gamma(\Phi)} \int_0^{\mathrm{I}} (\mathrm{I} - \wp)^{\Phi - 1} \left| \Theta_1(\wp) - \Theta_2(\wp) \right| \mathrm{d}\wp \tag{13}$$

Applying fractional Grönwall's inequality, we get:

$$\left\| \int \Theta_1 - \int \Theta_2 \right\| \le \omega \left\| \Theta_1 - \Theta_2 \right\| \tag{14}$$

where $\omega < 1$ under small $\int$. This proves that $\int$ is a contraction, and by Banach's Fixed-Point Theorem, there exists a unique solution. The paper's leftovers had been arranged as follows: The solutions to fractional differential equations and the existence of solutions to fractional differential equations and their uniqueness are presented in Section 2. The findings and discussion of the investigation are presented in Section 3.

2. Applications

In this section, we solve some fractional differential equations by NT to gain general solution of FDEs.

2.1. Problem 1

The Fractional exponential growth equation refers to an extension of the classical exponential growth model using fractional calculus, which deals with derivatives and integrals of non-integer (fractional) order. This is used when growth processes exhibit memory effects or anomalous dynamics, often seen in complex systems like biology, finance, or physics. Consider the following fractional exponential growth equation [13]:

$$\frac{\mathrm{d}^{\Phi}\Theta}{\mathrm{d}t^{\Phi}} = \Lambda \Theta(\mathrm{I}), \Theta(0) = \Theta_0, 0 < \Phi < 1, \quad \Lambda \text{ is the growth rate} \tag{15}$$

Applying NT on both sides of Equation (14), we have

$$N\left[\frac{d^{\Phi}\Theta}{dI^{\Phi}}\right]=\Lambda N\left[\Theta(I)\right]. \tag{16}$$

On simplifying

$$\frac{\wp^{\alpha}}{\lambda^{\alpha}}\Re(\wp,\lambda)-\frac{\wp^{\alpha-1}}{\lambda^{\alpha}}\Theta(0)=\Lambda\Re(\wp,\lambda) \tag{17}$$

i.e.,

$$\Re(\wp,\lambda)=\frac{\wp^{-1}}{1-\frac{\lambda^{\alpha}}{s^{\alpha}}\Lambda}\Theta_0 \tag{18}$$

Taking the inverse NT of Equation (17), we obtain

$$\Theta(I)=N^{-2}\left[\wp^{-1}\left(1-\frac{\lambda^{\alpha}}{\wp^{\alpha}}\Lambda\right)^{-1}\right]\Theta_0 \tag{19}$$

[14] obtained

$$N^{-1}\left[\wp^{-1}\left(1-\frac{\lambda^{\alpha}}{\wp^{\alpha}}\Lambda\right)^{-1}\right]=E_{\Phi}\left(\Lambda I^{\Phi}\right) \tag{20}$$

where $E_{\Phi}\left(\Lambda I^{\alpha}\right)=\sum_{\iota=0}^{\infty}\frac{\left(\Lambda I^{\Phi}\right)^{\iota}}{\Gamma(\Phi\iota+1)}$ Is the Mittag-Leffer function.

Then, the fractional solution of Equation (14) is

$$\Theta(I)=\Theta_0\left[1+\frac{\Lambda I^{\Phi}}{\Phi!}+\frac{\left(\Lambda I^{\Phi}\right)^2}{2\Phi!}+\frac{\left(\Lambda I^{\Phi}\right)^3}{3\Phi!}+\frac{\left(\Lambda I^{\Phi}\right)^4}{4\Phi!}+\cdots\right] \tag{21}$$

Taking the case $\Phi=1$ from Equation (21), we have the exact solution of Equation (15) obtained by [15]:

$$\Theta(I)=\Theta_0 e^{\Lambda I} \tag{22}$$

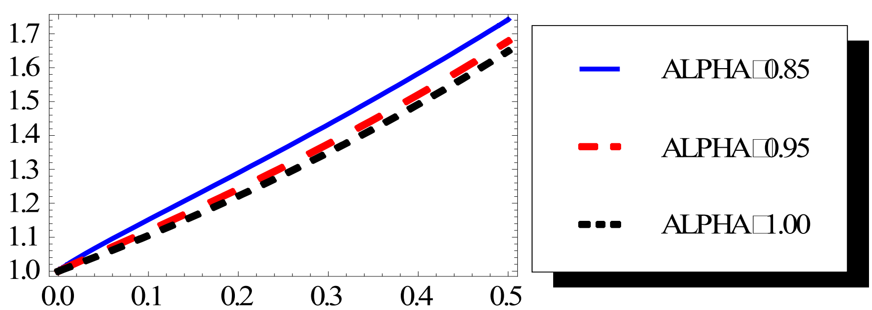

Figure 1. Plot of the NT solution of (15) at various values of Φ at $\Lambda=1,\Theta_0=1$ in comparing with the exact solution at $\Phi=1$ obtained via [15].

Now, we can prove the existence and uniqueness of the solution of Equation (14) using fixed-point theory as follows:

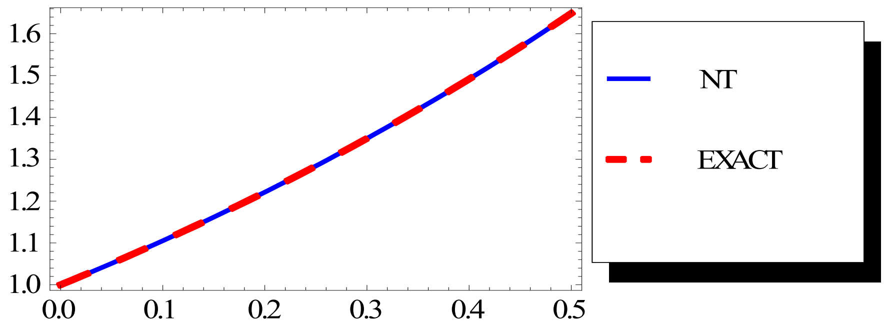

Figure 2. Plot of the NT solution of (15) at various values of Φ at $\Lambda = 1, \Theta_0 = 1$ in comparing with the exact solution obtained via [15].

Step 1: Consider the following fractional exponential growth equation

$$\frac{d^{\Phi}\Theta}{dI^{\Phi}} = \Lambda\Theta(I), \Theta(0) = \Theta_0, 0 < \Phi < 1, I \in [0, I] \tag{23}$$

Step 2: Convert Equation (22) into an integral equation

Caputo FDEs can be rewritten as a Valera-type integral equation:

$$\Theta(I) = \Theta_0 + \frac{\Lambda}{\Gamma(\Phi)} \int_0^I (I - \wp)^{\Phi - 1} \Theta(\wp) d\wp \tag{24}$$

Step 3: Define a fixed-point operator

Define an operator $\int$ as:

$$\left(\int \Theta\right)(I) = \Theta_0 + \frac{\Lambda}{\Gamma(\Phi)} \int_0^I (I - \wp)^{\Phi - 1} \Theta(\wp) d\wp \tag{25}$$

Step 4: Prove uniqueness using Banach's fixed-point theorem

Banach's theorem states that if an operator $\int$ is a contraction, then it has a unique fixed.

Step 4.1: Define a function space

Consider the Banach space C ([0, T]) with the norm:

$$\|\Theta\| \le \sup_{I \in [0,I]} |\Theta(I)| \tag{26}$$

Applying fractional Grönwall's inequality, we get:

$$\|\Theta\| \le \sup_{I \in [0,I]} |\Theta(I)| \tag{27}$$

Step 4.2: Show $\int$ is a contraction

Take two functions Θ_1 and Θ_2 compute:

$$\left|\left(\int \Theta_1\right)(I) - \left(\int \Theta_2\right)(I)\right| = \left|\Lambda \int_0^I \frac{(I - \wp)^{\Phi}}{\Gamma(\Phi)} \left(\Theta_2(\wp) - \Theta_1(\wp)\right) d\wp\right| \tag{28}$$

Applying the norm:

$$\left|\left(\int \Theta_1\right)(I) - \left(\int \Theta_2\right)(I)\right| \le \frac{\Lambda}{\Gamma(\Phi)} \sup_{\wp \in [0,I]} \left|\Theta_1(\wp) - \Theta_2(\wp)\right| \int_0^I (I - \wp)^{\Phi - 1} d\wp \tag{29}$$

Since

$$\int_0^{\mathrm{I}} (\mathrm{I} - \wp)^{\Phi-1} \mathrm{d}\wp = \frac{\mathrm{I}^{\Phi}}{\Phi} \tag{30}$$

we get:

$$\left| \left(\int \Theta_1 \right)(\mathrm{I}) - \left(\int \Theta_2 \right)(\mathrm{I}) \right| \le \frac{\Lambda \mathrm{I}^{\Phi}}{\Phi \Gamma(\Phi)} \left\| \Theta_1 - \Theta_2 \right\| \tag{31}$$

If

$$\omega = \frac{\Lambda \mathrm{I}^{\Phi}}{\Phi \Gamma(\Phi)} < 1 \tag{32}$$

Then $\int$ is a contraction, and by Banach's fixed-point. So, there exists a unique solution of the fractional exponential growth problem.

2.2. Problem 2

Temperature difference in any situation results from energy flow into a system or energy flow from a system to surroundings. The former leads to heating, whereas the latter leads to cooling. Newton's law of cooling states that the rate of change of temperature of the body is proportional to the difference between the temperature of the body and that of the surrounding medium. [16] Let $\Theta(\tau)$ be the temperature of the object at time τ, and η be the ambient temperature (constant). The fractional Newton's Law of Cooling is given by:

$$D_t^{\alpha} \Theta(\tau) = -\varepsilon\left(\Theta(\tau) - \eta\right), \quad 0 < \Phi \le 1, \tag{33}$$

D_t^{α} is the Caputo fractional derivative of order Φ, $\Theta(\tau)$ temperature of the object, $\varepsilon > 0$ is the cooling rate constant, and $\Theta(0) = \Theta_0$ is the initial temperature of the object. Applying NT on both sides of Equation (32), we have

$$\frac{\wp^{\Phi}}{\lambda^{\Phi}} \Re(\wp, \lambda) - \frac{\wp^{\Phi-1}}{\lambda^{\Phi}} \Theta(0) = -\varepsilon \Re(\wp, \lambda) + \frac{\varepsilon \eta}{\wp} \tag{34}$$

On simplifying

$$\left(\frac{\wp^{\Phi}}{\lambda^{\Phi}} + \varepsilon \right) \Re(\wp, \lambda) = \frac{\wp^{\Phi-1}}{\lambda^{\Phi}} \Theta_0 + \frac{\varepsilon \eta}{\wp} \tag{35}$$

i.e.,

$$\Re(\wp, \lambda) = \frac{1}{\lambda} \frac{1}{\frac{\wp}{\lambda}} \frac{\left(\frac{\wp}{\lambda}\right)^{\Phi}}{\left(\frac{\wp}{\lambda}\right)^{\Phi} + \frac{\lambda^{\Phi}}{\wp^{\Phi}} \varepsilon} \Theta_0 + \frac{1}{\lambda} \frac{1}{\frac{\wp}{\lambda}} \frac{\varepsilon}{\left(\frac{\wp}{\lambda}\right)^{\Phi} + \varepsilon} \eta \tag{36}$$

Taking the inverse NT of Equation (35), we obtain

$$\Theta(\tau) = \mathrm{N}^{-1} \left[\frac{1}{\lambda} \frac{1}{\frac{\wp}{\lambda}} \frac{\left(\frac{\wp}{\lambda}\right)^{\Phi}}{\left(\frac{\wp}{\lambda}\right)^{\Phi} + \frac{\lambda^{\Phi}}{\wp^{\Phi}} \varepsilon} \Theta_0 + \frac{1}{\lambda} \frac{1}{\frac{\wp}{\lambda}} \frac{\varepsilon}{\left(\frac{\wp}{\lambda}\right)^{\Phi} + \varepsilon} \eta \right] \tag{37}$$

[14] obtained

$$N^{-1}\left[\frac{1}{\lambda}\frac{1}{\frac{\wp}{\lambda}}\frac{\left(\frac{\wp}{\lambda}\right)^{\Phi}}{\left(\frac{\wp}{\lambda}\right)^{\Phi}+\frac{\lambda^{\Phi}}{\wp^{\Phi}}\varepsilon}\right]=E_{\Phi}\left(-\varepsilon\tau^{\Phi}\right), \quad (38)$$

$$N^{-1}\left[\frac{1}{\lambda}\frac{1}{\frac{\wp}{\lambda}}\frac{\varepsilon}{\left(\frac{\wp}{\lambda}\right)^{\Phi}+\varepsilon}\eta\right]=1-E_{\alpha}\left(-Kt^{\alpha}\right) \quad (39)$$

Then, the fractional solution of Equation (32) is

$$\Theta(\tau)=E_{\alpha}\left(-\varepsilon\tau^{\Phi}\right)\Theta_0+\left(1-E_{\alpha}\left(-\varepsilon\tau^{\alpha}\right)\right)\eta \quad (40)$$

Then,

$$\Theta(\tau)=\eta+\left(\Theta_0-\eta\right)E_{\alpha}\left(-\varepsilon\tau^{\alpha}\right) \quad (41)$$

$$\Theta(\tau)=\eta+\left(\Theta_0-\eta\right)e^{-\varepsilon\tau} \quad (42)$$

This is the exact solution obtained by [16].

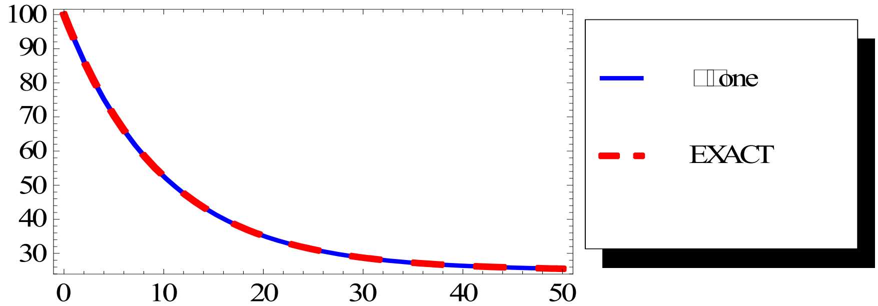

Figure 3. Plot of the NT solution of (32) at various values of Φ at $\varepsilon=0.1, \Theta_0=100, \eta=25$ in comparing with the exact solution at $\Phi=1$ obtained via [16].

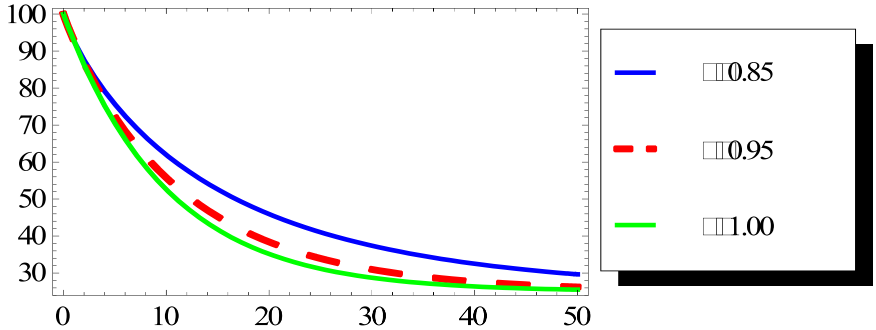

Figure 4. Plot of the NT solution of (32) at various values of Φ at $\varepsilon=0.1, \Theta_0=100, \eta=25$ in comparing with the solution via STVIM and the exact solution obtained via [16].

Now, we can prove the existence and uniqueness of the solution of Equation

(32) using fixed-point theory as follows:

Step 1: Consider the following fractional Newton's law of cooling is given by:

$$D_t^{\alpha}\Theta(\tau) = -\varepsilon\left(\Theta(\tau)-\eta\right),\quad 0<\Phi\le 1, \tag{43}$$

Step 2: Convert Equation (41) into an integral equation

Caputo FDEs can be rewritten as a Valera-type integral equation:

$$\Theta(\tau) = \Theta_0 - \frac{\varepsilon}{\Gamma(\Phi)}\int_0^{\tau}\left(\Theta(\wp)-\eta\right)(\tau-\wp)^{\Phi-1}\,\mathrm{d}\wp \tag{44}$$

This is a good starting point for applying fixed-point theory.

Step 3: Define a fixed-point operator

Define an operator $\perp$ as:

$$(\perp\Theta)(\tau) = \Theta_0 - \frac{\varepsilon}{\Gamma(\Phi)}\int_0^{\tau}\left(\Theta(\wp)-\eta\right)(\tau-\wp)^{\Phi-1}\,\mathrm{d}\wp \tag{45}$$

Step 4: Prove uniqueness using Banach's fixed-point. It states that if an operator $\perp$ is a contraction, then it has a unique fixed point, as follows:

Step 4.1: Define a function space

Consider the Banach space C ([0, R]) with the norm:

$$\|\Theta\| \le \sup_{\tau\in[0,\mathrm{I}]}\left|\Theta(\tau)\right| \tag{46}$$

Step 4.2: Show $\perp$ is a Contraction

Take two functions Θ_1 and Θ_2 compute:

$$\left|(\perp\Theta_1)(\tau)-(\perp\Theta_2)(\tau)\right| = \left|\varepsilon\int_0^{\tau}\frac{(\tau-\wp)^{\Phi}}{\Gamma(\Phi)}\left(\Theta_2(\wp)-\Theta_1(\wp)\right)\mathrm{d}\wp\right| \tag{47}$$

Applying the norm:

$$\begin{aligned}&\left|(\int\perp\Theta_1)(\tau)-(\perp\Theta_2)(\tau)\right|\\ &\le \frac{\varepsilon}{\Gamma(\Phi)}\sup_{\wp\in[0,\mathrm{I}]}\left|\Theta_1(\wp)-\Theta_2(\wp)\right|\int_0^{\mathrm{I}}(\mathrm{I}-\wp)^{\Phi-1}\,\mathrm{d}\wp\end{aligned} \tag{48}$$

Since

$$\int_0^{\mathrm{I}}(\mathrm{I}-\wp)^{\Phi-1}\,\mathrm{d}\wp = \frac{\mathrm{I}^{\Phi}}{\Phi} \tag{49}$$

we get:

$$\left|(\perp\Theta_1)(\tau)-(\perp\Theta_2)(\tau)\right| \le \frac{\varepsilon\mathrm{I}^{\Phi}}{\Phi\Gamma(\Phi)}\left\|\Theta_1-\Theta_2\right\| \tag{50}$$

If

$$\omega = \frac{\varepsilon\tau^{\Phi}}{\Phi\Gamma(\Phi)} < 1 \tag{51}$$

Then $\int$ is a contraction, and by Banach's Fixed-Point, there exists a unique solution of the fractional exponential growth problem.

3. Conclusion

This approach displayed a novel technique named the N-Transform (NT). It is a general algorithm of the Laplace transform and the Sumudu transform. It gives novel general solutions to fractional differential equations in different fields of science. The results give no vel generic fractional solutions to exponential growth equation and Newton's law of cooling. To clarify the generalization of the algorithm, we compared the results of the current methods with those of other methods like the Sumudu transform method ($\wp T$), the Laplace transform method (ℓT), and the exact solutions. At $\wp = 1$, in Equation (19) and Equation (37),we have the solutions via $\wp T$. At $\lambda = 1$, in Equation (19) and Equation (37), we have solutions via ℓT. At $\Phi \to 1$ we have the classical form solutions. Fractional exponential growth equation and Newton's law of cooling were studied analytically and numerically and novel general solutions were gained via the proposed algorithm. The existence and uniqueness of the solutions were proved using Banach's fixed-point theorem. Figures 1-4 clarify the simplicity, quality, and generalization of the proposed algorithm (NT). The outcomes showed the significance of the proposed method (NT) for solving various differential equations in different aspects of science. The NT method has many advantages compared with other methods. The NT method is more effective, efficient, accurate and can easily handle a wide class of linear or nonlinear differential equations. The NT method demonstrates fast convergence of the solution. It will be successfully used to handle most types of differential equations that appear in several physical models and scientific applications. Its method attacks the problem in a direct way and in a straightforward fashion without using linearization, perturbation, or any other restrictive assumption. Lastly, the NT method is a novel generic method that can be employed to solve problems in different aspects of science.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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