Journal of Applied Mathematics and Physics, 2025, 13(9), 3173-3185
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Polarization Simultons in CARS by Polaritons: Energy and Velocity

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How to cite this paper: Feshchenko, V. and Feshchenko, G. (2025) Polarization Simultons in CARS by Polaritons: Energy and Velocity. *Journal of Applied Mathematics and Physics*, **13**, 3173-3185.
https://doi.org/10.4236/jamp.2025.139180

Received: August 10, 2025
Accepted: September 25, 2025
Published: September 28, 2025

Abstract

This paper is the continuation of our previous study on the propagation of temporary simultons in dipole-active crystals in the case of Coherent Anti-Stokes Raman Scattering (CARS), in which the possibility of the formation of simultons at all frequencies of interacting waves is considered. The present research considered the energy carried by the waves (the Manley-Rowe relation) and its relationship with the simultons' velocity. The cause of such study is twofold: some crystals (for instance a cubic crystal) owing to deformation of the dielectric constant by the pumping field become anisotropic and as such proper polarization appear; since the soliton's velocity depends on initial conditions and initial conditions here are the stimulated Raman scattering, it is important to find the relationship between the one of the most significant factors of that scattering which is the Raman gain factor and soliton's velocity.

Keywords

Coherent Anti-Stokes Raman Scattering, Polaritons, Phonons, Solitons, Polarization, Manley-Rowe Relation

1. Introduction

Recent years have shown the growth of interest in the theory of optical solitons and their practical applications, such as optical fiber communication, fiber lasers, optoelectronic devices, etc. [1]-[4]. Significant progress was also made in both areas of optical fibers [5]-[11] and optical lasers [12]-[15]. For instance, [15] presented a detailed review of studies on optical solitons in fiber lasers. This study includes the results in the developing of fiber lasers that are considered to be the proper nonlinear systems for the experimental study of the evolution of the temporal solitons, since during the last decade, several kinds of the theoretically predicted solitons were observed experimentally such as bright solitons, dark solitons, vector solitons, dissipative solitons, dispersion-managed solitons, polarization domain wall solitons, and so on.

The last decade is also characterized by the study of such important characteristics of solitons as their polarization [16]-[31]. For example, in [26], the research on the polarization dynamics of ultrafast solitons in mode-locked fiber lasers is presented. It was found that when a stable soliton was generated, its state of polarization shifted toward a stable state, and when the soliton was generated with excess power levels, it experienced relaxation oscillations in its intensity and timing. On the other hand, when a soliton is generated in an unstable state of polarization, it either decays in intensity until it disappears or its temporal width decreases until it explodes into several solitons, and then it disappears. In that paper, it was also found that when two solitons were simultaneously generated close to each other, they attract each other until they collide and merge into a single soliton. When these two solitons were generated with different states of polarization, they shifted their state of polarization closer to each other until the polarization coincides when they collide. The findings were supported by numerical calculations of a non-Lagrangian approach by simulating the Ginzburg-Landau equation governing the dynamics of solitons in a laser cavity. Their model also predicts the relaxation oscillations of stable solitons and the two types of unstable solitons observed in the experimental measurements. In [27], it was pointed out that optical switching had important applications in optical information processing, optical computing, and optical communications. The long-term pursuit of optical switches aimed to achieve a short switching time and a large modulation depth. It was concluded that among various mechanisms, all-optical switching based on the Kerr effect represented a promising solution. However, it is usually difficult to compromise both the switching time and modulation depth of a Kerr-type optical switch. As a compromise between the switching time and modulation depth, symmetry-selective polarization switching via Second-Harmonic Generation (SHG) in nonlinear crystals was considered to be a solution. That is why, in this paper, SHG-based all-optical ultrafast polarization switching by using geometric phase-controlled nonlinear plasmonic meta-surfaces was demonstrated. A switching time of hundreds of femtoseconds and a modulation depth of 97% were experimentally demonstrated. The function of dual-channel all-optical switching was also demonstrated on a meta-surface, which consisted of spatially variant meta-atoms. Some applications of polarization techniques in biological and clinical research were considered in [28]. In this review, the summarizing methodologies and applications related to tissue polarimetry were considered, with an emphasis on the adoption of the Stokes–Mueller formalism. Several recent breakthroughs, development trends, and potential multimodal uses in conjunction with other techniques were also presented.

A thorough theoretical research on polarization-related problems was provided in [29]-[31]. The study presented in [29] explores the dynamics of highly dispersive optical solitons in Nonlinear Schrödinger Equations (NLSE) with non-local Self-Phase Modulation (SPM) and Polarization-Mode Dispersion (PMD) since these nonlinear effects significantly influence soliton propagation and stability in advanced optical communication systems. Employing the Improved Modified Extended Tanh-Function Method (IMETFM), the exact soliton solutions were derived, including bright, dark, singular, and combo solitons. [30] implements the improved version of the modified extended tanh—function approach to retrieve a few new exact soliton solutions of the model with differential group delay. The self-modulation structure was from the Kerr law of nonlinearity. The material presented in [31] investigated the numerical computation of cubic-quartic optical solitons in birefringent fibers by Kerr's law. Utilizing the Improved Adomian Decomposition Method (IADM), the study improved the solution of complex-valued nonlinear evolution equations. This method decomposed both linear and nonlinear differential equations into simpler sub-problems, enabling the extraction of approximate analytical solutions without the need for linearization or perturbation techniques.

Another promising area of study on solitons is their formation and propagation in crystals where the nonlinear interaction is provided by the Raman effect [32]-[34]. The effects of the polarization on the evolution of Raman simultons were considered [35] [36]. In [37], we considered the group of questions related to the energy of simultons resulting from the boundary conditions in case CARS in dipole-active anisotropic crystals, which were consistent with the experimental results. In the present paper, we theoretically consider a more general case: the possibility of the formation of polarization simultons (simultaneously propagating pulses not only at frequencies of the laser, anti-Stokes, Stokes, and polariton waves) but also being polarized in x- and y-directions perpendicular to each other.

2. Basic Principles and Equations

In this paper, we consider the nonlinear coherent nonstationary interaction of four electromagnetic waves: anti-Stokes, Stokes, laser pump, and polariton. The pump wave is a linearly polarized plane wave, whereas anti-Stokes and Stokes have two mutually perpendicular components (the nonlinear medium is assumed to be nonmagnetic and transparent at frequencies of anti-Stokes, Stokes, and laser waves). It is also assumed that the nonlinear interaction takes place in a nonlinear medium in the form of a layer bounded by infinite planes located at $z = 0$ and $z = L$. The pump wave

$$\vec{\boldsymbol{E}}_l\left(\vec{\boldsymbol{r}},t\right)=\hat{\boldsymbol{e}}_l A_l\left(z,t\right)\exp\left[i\left(k_l^z z-\omega_l t\right)\right]+c.c. \tag{1}$$

propagates along the z-axis. The subscripts a, l, s, and p henceforth denote the anti-Stokes, laser, Stokes, and polariton waves at the frequencies $\omega_{a,l,s,p}$. We use the expressions for the anti-Stokes, Stokes, and polariton fields in the form

$$\vec{\boldsymbol{E}}_a\left(\vec{\boldsymbol{r}},t\right)=\sum_{\mu=1,2}\hat{\boldsymbol{e}}_a^{(\mu)}A_a^{(\mu)}\left(z,t\right)\exp\left[i\left(\vec{\boldsymbol{k}}_a^{(\mu)}\cdot\vec{\boldsymbol{r}}-\omega_a t\right)\right]+c.c. \tag{2}$$

$$\vec{E}_s(\vec{r},t)=\sum_{\mu=1,2}\hat{e}_s^{(\mu)}A_s^{(\mu)}(z,t)\exp\left[i\left(\vec{k}_s^{(\mu)}\cdot\vec{r}-\omega_s t\right)\right]+c.c. \tag{3}$$

$$\vec{E}_p(\vec{r},t)=\sum_{\sigma=1,2,3}\hat{e}_p^{(\sigma)}A_p^{(\sigma)}(z,t)\exp\left[i\left(\vec{W}^{(\mu)}\cdot\vec{r}-\omega_p t\right)\right]+c.c. \tag{4}$$

where $k_{a,s,p}^{(\mu)}=q_{a,s,p}n_{a,s,p}^{(\mu)}$; $k_l=q_l n_l$; $n_{a,s,p}^{(\mu)},n_l$ $k_{a,s,p}^{(\mu)},k_l$ are the refractive indices and the magnitude of wave vectors in the unpumped medium; $\hat{e}_l,\hat{e}_{a,s,p}^{(\mu)}$ are the real unit vectors of corresponding electromagnetic fields.

Since we consider the non-resonant frequencies, the longitudinal components of the anti-Stokes and Stokes waves can be neglected, but this cannot be done for the polariton wave in the vicinity of the phonon resonance. As it was shown in [38], with a further advance towards this region, the amplitudes of all three polariton waves $A_p^{(\sigma)}$ become comparable at first, then $A_p^{(3)}$ (longitudinal component becomes dominant (of course, if such excitation is allowed by the selection rules). The phase of the polariton wave is determined by the vector $\vec{W}^{(\mu)}$ (not by $\vec{k}_p^{(\mu)}$) $k_p^{(\mu)}=q_p\sqrt{\varepsilon_p^{(\mu)}}$, $\varepsilon_p^{(\mu)}=\varepsilon_p'^{(\mu)}+i\varepsilon_p''^{(\mu)}$ which is the dielectric constant at the polariton frequency ω_p.

The nonlinear interaction of the electromagnetic waves $\omega_{l,s}$ with further generation of anti-Stokes and polariton waves is described by the nonlinear parts of the corresponding polarizations ($\mu=1,2$):

$$\begin{gathered}
P_a^{(\mu)}=\chi_a^{\mu\sigma}A_l A_p^{(\sigma)}\mathrm{e}^{-i\Delta k^z z}+\gamma_{a2}^{\mu\mu'}\left|A_l\right|^2A_a^{(\mu')}+\gamma_{a2}^{\mu\mu'\mu''}A_s^{(\mu)}A_s^{(\mu')}A_a^{(\mu'')},\\
P_l=\chi_{l1}^{\mu\sigma}A_s^{(\mu)}A_p^{(\sigma)}+\chi_{l2}^{\mu\sigma}A_a^{(\mu)}A_p^{(\sigma)*}\mathrm{e}^{i\Delta k^z z},\\
P_s^{(\mu)}=\chi_s^{\mu\sigma}A_l A_p^{(\sigma)*}+\gamma_{s1}^{\mu\mu'}\left|A_l\right|^2A_s^{(\mu')}+\gamma_{s2}^{\mu\mu'\mu''}A_a^{(\mu)}A_a^{(\mu')}A_s^{(\mu'')},\\
P_p^{(\sigma,\mu)}=\chi_{p1}^{\mu\sigma}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu\sigma}A_l A_a^{(\mu)*}\exp\left(-i\Delta k^z z\right)\quad(\sigma=1,2)\\
P_p^{(3)(\mu)}=\chi_{p1}^{\mu 3}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu 3}A_l A_a^{(\mu)*}\exp\left(-i\Delta k^z z\right),
\end{gathered} \tag{5}$$

where $\chi_a^{\mu\sigma},\chi_{l1}^{\mu\sigma},\chi_{l2}^{\mu\sigma},\chi_s^{\mu\sigma},\chi_{p1}^{\mu\sigma},\chi_{p2}^{\mu\sigma},\chi_{p1}^{\mu3},\chi_{p2}^{\mu3},\gamma_{a2}^{\mu\mu'},\gamma_{a2}^{\mu\mu'\mu''},\gamma_{s1}^{\mu\mu'},\gamma_{s2}^{\mu\mu'\mu''}$ are the corresponding tensor contractions of the non-resonance quadratic and cubic nonlinear polarizabilities with unit vectors of interacting waves, $\Delta k^z\equiv k_l^z+W^z-k_a^z$.

The system of shortened equations for the amplitudes $A_{a,l,s,p}$ is obtained from Maxwell's equations by using the standard method of getting shortened equations by applying the approximation of slowly varying amplitudes [39] ($\mu=1,2$, $\sigma=1,2,3$)

$$\begin{aligned}
\frac{\partial A_a^{(\mu)}}{\partial z}+\frac{1}{v_a^{z(\mu)}}\frac{\partial A_a^{(\mu)}}{\partial t}=i\frac{2\pi\omega_a}{cn_a^{(\mu)}\cos\theta_a^{z(\mu)}}&\left\{\chi_a^{\mu\sigma}A_l A_p^{(\sigma)*}\mathrm{e}^{-i\Delta k^z z}\right.\\
&\left.+\gamma_{a2}^{\mu\mu'}\left|A_l\right|^2A_a^{(\mu')}+\gamma_{a2}^{\mu\mu'\mu''}A_s^{(\mu)}A_s^{(\mu')*}A_a^{(\mu'')}\right\},
\end{aligned} \tag{6}$$

$$\frac{\partial A_l}{\partial z}+\frac{1}{v_l^z}\frac{\partial A_l}{\partial t}=i\frac{2\pi\omega_l}{cn_l\cos\theta_l^z}\left\{\chi_{l1}^{\mu\sigma}A_s^{(\mu)}A_p^{(\sigma)}+\chi_{l2}^{\mu\sigma}A_a^{(\mu)}A_p^{(\sigma)*}\mathrm{e}^{i\Delta k^z z}\right\}, \tag{7}$$

$$\begin{aligned}
\frac{\partial A_s^{(\mu)}}{\partial z}+\frac{1}{v_s^{z(\mu)}}\frac{\partial A_s^{(\mu)}}{\partial t}=i\frac{2\pi\omega_s}{cn_s^{(\mu)}\cos\theta_s^z}&\left\{\chi_s^{\mu\sigma}A_l A_p^{(\sigma)*}+\gamma_{s1}^{\mu\mu'}\left|A_l\right|^2A_s^{(\mu')}\right.\\
&\left.+\gamma_{s2}^{\mu\mu'\mu''}A_a^{(\mu)}A_a^{(\mu'')*}A_s^{(\mu'')}\right\},\quad\sigma=1,2
\end{aligned} \tag{8}$$

$$\begin{aligned}&2iW^{z}\frac{\partial A_p^{(\sigma)*}}{\partial z}-iWe_p^{(\sigma)z}\frac{\partial A_p^{(3)*}}{\partial z}+i\frac{2\omega_p\varepsilon_p^{(\sigma)*}}{c^2}\frac{\partial A_p^{(\sigma)*}}{\partial t}+\left(W^2-k_p^{2*}\right)A_p^{(\sigma)*}\\&=4\pi q_p^2\left\{\chi_{p1}^{\mu\sigma}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu\sigma}A_lA_a^{(\mu)*}\mathrm{e}^{-i\Delta k^z z}\right\}\end{aligned} \tag{9}$$

$$\begin{aligned}&-iW\left(e_p^{(1)z}\frac{\partial A_p^{(1)*}}{\partial z}+e_p^{(2)z}\frac{\partial A_p^{(2)*}}{\partial z}\right)+i\frac{\mathrm{d}A_p^{(3)*}}{\mathrm{d}z}\left(W^z-We_p^{(3)z}\right)\\&+i\frac{2\omega_p\varepsilon_p^{(3)*}}{c^2}\frac{\partial A_p^{(3)*}}{\partial t}-k_p^{2*}A_p^{(3)*}=4\pi q_p^2\left\{\chi_{p1}^{\mu 3}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu 3}A_lA_a^{(\mu)*}\mathrm{e}^{-i\Delta k^z z}\right\}\end{aligned} \tag{10}$$

Provided the strong polariton absorption we have [38]

$$\left|W\left(A_p^{(\sigma)}\right)^{-1}\frac{\partial A_p^{(\sigma)}}{\partial z}\right|\approx\left|\frac{\omega_p}{c^2}\left(A_p^{(\sigma)}\right)^{-1}\frac{\partial A_p^{(\sigma)}}{\partial t}\right|\ll\left|W^2-k_p^{2*}\right|, \tag{11}$$

We can neglect (9) and (10) with the derivatives, so that we can directly obtain the expressions for $A_p^{(\sigma)}$ $(\sigma=1,2)$ and $A_p^{(3)}$:

$$A_p^{(\sigma)*}=\frac{4\pi}{s^2-\varepsilon_p^*}\left\{\chi_{p1}^{\mu\sigma}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu\sigma}A_lA_a^{(\mu)*}\mathrm{e}^{-i\Delta k^z z}\right\},(\sigma=1,2), \tag{12}$$

and

$$A_p^{(3)*}=-\frac{4\pi}{\varepsilon_p^*}\left\{\chi_{p1}^{\mu 3}A_l^*A_s^{(\mu)}+\chi_{p2}^{\mu 3}A_lA_a^{(\mu)*}\mathrm{e}^{-i\Delta k^z z}\right\}, \tag{13}$$

where $s=\dfrac{W}{q_p}$.

The substitution of the obtained expressions (12) and (13) for the amplitudes of polariton waves in (6)-(10) results in a new system of differential equations for $A_{a,l,s}$ as follows:

$$\begin{aligned}\frac{\partial A_a^{(\mu)}}{\partial z}+\frac{1}{v_a^{z(\mu)}}\frac{\partial A_a^{(\mu)}}{\partial t}=i\frac{2\pi\omega_a}{cn_a^{(\mu)}\cos\theta_a^{z(\mu)}}&\left\{\overline{\gamma}_{a1}^{\mu\mu'\sigma}A_l^2A_s^{(\mu')*}\mathrm{e}^{-i\Delta k^z z}\right.\\&\left.+\overline{\gamma}_{a2}^{\mu\mu'\sigma}\left|A_l\right|^2A_a^{(\mu')}+\gamma_{a2}^{\mu\mu'\mu''}A_s^{(\mu)}A_s^{(\mu')*}A_a^{(\mu'')}\right\}\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial A_l}{\partial z}+\frac{1}{v_l^z}\frac{\partial A_l}{\partial t}=i\frac{2\pi\omega_l}{cn_l\cos\theta_l^z}&\left\{\overline{\gamma}_{l11}^{\mu\mu'\sigma}A_lA_s^{(\mu)}A_s^{(\mu')*}+\overline{\gamma}_{l12}^{\mu\mu'\sigma}A_l^*A_s^{(\mu)}A_a^{(\mu')}\mathrm{e}^{i\Delta k^z z}\right.\\&\left.+\overline{\gamma}_{l21}^{\mu\mu'\sigma}A_l^*A_s^{(\mu')}A_a^{(\mu)}\mathrm{e}^{i\Delta k^z z}+\overline{\gamma}_{l22}^{\mu\mu'\sigma}A_lA_a^{(\mu)}A_a^{(\mu')*}\right\}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial A_s^{(\mu)}}{\partial z}+\frac{1}{v_s^{z(\mu)}}\frac{\partial A_s^{(\mu)}}{\partial t}=i\frac{2\pi\omega_s}{cn_s^{(\mu)}\cos\theta_s^z}&\left\{\overline{\gamma}_{s1}^{\mu\mu'\sigma}\left|A_l\right|^2A_s^{(\mu')}\right.\\&\left.+\overline{\gamma}_{s2}^{\mu\mu'\sigma}A_l^2A_a^{(\mu')*}\mathrm{e}^{-i\Delta k^z z}+\gamma_{s2}^{\mu\mu'\mu''}A_a^{(\mu')}A_a^{(\mu'')*}A_s^{(\mu''')}\right\}\end{aligned} \tag{16}$$

where

$$\overline{\gamma}_{a1}^{\mu\mu'\sigma}\equiv 4\pi\left(\frac{\chi_a^{\mu\sigma}\chi_{p1}^{\mu'\sigma}}{s^2-\varepsilon_p}-\frac{\chi_a^{\mu 3}\chi_{p1}^{\mu' 3}}{\varepsilon_p}\right),\overline{\gamma}_{a2}^{\mu\mu'\sigma}\equiv 4\pi\left(\frac{\chi_a^{\mu\sigma}\chi_{p1}^{\mu'\sigma}}{s^2-\varepsilon_p}-\frac{\chi_a^{\mu 3}\chi_{p2}^{\mu' 3}}{\varepsilon_p}\right)+\gamma_{a2}^{\mu\mu'},$$

$$\overline{\gamma}_{l11}^{\mu\mu'\sigma}\equiv 4\pi\chi_{l1}^{\mu\sigma}\left(\frac{\chi_{p1}^{\mu'\sigma}}{s^2-\varepsilon_p}-\frac{\chi_{p1}^{\mu' 3}}{\varepsilon_p}\right),\overline{\gamma}_{l12}^{\mu\mu'\sigma}\equiv 4\pi\chi_{l1}^{\mu\sigma}\left(\frac{\chi_{p2}^{\mu'\sigma}}{s^2-\varepsilon_p}-\frac{\chi_{p2}^{\mu' 3}}{\varepsilon_p}\right),$$

$$\overline{\gamma}_{l21}^{\mu\mu'\sigma} \equiv 4\pi\chi_{l2}^{\mu\sigma}\left(\frac{\chi_{p1}^{\mu'\sigma}}{s^2-\varepsilon_p^*}-\frac{\chi_{p1}^{\mu'3}}{\varepsilon_p^*}\right), \overline{\gamma}_{l22}^{\mu\mu'\sigma} \equiv 4\pi\chi_{l2}^{\mu\sigma}\left(\frac{\chi_{p2}^{\mu'\sigma}}{s^2-\varepsilon_p^*}-\frac{\chi_{p2}^{\mu'3}}{\varepsilon_p^*}\right),$$

$$\overline{\gamma}_{s1}^{\mu\mu'\sigma} \equiv \gamma_{s1}^{\mu\mu'} + 4\pi\chi_{s}^{\mu\sigma}\left(\frac{\chi_{p1}^{\mu'\sigma}}{s^2-\varepsilon_p^*}-\frac{\chi_{p1}^{\mu'3}}{\varepsilon_p^*}\right), \overline{\gamma}_{s2}^{\mu\mu'\sigma} \equiv 4\pi\chi_{s}^{\mu\sigma}\left(\frac{\chi_{p2}^{\mu'\sigma}}{s^2-\varepsilon_p^*}-\frac{\chi_{p2}^{\mu'3}}{\varepsilon_p^*}\right).$$

The system (14)-(16) can also be simplified if we introduce new variables as

$$A_a'^{(\mu)} \equiv A_a^{(\mu)} e^{i\Delta k^z z/2}, \quad A_s'^{(\mu)} \equiv A_s^{(\mu)} e^{i\Delta k^z z/2} \tag{17}$$

Assuming the "week" wave mismatch between waves at Stokes and anti-Stokes frequencies, that is

$$\left|\frac{\partial A_{a,s}'^{(\mu)}}{\partial z}+\frac{1}{v_{a,s}^{z(\mu)}}\frac{\partial A_{a,s}'^{(\mu)}}{\partial t}\right| \gg \frac{\Delta k^z}{2} A_{a,s}'^{(\mu)}, \tag{18}$$

and after bringing all variables to the unitless form, the system of nonstationary equations simulating CARS can be rewritten as follows:

$$\frac{\partial \tilde{A}_a'^{(\mu)}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_a^{z(\mu)}}\frac{\partial \tilde{A}_a'^{(\mu)}}{\partial \tilde{t}} = i\Big\{C_{a1}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}+C_{a2}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')} + C_{a2}^{\mu\mu'\mu''}\tilde{A}_s'^{(\mu)}\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu'')}\Big\} \tag{19}$$

$$\frac{\partial \tilde{A}_l}{\partial \tilde{z}}+\frac{1}{\tilde{v}_l^{z}}\frac{\partial \tilde{A}_l}{\partial \tilde{t}} = i\Big\{C_{l11}^{\mu\mu'}\tilde{A}_l\tilde{A}_s'^{(\mu)}\tilde{A}_s'^{(\mu')*}+C_{l12}^{\mu\mu'}\tilde{A}_l^*\tilde{A}_s'^{(\mu)}\tilde{A}_a'^{(\mu')} + C_{l21}^{\mu\mu'}\tilde{A}_l^*\tilde{A}_s'^{(\mu')}\tilde{A}_a'^{(\mu)}+C_{l22}^{\mu\mu'}\tilde{A}_l\tilde{A}_a'^{(\mu)}\tilde{A}_a'^{(\mu')*}\Big\} \tag{20}$$

$$\frac{\partial \tilde{A}_s'^{(\mu)}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_s^{z(\mu)}}\frac{\partial \tilde{A}_s'^{(\mu)}}{\partial \tilde{t}} = i\Big\{C_{s1}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_s'^{(\mu')}+C_{s2}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_a'^{(\mu')*} + C_{s2}^{\mu\mu'\mu''}\tilde{A}_a'^{(\mu)}\tilde{A}_a'^{(\mu')*}\tilde{A}_s'^{(\mu'')}\Big\} \tag{21}$$

where $\tilde{A}_{a,s}'^{(\mu)} \equiv A_{a,s}'^{(\mu)}/A_0$, $\tilde{A}_l \equiv A_l/A_0$, $\tilde{t} \equiv t/\tau_0$ (A_0 and τ_0 are the peak amplitude and characteristic pulse duration of the pump, $z_0 = c\tau_0$, c is the speed of light in vacuum,

$$C_{a1}^{\mu\mu'} \equiv \frac{2\pi\omega_a z_0}{cn_a^{(\mu)}\cos\theta_a^{z(\mu)}}\overline{\gamma}_{a1}^{\mu\mu'\sigma}A_0^2;\ C_{a2}^{\mu\mu'} \equiv \frac{2\pi\omega_a z_0}{cn_a^{(\mu)}\cos\theta_a^{z(\mu)}}\overline{\gamma}_{a2}^{\mu\mu'\sigma}A_0^2;$$

$$C_{a2}^{\mu\mu'\mu''} \equiv \frac{2\pi\omega_a z_0}{cn_a^{(\mu)}\cos\theta_a^{z(\mu)}}\overline{\gamma}_{a2}^{\mu\mu'\mu''}A_0^2;\ C_{l11}^{\mu\mu'} \equiv \frac{2\pi\omega_l z_0}{cn_l\cos\theta_l^{z}}\overline{\gamma}_{l11}^{\mu\mu'\sigma}A_0^2;$$

$$C_{l12}^{\mu\mu'} \equiv \frac{2\pi\omega_l z_0}{cn_l\cos\theta_l^{z}}\overline{\gamma}_{l12}^{\mu\mu'\sigma}A_0^2;\ C_{l21}^{\mu\mu'} \equiv \frac{2\pi\omega_l z_0}{cn_l\cos\theta_l^{z}}\overline{\gamma}_{l21}^{\mu\mu'\sigma}A_0^2; \tag{22}$$

$$C_{l22}^{\mu\mu'} \equiv \frac{2\pi\omega_l z_0}{cn_l\cos\theta_l^{z}}\overline{\gamma}_{l22}^{\mu\mu'\sigma}A_0^2;\ C_{s1}^{\mu\mu'} \equiv \frac{2\pi\omega_s z_0}{cn_s^{(\mu)}\cos\theta_s^{z}}\overline{\gamma}_{s1}^{\mu\mu'\sigma}A_0^2;$$

$$C_{s2}^{\mu\mu'} \equiv \frac{2\pi\omega_s z_0}{cn_s^{(\mu)}\cos\theta_s^{z}}\overline{\gamma}_{s2}^{\mu\mu'\sigma}A_0^2;\ C_{s2}^{\mu\mu'\mu''} \equiv \frac{2\pi\omega_s z_0}{cn_s^{(\mu)}\cos\theta_s^{z}}\overline{\gamma}_{s2}^{\mu\mu'\mu''}A_0^2;$$

3. The Manley-Rowe Relation for Simultaneously Propagating Waves at Frequencies $\omega_{a,l,s}$

We multiply each of the Equations (19)-(21) by the corresponding c.c. amplitude and add with its c.c. counterpart:

$$\tilde{A}_a'^{(\mu')*}\left(\frac{\partial \tilde{A}_a'^{(\mu)}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_a^{z(\mu)}}\frac{\partial \tilde{A}_a'^{(\mu)}}{\partial \tilde{t}}\right)=i\tilde{A}_a'^{(\mu')*}\Big\{C_{a1}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}+C_{a2}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')} + C_{a2}^{\mu\mu'\mu''}\tilde{A}_s'^{(\mu)}\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu'')}\Big\} \quad (23)$$

$$\tilde{A}_a'^{(\mu')}\left(\frac{\partial \tilde{A}_a'^{(\mu')*}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_a^{z(\mu)}}\frac{\partial \tilde{A}_a'^{(\mu')*}}{\partial \tilde{t}}\right)=-i\tilde{A}_a'^{(\mu')}\Big\{C_{a1}^{\mu\mu'*}\tilde{A}_l^{2*}\tilde{A}_s'^{(\mu')}+C_{a2}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')*} + C_{a2}^{\mu\mu'\mu''*}\tilde{A}_s'^{(\mu')*}\tilde{A}_s'^{(\mu')}\tilde{A}_a'^{(\mu'')*}\Big\} \quad (24)$$

$$\tilde{A}_l^*\left(\frac{\partial \tilde{A}_l}{\partial \tilde{z}}+\frac{1}{\tilde{v}_l^z}\frac{\partial \tilde{A}_l}{\partial \tilde{t}}\right)=i\tilde{A}_l^*\Big\{C_{l11}^{\mu\mu'}\tilde{A}_l\tilde{A}_s'^{(\mu)}\tilde{A}_s'^{(\mu')*}+C_{l12}^{\mu\mu'}\tilde{A}_l^*\tilde{A}_s'^{(\mu)}\tilde{A}_a'^{(\mu')} + C_{l21}^{\mu\mu'}\tilde{A}_l^*\tilde{A}_s'^{(\mu')}\tilde{A}_a'^{(\mu)}+C_{l22}^{\mu\mu'}\tilde{A}_l\tilde{A}_a'^{(\mu)}\tilde{A}_a'^{(\mu')*}\Big\} \quad (25)$$

$$\tilde{A}_l\left(\frac{\partial \tilde{A}_l^*}{\partial \tilde{z}}+\frac{1}{\tilde{v}_l^z}\frac{\partial \tilde{A}_l^*}{\partial \tilde{t}}\right)=-i\tilde{A}_l\Big\{C_{l11}^{\mu\mu'*}\tilde{A}_l^*\tilde{A}_s'^{(\mu)*}\tilde{A}_s'^{(\mu')}+C_{l12}^{\mu\mu'*}\tilde{A}_l\tilde{A}_s'^{(\mu)*}\tilde{A}_a'^{(\mu')*} + C_{l21}^{\mu\mu'*}\tilde{A}_l\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu)*}+C_{l22}^{\mu\mu'*}\tilde{A}_l^*\tilde{A}_a'^{(\mu)*}\tilde{A}_a'^{(\mu')}\Big\} \quad (26)$$

$$\tilde{A}_s'^{(\mu)*}\left(\frac{\partial \tilde{A}_s'^{(\mu)}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_s^{z(\mu)}}\frac{\partial \tilde{A}_s'^{(\mu)}}{\partial \tilde{t}}\right)=i\tilde{A}_s'^{(\mu)*}\Big\{C_{s1}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_s'^{(\mu')}+C_{s2}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_a'^{(\mu')*} + C_{s2}^{\mu\mu'\mu''}\tilde{A}_a'^{(\mu)}\tilde{A}_a'^{(\mu')*}\tilde{A}_s'^{(\mu'')}\Big\} \quad (27)$$

$$\tilde{A}_s'^{(\mu)}\left(\frac{\partial \tilde{A}_s'^{(\mu)*}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_s^{z(\mu)}}\frac{\partial \tilde{A}_s'^{(\mu)*}}{\partial \tilde{t}}\right)=-i\tilde{A}_s'^{(\mu)}\Big\{C_{s1}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_s'^{(\mu)*}+C_{s2}^{\mu\mu'*}\tilde{A}_l^{2*}\tilde{A}_a'^{(\mu')} + C_{s2}^{\mu\mu'\mu''*}\tilde{A}_a'^{(\mu)*}\tilde{A}_a'^{(\mu')}\tilde{A}_s'^{(\mu'')*}\Big\} \quad (28)$$

When we add those equations together, the right part yields 0, which means that our equations adequately describe the processes under consideration:

$$\frac{\partial\left|\tilde{A}_a'^{(\mu)}\right|^{2'}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_a^{z(\mu)}}\frac{\partial\left|\tilde{A}_a'^{(\mu)}\right|^2}{\partial \tilde{t}}+\frac{\partial\left|\tilde{A}_l'\right|^{2'}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_l^z}\frac{\partial\left|\tilde{A}_l'\right|^2}{\partial \tilde{t}} +\frac{\partial\left|\tilde{A}_s'^{(\mu)}\right|^{2'}}{\partial \tilde{z}}+\frac{1}{\tilde{v}_s^{z(\mu)}}\frac{\partial\left|\tilde{A}_s'^{(\mu)}\right|^2}{\partial \tilde{t}}=0. \quad (29)$$

We were also assuming that:

$$C_{a1}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu')*}+C_{s2}^{\mu\mu'}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu')*} = C_{l12}^{\mu\mu'*}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu')*}+C_{l21}^{\mu\mu'*}\tilde{A}_l^2\tilde{A}_s'^{(\mu')*}\tilde{A}_a'^{(\mu)*}.$$

$$C_{a2}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')}\tilde{A}_a'^{(\mu')*}+C_{l22}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu)}\tilde{A}_a'^{(\mu')*} = C_{a2}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')}\tilde{A}_a'^{(\mu')*}+C_{l22}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_a'^{(\mu')}\tilde{A}_a'^{(\mu)*}. \quad (30)$$

$$C_{a2}^{\mu\mu'\mu''}\tilde{A}_s^{\prime(\mu)}\tilde{A}_s^{\prime(\mu')*}\tilde{A}_a^{\prime(\mu')*}\tilde{A}_a^{\prime(\mu'')}+C_{s2}^{\mu\mu'\mu''}\tilde{A}_s^{\prime(\mu'')}\tilde{A}_s^{\prime(\mu)*}\tilde{A}_a^{\prime(\mu')*}\tilde{A}_a^{\prime(\mu)}$$
$$=C_{a2}^{\mu\mu'\mu''*}\tilde{A}_s^{\prime(\mu')}\tilde{A}_s^{\prime(\mu)*}\tilde{A}_a^{\prime(\mu'')*}\tilde{A}_a^{\prime(\mu')}+C_{s2}^{\mu\mu'\mu''*}\tilde{A}_s^{\prime(\mu)}\tilde{A}_s^{\prime(\mu'')*}\tilde{A}_a^{\prime(\mu)*}\tilde{A}_a^{\prime(\mu')}.$$

$$C_{l12}^{\mu\mu'}\tilde{A}_l^{2*}\tilde{A}_s^{\prime(\mu)}\tilde{A}_a^{\prime(\mu')}+C_{l21}^{\mu\mu'}\tilde{A}_l^{2*}\tilde{A}_s^{\prime(\mu')}\tilde{A}_a^{\prime(\mu)}$$
$$=C_{a1}^{\mu\mu'*}\tilde{A}_l^{2*}\tilde{A}_s^{\prime(\mu')}\tilde{A}_a^{\prime(\mu')}+C_{s2}^{\mu\mu'*}\tilde{A}_l^{2*}\tilde{A}_s^{\prime(\mu)}\tilde{A}_a^{\prime(\mu')}.$$

$$C_{s1}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_s^{\prime(\mu')}\tilde{A}_s^{\prime(\mu)*}+C_{l11}^{\mu\mu'}\left|\tilde{A}_l\right|^2\tilde{A}_s^{\prime(\mu)}\tilde{A}_s^{\prime(\mu')*}$$
$$=C_{s1}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_s^{\prime(\mu)}\tilde{A}_s^{\prime(\mu')*}+C_{l11}^{\mu\mu'*}\left|\tilde{A}_l\right|^2\tilde{A}_s^{\prime(\mu)*}\tilde{A}_s^{\prime(\mu')}.$$

If we introduce the energy per area delivered by any wave as

$$W_{a,s}^{(\mu)}\equiv\int_{-\infty}^{\infty}\left|\tilde{A}_{a,s}^{\prime(\mu)}\right|^2\mathrm{d}\tilde{t},W_l\equiv\int_{-\infty}^{\infty}\left|\tilde{A}_l\right|^2\mathrm{d}\tilde{t}\ . \tag{31}$$

Then it can be easily shown (after integration over time of (29)) that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\tilde{z}}\left(W_a^{(\mu)}+W_l+W_s^{(\mu)}\right)=0}, \tag{32}$$

which means that the electromagnetic energy of solitary traveling pulses is conserved when traveling in a nonlinear medium.

4. Polarization Simultons Speed in the Case of CARS by Polaritons

To do that, we will analyze the system of nonlinear equations found in [36]

$$\frac{\mathrm{d}Q}{\mathrm{d}\tilde{\xi}}=\alpha Q^2\sin\Phi\ , \tag{33}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde{\xi}}=2\alpha Q\cos\Phi+\beta Q\ , \tag{34}$$

where $\lambda_a^{(\mu)2}\equiv-\kappa_a^{(\mu)}C_{a1}^{\mu\mu}$, $\lambda_l^2\equiv\kappa_l\left(C_{l12}^{\mu\mu}+C_{l21}^{\mu\mu}\right)$,

$$\lambda_s^{(\mu)2}\equiv-\kappa_s^{(\mu)}C_{s2}^{\mu\mu},\quad \alpha\equiv2\lambda_a^{(\mu)}\lambda_l^2\lambda_s^{(\mu)},$$

$$\beta\equiv\left(2\kappa_lC_{l22}^{\mu\mu}-\kappa_s^{(\mu)}C_{s2}^{\mu\mu\mu}\right)\lambda_a^{(\mu)2}-\left(\kappa_s^{(\mu)}C_{s1}^{\mu\mu}+\kappa_a^{(\mu)}C_{a2}^{\mu\mu}\right)\lambda_l^2$$
$$+\left(2\kappa_lC_{l11}^{\mu\mu}-\kappa_a^{(\mu)}C_a^{\mu\mu\mu}\right)\lambda_s^{(\mu)2}.$$

$$\tilde{A}_{a,s}^{\prime(\mu)}\left(\tilde{z},\tilde{t}\right)\equiv B_{a,s}^{(\mu)}\left(\tilde{\xi}\right)\mathrm{e}^{i\Phi_{a,s}^{(\mu)}\left(\tilde{\xi}\right)} \tag{35}$$

$\tilde{A}_l\left(\tilde{z},\tilde{t}\right)\equiv B_l\left(\tilde{\xi}\right)\mathrm{e}^{i\Phi_l\left(\tilde{\xi}\right)}$, $Q\equiv\frac{B_a^{(\mu)2}}{\lambda_a^{(\mu)2}}=\frac{B_l^2}{\lambda_l^2}=\frac{B_s^{(\mu)2}}{\lambda_s^{(\mu)2}}$, $\kappa_{a,s}^{(\mu)}\equiv\tilde{v}_{a,s}^{z(\mu)}\tilde{v}^z\Big/\left(\tilde{v}^z-\tilde{v}_{a,s}^{z(\mu)}\right)$,

$\kappa_l\equiv\tilde{v}_l^z\tilde{v}^z\Big/\left(\tilde{v}^z-\tilde{v}_s^z\right)$, $\Phi\equiv2\Phi_l-\Phi_s^{(\mu)}-\Phi_a^{(\mu)}$. $\tilde{\xi}\equiv\tilde{t}-\tilde{z}\big/\tilde{v}^{z(\mu)}$; $\tilde{v}^{z(\mu)}$ is the velocity of simultons at the frequencies $\omega_{a,l,s}$; $B_l, B_{a,s}^{(\mu)}$ and $\Phi_l,\Phi_{a,s}^{(\mu)}$ are the real amplitudes and phases of the interacting waves, respectively. Q is the simultons intensity, α and β are the combinations of cubic nonlinear polarizabilities in the unitless form needed to form CARS.

The system (33)-(34) can be rewritten as,

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = Q^2 \sin\Phi, \tag{36}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = Q\left(\tilde{\beta} + 2\cos\Phi\right), \tag{37}$$

where $x = \alpha\tilde{\xi}, \tilde{\beta} \equiv \beta/\alpha$. We can reduce the number of equations by using the integral of motion

$$Q\left(\tilde{\xi}\right) = \frac{Const}{\sqrt{2\cos\Phi\left(\tilde{\xi}\right) + \tilde{\beta}}}, \tag{38}$$

where $Q > 0$, $\tilde{\beta} > 2$ ($\tilde{\beta} \equiv \beta/\alpha$).

It is easy to show (by using the system of equations with the integral of motion) that

$$\int_{-\infty}^{\infty} Q\left(\tilde{\xi}\right)\mathrm{d}\tilde{\xi} = \frac{2\pi}{\sqrt{(\beta+2\alpha)(\beta-2\alpha)}} \tag{39}$$

On the other hand, we can introduce the ratio of the energy (per unit area) to the energy (per unit area) for the laser pump as

$$\tilde{W}_l \equiv \int_{-\infty}^{\infty} \tilde{B}_l^2 \mathrm{d}\tilde{\xi}, \tilde{W}_{a,s}^{(\mu)} \equiv \int_{-\infty}^{\infty} \tilde{B}_{a,s}^{(\mu)2} \mathrm{d}\tilde{\xi}, \tag{40}$$

and the energy conservation relationship as

$$\tilde{W}_a^{(\mu)} + \tilde{W}_l + \tilde{W}_s^{(\mu)} = \tilde{W}_0, \tag{41}$$

where $\tilde{W}_0$ is the total energy per unit area of all interacting electromagnetic waves at the input to the nonlinear media.

Consequently, when we consider the left part of (41), we can get

$$\begin{aligned}\tilde{W}_a^{(\mu)} + \tilde{W}_l + \tilde{W}_s^{(\mu)} &= \int_{-\infty}^{\infty} B_a^{(\mu)2}\mathrm{d}\tilde{\xi} + \int_{-\infty}^{\infty} B_l^2 \mathrm{d}\tilde{\xi} + \int_{-\infty}^{\infty} B_s^{(\mu)2}\mathrm{d}\tilde{\xi} \\ &= \lambda_a^{(\mu)2}\int_{-\infty}^{\infty} Q\left(\tilde{\xi}\right)\mathrm{d}\tilde{\xi} + \lambda_l^2 \int_{-\infty}^{\infty} Q\left(\tilde{\xi}\right)\mathrm{d}\tilde{\xi} + \lambda_s^{(\mu)2}\int_{-\infty}^{\infty} Q\left(\tilde{\xi}\right)\mathrm{d}\tilde{\xi} \\ &= \left(\lambda_a^{(\mu)2} + \lambda_l^2 + \lambda_s^{(\mu)2}\right)\int_{-\infty}^{\infty} Q\left(\tilde{\xi}\right)\mathrm{d}\tilde{\xi}\end{aligned} \tag{42}$$

Finally, when we equate (39) and (42), we get the relationship between the boundary conditions and the simultons' speed.

$$\frac{2\pi\left(\lambda_a^{(\mu)2} + \lambda_l^2 + \lambda_s^{(\mu)2}\right)}{\sqrt{(\beta+2\alpha)(\beta-2\alpha)}} = \tilde{W}_0 \tag{43}$$

The last equation, in the case of weak dispersion, can be reduced to (see [37])

$$\frac{1}{\tilde{v}^{z(\mu)}} \approx \frac{1}{\tilde{v}_{em}^{z(\mu)}} + g\tilde{W}_0 \tag{44}$$

where $\tilde{v}_{em}^{z(\mu)} = \tilde{v}_l^z \approx \tilde{v}_a^{z(\mu)} \approx \tilde{v}_s^{z(\mu)}$ g is the gain factor of Raman scattering $g \approx 8\pi^2\omega z_0 \chi^2 A_0^2/(cn)$ ([36]). (**Figure 1**)

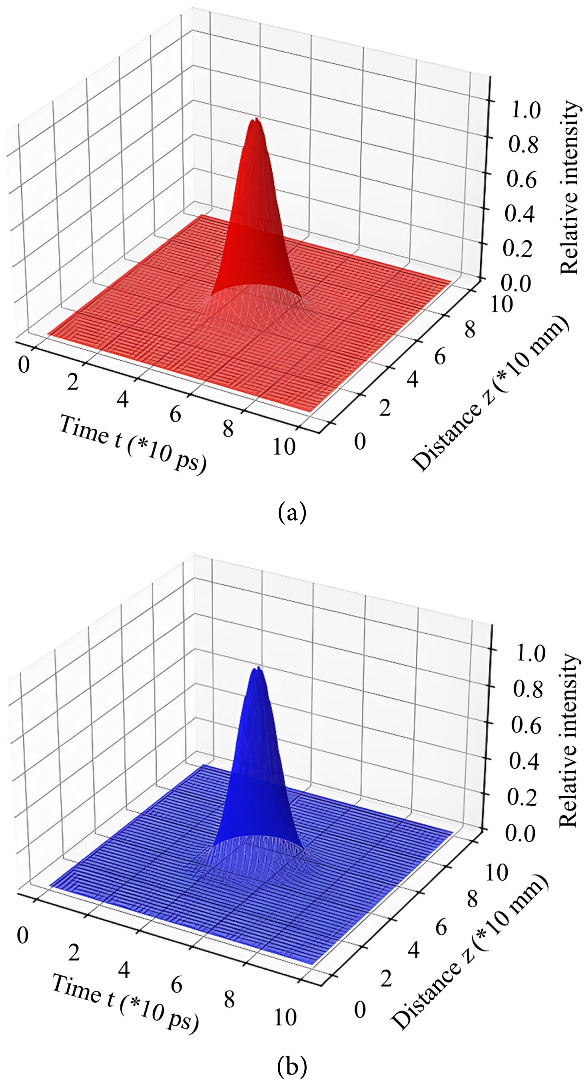

Figure 1. (a) Simultons traveling in the z-direction and being polarized in the x-direction, (b) Simultons traveling in the z-direction and being polarized in the y-direction.

5. Conclusion

In this research paper, we theoretically investigated energy relationships in the case of polarization simultons in CARS. It was proven that the derived system of differential equations, which simulates CARS of pulses propagating with different polarizations, obeys the Manley-Rowe relation. The relationship between simultons' velocity, energy, and the gain factor of Raman scattering was also found. This can be used in optoelectronics to create polarization filters and optical switches. The results presented in this paper might be used in spectroscopy to study the characteristics of anisotropic structures with the stable ultrashort pulses of a certain polarization. Also, they may be used in such optoelectronic devices as polarization filters when the electromagnetic waves of certain polarization become suppressed due to the nonlinear properties of the medium. Lastly, some applications can be foreseen when developing delay (compensatory) lines in communications.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Abbreviations

SRS	Stimulated Raman Scattering
CARS	Coherent Anti-Stokes Raman Scattering