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A New Efficient *h*-Type Adaptive Analysis Procedure for Finite Element Method

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Abstract

This paper focuses on constructing the adaptive analysis model based on the Finite Element Method (FEM) with an unstructured mesh, which includes taking full advantage of the two-dimensional triangle's ability to discretize any complex model. The error indicator and h-type refinement scheme for FEM has been presented by providing a simple error indicator based on the difference value of nodal fundamental physical variables. It provides an effective measure of the error in the numerical solution, which can lead to an adaptive refinement scheme, determining the optimal location of new inserted nodes that are mainly distributed in the local high stress gradient domain. A numerical experiment is presented to demonstrate that the proposed error indicator reasonably tracks the distribution of the true error, and the effectiveness of the proposed adaptive analysis procedure, which achieves significantly higher convergence rates compared to the uniform mesh.

Keywords

Adaptive Analysis, Error Indicator, FEM, H-Type Refinement, Delaunay Triangulation

1. Introduction

The Finite Element Method (FEM) has a well-established theoretical basis, and has been evolving rapidly with the advancement of computer technology, playing a crucial role in various fields, including machinery, metallurgy, automobile, civil engineering, aerospace, and materials [1]-[3]. After many years of unremitting efforts of several generations of researchers, FEM has developed into the most widely used and successful numerical method in the field of engineering simulation applications.

With the rapid development of computer technology, the finite element method has been widely studied and applied. Now it has become the most common numerical method in the field of engineering computation and simulation. For engineering applications, the most concerning problem is the reliability and effectiveness of numerical algorithms. Therefore, the research on the reliability and efficiency of finite element analysis technology has always been a research hotspot for researchers [4]-[6]. As engineering problems become more and more complex, the discrete models that need to be calculated using numerical methods become increasingly huge. Modern numerical analysis technology is developing towards high efficiency, high precision, low cost, and high performance, while the traditional analysis technology has been unable to meet the requirements [7] [8]. In order to solve this problem, under the existing hardware resources, it is necessary to develop efficient and high-precision numerical methods to analyze engineering problems, which not only improve the computational accuracy but also take into account the computational efficiency, so as to guide the direction of engineering improvement and optimization [9] [10].

Adaptive analysis technique is an effective way to improve the efficiency and accuracy of numerical methods by estimating the error of some intermediate results of numerical methods and refining the mesh in the region where the gradient of field variables changes sharply, with the ultimate goal of obtaining the highest computational accuracy by minimizing the computational cost [11] [12]. Adaptive analysis based on the finite element method has many advantages, and it is a worthy topic for further study. So many researchers devote much attention to the development of the adaptive finite element method [13]-[15].

2. Finite Element Method

Finite element analysis can be divided into three stages: pre-processing, calculation, and post-processing. The pre-processing is to establish the finite element model and complete the mesh generation; calculation and solution refer to the use of a finite element solver, which includes the derivation of finite element formulations, including the selection of a reasonable element coordinate system, the establishment of element trial functions, and the formation of the stiffness matrix.

Traditional FEM can select the basis functions in each element and using the linear combination of the element basis functions to approximate the true solution in the element, the overall basis function in the whole computational domain can be regarded as consisting of each element basis function, then the solution in the whole computational domain can be regarded as consisting of the approximate solutions in all elements. The unit is assembled into a discrete domain of the total matrix equation, and finally, it is the solution of the simultaneous equations. Post-processing is to collect and process the analysis results, including visual display, so that users can have a more intuitive understanding of the calculation results. The analysis flow of the finite element method is shown in **Figure 1**.

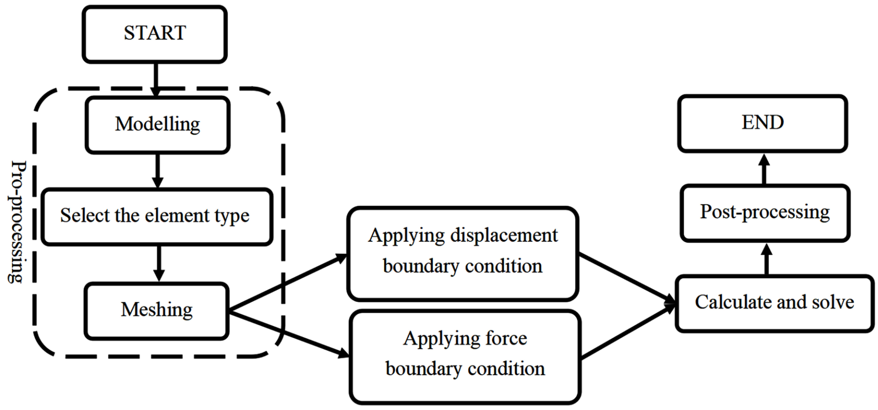

Figure 1. FEM analysis flowchart.

3. Adaptive Scheme

The finite element adaptive analysis technique guides the refinement of the mesh in the region where the gradient of the field variables is sharp by estimating the error of some intermediate results, while the mesh is kept relatively sparse in the region where the gradient is not obvious. The intelligent arrangement of mesh density can improve the computational efficiency while ensuring a high-precision solution.

The adaptive finite element analysis technique is used to refine the mesh in the area where the gradient of the field variable changes dramatically by estimating the error of some intermediate results of the finite element method, while the mesh in the area where the gradient of the field variable does not change significantly is kept relatively sparse. This intelligent arrangement of mesh density can improve the computational efficiency on the premise of ensuring a high-precision solution. It is particularly noted that any geometric model is ultimately represented by a set of elements, and the curve part of a two-dimensional problem is usually approximated by a piecewise straight line. For example, the edges of triangular elements are used to approximate the curve at the boundary of the model, and the accuracy of curve approximation is directly related to the number of elements used. The more elements used, the denser the elements, so that the curve part approximated by straight line segments is smoother and closer to the actual model.

The physical problems in the modeling research and calculation of practical problems are mostly described mathematically by differential equations. Among all kinds of numerical calculation methods to solve these equations, the finite element method is the most widely used and the most mature. The mathematical proof of the finite element method shows that in the process of using the finite element method to construct the element, the solution can infinitely converge to the exact solution of the original differential equation with the continuous refinement of the mesh. Therefore, the uniform refinement method is convenient to generate uniform fine meshes that meet the accuracy requirements in the whole solution domain, but in the actual model, the high stress or high strain area is only concentrated in a small part of the area, and each additional node corresponds to an equation, so the overall uniform refinement will bring unnecessary increase in calculation, which is not the most ideal method. The difference between uniform refinement and adaptive refinement is clearly shown in Figure 2.

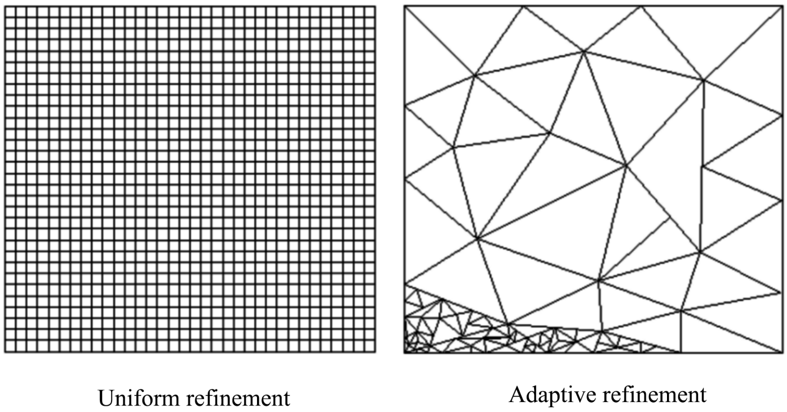

Figure 2. Comparison of uniform refinement and adaptive refinement.

In engineering analysis, structural damage is often caused by excessive local stress in a certain part of the structure. Therefore, under certain boundary conditions, we hope to get the accurate stress distribution of the structure, especially the stress of the key parts. The optimization mesh should be in the key parts of the analysis domain, the mesh density should be large enough, and the finite element solution should be accurate enough; in the non-key parts, the mesh density should be appropriate, and the accuracy requirement can be relaxed. In practical work, in order to obtain a high-precision solution and spend the minimum computational cost, the experience of the analyst is very important, but it is very difficult to determine the mesh density distribution only according to the experience of the analyst and to guess out of thin air. It is also difficult to give an optimal discrete model that can correctly reflect the true distribution of errors, which often leads to low analysis efficiency and unsatisfactory solution accuracy. Adaptive analysis technology is one of the ideal ways to solve this contradiction, which can use computers to replace analysts to automatically arrange the density distribution of nodes and elements. The ultimate goal of adaptive analysis is to obtain the most satisfactory accuracy through the minimum computational cost, which is an effective way to improve the efficiency and accuracy of numerical methods.

3.1. Framework and Flow Chat

The design of this procedure strictly follows the basic framework and flow of finite element adaptive analysis, and combined with the programming idea of functional modularization, the software is divided into five functional modules: finite element solver, error estimation, local mesh refinement critical, H-type refinement, subdivision of triangular elements, Delaunay triangle remesh, as depicted in Figure 3.

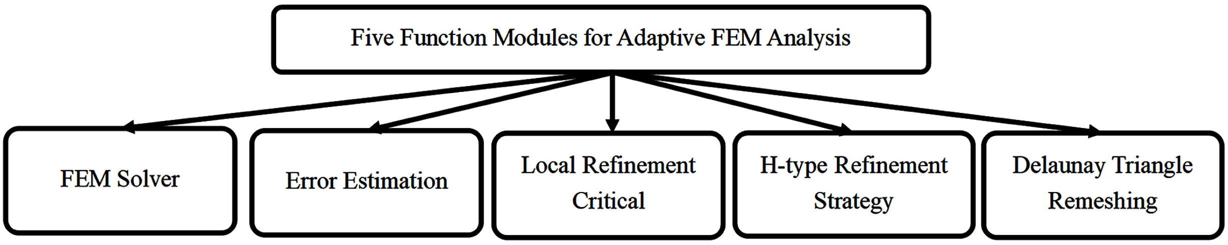

Figure 3. Five modules for adaptive FEM analysis.

Firstly, a finite element solver is used to read the initial mesh of the model to be analyzed, and force boundary conditions and displacement boundary conditions are applied. Error analysis is carried out according to the basic physical variables needed in the calculation process, and the real error distribution of the model is indicated according to the error indicator designed in the error analysis module. The local refinement critical is called to accurately locate the high error area where the gradient of the physical quantity changes dramatically, and determine the element set that needs to add points. The module responsible for H-type refinement and subdivision of triangular elements located in high-error areas. The new node information is obtained to delete the duplicated nodes, and the mesh refinement is used to regenerate the discrete model after the nodes are refined through the mesh reconstruction technology. At this point, a complete adaptive analysis is finished, the newly obtained model mesh information is imported into the finite element solver to restart the second step of adaptive procedure, when the number of iterative calculation steps of the adaptive procedure reaches the initially set maximum value, the calculation is immediately stopped, the iteration is jumped out, the whole adaptive analysis process is finished, and the stage of data collation and analysis is carried out. The basic analysis process of the finite element adaptive analysis technique is shown in Figure 4 below.

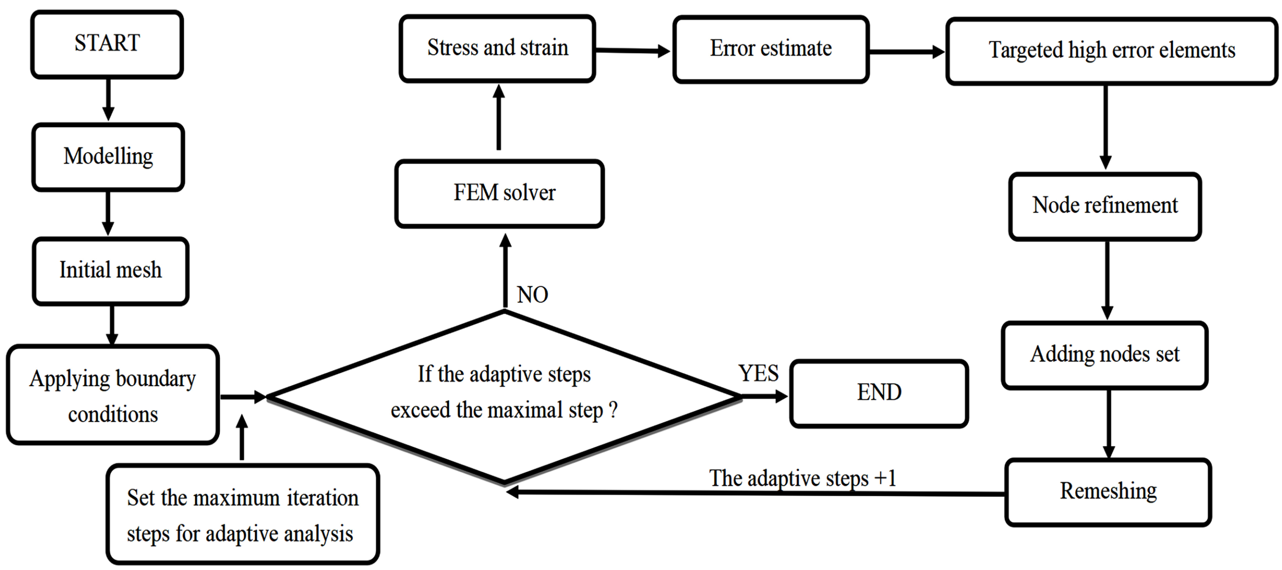

Figure 4. Adaptive FEM analysis flowchart.

3.2. Error Estimation

Error estimation is an effective error analysis of the intermediate results of the basic physical quantities of the discrete model through mathematical methods. It is an important part of adaptive analysis, and the design of error estimation will directly affect the success of the whole adaptive analysis. Its purpose is to find the error distribution as close as possible to the true one by means of error analysis, and to guide the density distribution of nodes in discrete models according to the error distribution. It is difficult to accurately judge the physical field distribution of the model before the calculation is completed, so it is impossible to accurately determine the density distribution of the meshes, and the mesh refinement can only be carried out in the parts where the gradient may be large by experience, which will inevitably make the grid refinement with a certain degree of blindness and cause a waste of resources.

In engineering, it is required to know the specific error value of numerical calculation in order to judge the reliability of the approximate solution, and objectively, it is required to know the characteristic information of the solution process in order to determine the error distribution of the calculation results. This kind of error estimation is organically combined with finite element mesh generation and improvement to form a progressive analysis process, so that the finite element solution and mesh meeting the accuracy requirements can be generated simultaneously. For each three-node tetrahedral element, we can obtain the difference value of nodal fundamental physical variables between node 1 and node 2 by the following equations.

$$\begin{aligned}
&\Delta\sigma_{xx}^{(12)} = \sigma_{xx}^{(1)} - \sigma_{xx}^{(2)} \quad && \Delta\varepsilon_{xx}^{(12)} = \varepsilon_{xx}^{(1)} - \varepsilon_{xx}^{(2)} \\
&\Delta\sigma_{yy}^{(12)} = \sigma_{yy}^{(1)} - \sigma_{yy}^{(2)} \quad && \Delta\varepsilon_{yy}^{(12)} = \varepsilon_{yy}^{(1)} - \varepsilon_{yy}^{(2)} \\
&\Delta\tau_{xy}^{(12)} = \tau_{xy}^{(1)} - \tau_{xy}^{(2)} \quad && \Delta\varepsilon_{xy}^{(12)} = \varepsilon_{xy}^{(1)} - \varepsilon_{xy}^{(2)}
\end{aligned}$$

High-error elements needing to be refined are screened out after all the elements' error indicators are arranged in descending order. The mesh refinement rate needs to be set artificially; it is defined as the ratio of the meshes that need to be refined to all the meshes. The mesh refinement ratio directly determines the number of elements needed to be refined in each step of adaptive analysis. Users can change the value of the refinement rate to control the adaptation and obtain results with desired accuracy. The larger the mesh refinement ratio value, the more elements are refined by adding more new nodes. On the contrary, the smaller the mesh refinement ratio value, the fewer elements are refined by adding fewer new nodes. However, when the value of the refinement ratio is 1, it becomes the case that the whole problem domain is uniformly refined.

3.3. H-Type Refinement Strategy

According to the result of error analysis in the previous step, the H-type refinement strategy adds nodes directly to elements located in the high error area in some way, so that the element is gradually refined to reduce the size of the element to achieve the purpose of adaptation, improve the accuracy of the solution, and make the calculation result approach the exact solution without changing the order of the element shape function. This method is simple, easy to implement, and has high reliability and stability, which makes it the most widely used in adaptive procedures.

The meshes used for scientific computing can be divided into two categories: one is a structured mesh, and the other is an unstructured mesh [16]. Unstructured meshes can be applied to models with complex geometric structures, but they are difficult to generate, and there is no mapping association between nodes and mesh elements; triangles belong to unstructured meshes. Triangular mesh generation is the most flexible method of automatic mesh generation. The triangle element has a complete mathematical theory basis and is relatively simple to obtain. Triangular meshes can simulate any complex geometric model or complex boundary. Although the calculation accuracy of quadrilateral elements is higher than that of triangular elements, quadrilateral elements can only be applied to some simple and regular models, and are still powerless for many complex problems in engineering applications. The curve part in two-dimensional problems is usually approximated by a piecewise straight line; for example, the edges of triangular elements are used to approximate the curve at the boundary of the model. The accuracy of curve approximation is directly related to the number of elements; the more elements are used, the denser the elements are, so the curve approximated by a straight line segment is smoother and closer to the actual model.

The specific h-type refinement strategy of adding points is shown in **Figure 5**. The corresponding edges of the angle A, the angle B and the angle C are respectively marked as the edge a, the edge b and the edge c, and a node is added at the midpoint position of the edge a, the edge b and the edge c respectively, which are respectively marked as the node 1, the node 2 and the node 3. The coordinates of the known vertices are, respectively, and the coordinates of the three new nodes can be obtained directly by using the midpoint formula: the coordinates of node 1 are $\left(\frac{x_B+x_C}{2},\frac{y_B+y_C}{2}\right)$, the coordinates of node 2 are $\left(\frac{x_A+x_C}{2},\frac{y_A+y_C}{2}\right)$, and the coordinates of node 3 are $\left(\frac{x_A+x_B}{2},\frac{y_A+y_B}{2}\right)$.

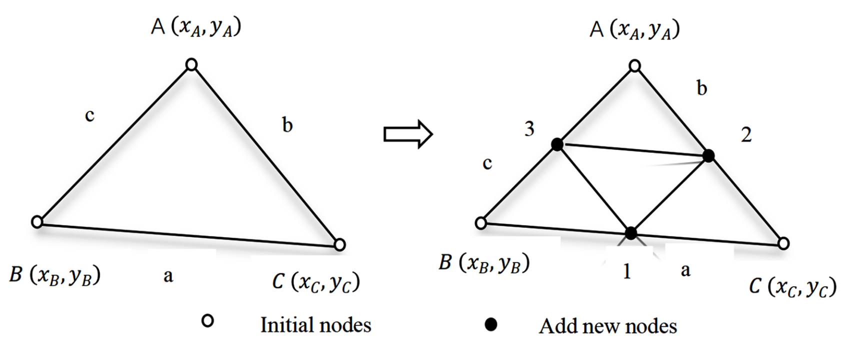

Figure 5. Illustration of the H-type refinement strategy for triangle.

4. Numerical Example

In order to show the effectiveness of the proposed adaptive FEM procedure, we consider the following benchmark example, which is a short two-dimensional cantilever plate. It is subjected to unit pressure along the upper edge and fixed along the left side, as shown in Figure 6. This case is considered a plane strain problem and the parameters are taken as Yong's modulus $E=1.0$, Poisson's ratio $\upsilon=0.3$, $a=1.0$ and $P=1.0$. The reference value of the strain energy of this problem is known as 0.951848 [17].

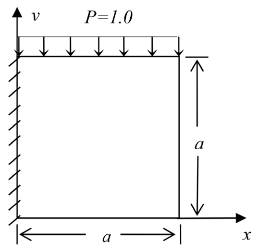

Figure 6. Short cantilever plate.

In order to demonstrate the performance of the proposed adaptive FEM, we studied this problem by comparing adaptive with uniform models. The uniform refined models with 220, 1090, 2402, and 6912 DOF, respectively. The coarse mesh with 220 DOF was used as the initial mesh for adaptive analysis, which was conducted for 6 steps with a refinement rate of 0.25. As shown in Figure 7, the solutions of strain energy are plotted against the DOF for both uniform and adaptive models. The results illustrate that all the results give a lower bound in energy norm to the exact one. One can also see that the computed strain energy obtained using the adaptive scheme converges to the reference solution much faster than that obtained using uniform refinement models. The number of nodes in the last step of adaptive analysis is only one-third of the number of nodes in the last step of the uniform refinement model, but the accuracy of the strain energy solution obtained is almost the same. These results have demonstrated the effectiveness of the present adaptive procedure.

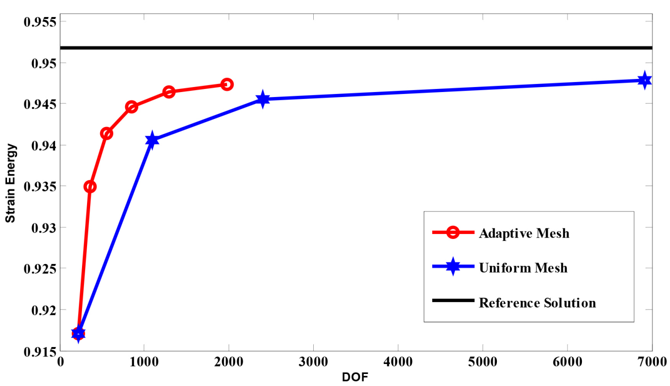

Figure 7. Comparison of convergence of strain energy solutions between uniform and adaptive models.

5. Conclusions

The paper has presented an adaptive procedure within the framework of the finite element method based on triangular meshes. The energy norm of the difference stress and strain between neighbour nodes in the same element is used as the error indicator. A simple H-type adding point strategy and refinement scheme for two-dimensional triangular elements has also been presented. Refinement rate can control the number of refined elements in each step of the adaptive analysis. The numerical benchmark example considered shows that the proposed error indicator reasonably tracks the distribution of the true error and that it provides an effective measure of the error in the numerical solution, which can lead to an adaptive refinement scheme. Numerical example has demonstrated that the proposed adaptive procedure provides a much higher convergence rate than uniform refinement.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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