Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(9), 3054-3091
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

# Integrated Mathematical Modelling and AI-Driven Simulation of Nonlinear Quantum Dynamics under Complex Electromagnetic Fields

**Kikmo Wilba Christophe*, Mbogba Guy Leopold, Mah Charitos Serge, Batambock Samuel, Ngoma Jean Pierre, Abanda Andre**

National Higher Polytechnic School of Douala, University of Douala, Douala, Cameroon
Email: *christopherkikmo@gmail.com

**How to cite this paper:** Christophe, K.W., Leopold, M.G., Serge, M.C., Samuel, B., Pierre, N.J. and Andre, A. (2025) Integrated Mathematical Modelling and AI-Driven Simulation of Nonlinear Quantum Dynamics under Complex Electromagnetic Fields. *Journal of Applied Mathematics and Physics*, **13**, 3054-3091.
https://doi.org/10.4236/jamp.2025.139173

**Received:** August 8, 2025
**Accepted:** September 20, 2025
**Published:** September 23, 2025



## Abstract

We present a novel integrated mathematical and numerical framework for the nonlinear Schrödinger equation in open quantum systems under electromagnetic fields, with a particular focus on Bose-Einstein condensates. To overcome limitations in modeling nonlinear dissipative effects and exogenous influences, we propose a non-Hermitian NLS formulation incorporating localized dissipation and complex electromagnetic couplings. Analytical tools, including generalized dissipative Strichartz estimates, variational methods, and spectral theory for non-self-adjoint operators, enable precise characterization of stability and dynamical behavior. A high-precision numerical platform, combining spectral discretization with semi-implicit Crank-Nicolson schemes and perfectly matched absorbing layers, was implemented to simulate multi-dimensional regimes. Machine learning modules, including convolutional neural networks, variational autoencoders, and sequential models (LSTM/Transformer), were employed to automatically explore critical regimes, predict temporal evolution, and identify phase transitions. Simulations revealed the formation of stable dissipative solitons and vortices, robust under environmental losses, and demonstrated that spatially structured dissipation combined with oscillating electromagnetic fields can stabilize quasi-stationary states. Quantitative analysis shows that topologically protected modes persist over long timescales and can be dynamically controlled via external fields, confirming the feasibility of encoding qubits and implementing basic quantum gates in dissipative quantum computing frameworks. By integrating nonlinear, dissipative, and topological effects within a unified framework, this approach provides new insights into the stability, coherence, and controllability of open quantum systems, offering a practical pathway for nanoscale device engineering and robust quantum state manipulation.

**Keywords**

Nonlinear Schrödinger Equation, Open Quantum Systems, Electromagnetic Fields, Bose-Einstein Condensates, Advanced Numerical Simulation

## 1. Introduction

Quantum dynamics are modelled variously and this issue sits at heart of theoretical physics research particularly in Bose-Einstein condensate studies. Systems evolve at extremely low temperatures and exhibit high correlation [1] [2]. Nonlinear Schrödinger equation plays a pivotal role as a reference model describing diverse complex phenomena in various mathematical contexts quite frequently nowadays. Nonlinear effects are incorporated via Gross-Pitaevskii terms in this model enabling capture of particle interactions rather nicely in condensed states [3] [4]. This equation models closed systems isolated from environment in its canonical form quite rigorously nowadays. Energy exchanges with external baths and dissipative interactions incur inherent information losses that are not taken into account properly over time. Experimental reality demands accounting for open quantum systems namely those subject extremely subtly to environmental decoherence and quite vigorous energy exchange [5]. Effects under discussion are frequently non-Hermitian in mathematical formulation and induce increased complexity analytically and also quite numerically. Technological advancements have made it fairly feasible now to manipulate such systems using controlled electromagnetic fields quite effectively nowadays [6] [7]. New avenues have consequently opened up for precise modulation of interparticle interactions and optical trapping alongside quantum state control simultaneously. Coupling open systems and nonlinear effects with electromagnetic fields poses a significant challenge requiring formulation of a coherent theoretical framework simultaneously models these dimensions.

Numerous studies scrutinize these components separately including analysis of NLS equations under stark non-linear conditions and introduction of magnetic fields [8]. These approaches remain predominantly fragmented and compartmentalised thereby failing miserably to integrate myriad cross-interactions that typify complex real-world systems. A significant methodological gap exists due to a lack of a unified mathematical framework rigorously addressing combined nonlinearity effects and environmental dissipation on CBE dynamics. Theoretical lacuna hinders comprehension of critical phenomena like formation of localised dissipative solitons and noise-induced phase transitions in coupled systems. Present study innovatively contributes by developing an integrated model rooted in a generalised nonlinear Schrödinger equation incorporating electromagnetic fields and system opening effects coherently. Advanced tools like Strichartz estimates and variational techniques under constraints get combined with high-precision numerical exploration based on adapted schemes. This study has a twofold objective essentially. It seeks to characterise asymptotic behaviours and emerging structures deeply within a novel theoretical framework very slowly. It aims at simulating effects of complex coupling on temporal dynamics and stability of CBEs subjected rather heavily to controlled external fields. This rigorous multidimensional approach ostensibly enables research to fill a critical gap in current literature and opens up novel vistas for quantum engineering. Perspectives gained will be particularly relevant for controlling coherent states and designing robust quantum devices amidst decoherence mechanisms in severely structured environments. A substantial enhancement manifests in modelling intricate quantum systems within non-conservative frameworks bearing fundamental and applied ramifications quite profoundly.

## 2. Mathematical Modelling of the Open Quantum System

### 2.1. Physical Setting and Assumptions

A macroscopic quantum system representative of a Bose-Einstein condensate at near-zero temperature evolves in partially controlled environment subjected to variable external electromagnetic fields spatially and temporally. System interacts significantly with surrounding quantum bath or macroscopic dissipation structure being inherently open and dynamically coupled [9] [10]. Openness gives rise emergently to non-conservative terms within effective Hamiltonian formulations surprisingly in most physical systems under consideration nowadays. A non-Hermitian generalisation of dynamics becomes imperative consequently [11] [12]. Model formulation occurs within spatial domain $\Omega \subset \mathbb{R}^d$ where $d = 1, 2, 3$ over finite time horizon $t \in [0, T]$. A normalised complex wave function denoted by $\psi = \psi(x,t)$ represents macroscopic quantum state of system belonging to complex Hilbert space $C$. Norm of this wave function defined as $\|\psi(x,t)\|_{L^2(\Omega)} = \int_\Omega |\psi(x,t)|^2 \mathrm{d}x \le 1$. Incorporation of dissipative dynamics occurs via decay of $\mathcal{L}^2$ norm as a function of time thereby enabling such complex behaviors naturally.

Fundamental physical underpinnings underlying model formulation are inherently quite simplistic:

- First hypothesis H1 pertains largely to open system somehow. Condensate interacts with quantum environment inducing irreversible losses of mass phase or coherence quite rapidly in most cases. Non-self-adjoint dissipative operators can model such phenomena effectively under certain conditions quite frequently in various mathematical contexts.
- Electromagnetic coupling denoted H2 emerges as second topic rather quietly. External electromagnetic field $A(x,t)$ and scalar potential $\Phi(x,t)$ modulate dynamics via minimal coupling in domain Ω over time interval [0, T] pretty regularly.

- Condensed nonlinear interaction is examined via third hypothesis H3 quite thoroughly. Interactions between atoms are represented by a local nonlinear potential $V_{int}\left(|\psi|^2\right)=g|\psi|^{2\sigma}$ pretty effectively with $g\in\mathbb{R}$ and $\sigma>0$.
- Hypothesis four H4 pertains largely to boundary conditions and initial states under somewhat specific circumstances typically found in various related studies. Domain subject to either periodic boundary conditions or homogeneous Dirichlet boundary conditions $\psi|_{\partial\Omega}=0$ and assumed given initial condition namely $\psi_0(x)\in H^1(\Omega)$ at $t=0$.

Mathematically formulation unfolds thus: under electromagnetic coupling a dissipative nonlinear Schrödinger equation emerges with its form manifesting subsequently:

$$i\hbar\partial t\psi(x,t)=\left[-\frac{\hbar}{2m}\left(\nabla-i\frac{\hbar}{q}A(x,t)\right)^2+q\Phi(x,t)+g|\psi(x,t)|^{2\sigma}\right]\psi(x,t)$$
$$-i\Gamma(x,t,\psi)\psi(x,t),$$

where the external vector potential (*i.e.*, the magnetic effect) is represented by the function $A(x,t)\in\mathbb{R}^d$.

The external scalar potential (electric effect) is denoted by $\Phi(x,t)\in\mathbb{R}$.

The parameter known as $g\in\mathbb{R}$ is the nonlinear interaction parameter. The value of $g\in\mathbb{R}$ can be positive, denoting a repulsive interaction, or negative, denoting an attractive interaction.

The quantity $\Gamma(x,t,\psi)\geq 0$ is defined as a modulated dissipation term, which represents losses that are associated with the process of coupling with the environment. These losses may be exemplified by non-linear and non-Markovian decay rates [13]-[15].

It is important to note that $\hbar$ and $q$ are, respectively, the reduced Planck constant and the effective charge of the particle.

Discussion proceeds via examination of functional notation within a mathematical framework slowly unfolding. A particular functional framework gets employed here.

$\psi(x,t)\in C\left([0,T];H^1(\Omega)\right)\cap C^1\left((0,T);H^{-1}(\Omega)\right)$, the potentials
$A,\Phi\in C^1\left(\bar{\Omega}\times[0,T]\right)$;

Term $\Gamma(x,t,\psi)$ assumed locally Lipschitz with respect to $\psi$ and measurable with respect to x and t simultaneously. Effective Hamiltonian becomes profoundly non-self-adjoint consequently:

$$\mathcal{H}_{eff}(t)=-\frac{\hbar}{2m}\left(\nabla-i\frac{\hbar}{q}A(x,t)\right)^2+q\Phi(x,t)+g|\psi(x,t)|^{2\sigma}-i\Gamma(x,t,\psi),$$

Quantum state dynamics unfold amidst nonlinear interactions and broken symmetry under modulatable fields and exogenous structural dissipation mechanisms fairly regularly now.

Modelling aims at deducing existence and stability of solutions $\psi(x,t)$ and characterising partial energy conservation in non-Hermitian regimes thoroughly meanwhile studying formation of localised stationary structures bifurcation phenomena and Γ-induced phase transitions quite extensively.

## 2.2. Generalised Non-Linear Schrödinger Equation

A non-Hermitian non-linear generalisation of Schrödinger equation describes behaviour of Bose-Einstein condensate pretty dynamically in an open environment under spatially and temporally modulated electromagnetic fields [16] [17]. This generalisation accounts for dissipation effects and electromagnetic coupling alongside condensed interparticle interactions within a rather complex system [18]. A complex, non-autonomous effective Hamiltonian forms the foundation of the proposed formulation. This framework emerges from a rigorous and consistent coupling between physical and topological mechanisms, ensuring both mathematical coherence and physical fidelity. The resulting governing equation can thus be formally expressed as:

$$i\hbar\partial t\psi(x,t)=\left[-\frac{\hbar}{2m}\left(\nabla-i\frac{\hbar}{q}A(x,t)\right)^2+q\Phi(x,t)+g\left|\psi(x,t)\right|^{2\sigma}+V_{ext}(x,t)\right]\psi(x,t)$$
$$-i\Lambda(x,t,\psi)\psi(x,t),$$

Several individuals intricately involved in project matters include people from various backgrounds and with diverse skill sets apparently:

The wave function of the condensate, designated as $\Psi(x,t)\in\mathbf{C}$.

The reduced Planck constant is denoted by $\hbar>0$.

In the case where $m>0$, the quantity being measured is the mass of the bosons.

The effective charge of the system is denoted by $q$.

It is asserted that $\mathrm{A}(\mathrm{x},\mathrm{t})\in\mathbb{R}^{\mathrm{d}}$: the vector potential that describes the external magnetic field.

The function $\Phi(x,t)\in\mathbb{R}$ is defined as the scalar potential that describes the external electric field.

$V_{ext}(x,t)\in\mathbb{R}$: the concept of trapping potential or geometric architecture (e.g., optical lattices, harmonic confinement, etc.) is employed.

In the case of $g\in\mathbb{R}$ and $\sigma>0$, the parameters are indicative of nonlinearity (*i.e.*, a generalized Gross-Pitaevskii interaction).

The function $\Lambda(x,t,\psi)$ is defined as nonlinear and possibly non-local dissipation function, with the capacity to induce a loss of normativity:

Thus primary differential operator stands out starkly:

$$\mathcal{H}(x,t,\psi):=-\frac{\hbar}{2m}\left(\nabla-i\frac{\hbar}{q}A(x,t)\right)^2+q\Phi(x,t)+g\left|\psi(x,t)\right|^{2\sigma}$$
$$+V_{ext}(x,t)-i\Lambda(x,t,\psi)$$

$\mathcal{H}_{eff}(t,\psi)=\mathcal{H}(x,t,\psi)-i\Lambda(x,t,\psi)$ demonstrably serves as non-Hermitian evolution generator defining a weirdly mixed linear-nonlinear operator.

Caractéristiques structurelles de l'équation

- ✓ Structural characteristics of equation are as follows: term generalised power-type non-linearity denotes specific kind of non-linearity characterised by exhibiting power-type behaviour quite remarkably $g|\psi|^{2\sigma}\psi$ has emerged as crucial factor in studying average interactions between bosonic particles under various conditions [19] [20]. Cubic models typically necessitate $\sigma=1$ for global existence of solutions pretty much always under certain conditions. In subcritical models with dimension $d\le 3$, $\sigma=1$ isn't enough and other parameters like $\sigma\in(0,2)$ must be taken into account carefully.
- ✓ Coupling $\nabla - i\frac{\hbar}{q}A(x,t)$ induces a connection with gauge invariance $A\mapsto A+\nabla\chi \quad \Phi\mapsto\Phi-\partial t\chi$ ensuring formulation quite physically consistent.
- ✓ Dissipation sans Hermiticity occurs via a non-Hermitian process characterized mostly by rather unusual energy loss mechanisms somehow. $i\Lambda(x,t,\psi)\psi$ serves largely to generalise Lindblad models or loss terms exhibiting highly amplitude-dependent rates quite effectively meanwhile. Consider following example now:

$$\Lambda(x,t,\psi)=\gamma(x,t)+\mu|\psi(x,t)|^{2\eta},$$

In the event that $\gamma\ge 0,\mu\ge 0$, and $\eta>0$, it can be deduced that a non-linear and non-conservative dissipation, modulated by the local probability density, is obtained.

- ✓ Breaking of unitary symmetry is explored in this study fairly thoroughly with intriguing results forthcoming from varied intricate analyses subsequently. Effective non-self-adjoint Hamiltonian $\mathcal{H}_{eff}$ induces a decidedly non-unitary flow quite freely. Evidently aforementioned equation blatantly contravenes fundamental principle of conservation of classical $\mathcal{L}^2$ norm pretty badly in most cases.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2_{L^2(\Omega)}=-2\int_\Omega \Lambda(x,t,\psi(x,t))|\psi(x,t)|^2\,\mathrm{d}x\le 0.$$

A wide range of concepts and methodologies is encompassed within complex spectral frameworks largely as a thoroughly theoretical construct. Operator $H_{eff}$ potentially admits a complex spectrum denoted by $\sigma(\mathcal{H}_{eff})$ within complex plane $\mathbb{C}$ quite remarkably [21] [22]. This complex spectrum induces a multitude of phenomena such as unstable regimes and amplified oscillations or damped spectral bifurcations.

## 2.3. Structural Properties of the Model

Generalised nonlinear Schrödinger equation model previously outlined comprises a gnarled mathematical structure integrating non-self-adjoint differential operators and nonlinear terms of disparate variable power explicitly [23]. Within a functional framework we undertake rigorous formal analysis of its fundamental properties thoroughly and with great structural detail

$\psi \in C([0,T];H^1(\Omega)) \cap C^1((0,T);H^{-1}(\Omega))$; where $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, and $H^1(\Omega)$, the Sobolev space is the standard reference point.

Generalised energy dissipation property

Generalised instantaneous energy of a system defined by some rather obscure functional will be considered in this somewhat esoteric investigation:

$$\mathcal{E}[\psi(t)] := \int_\Omega \left[ -\frac{\hbar}{2m} \left[ \left( \nabla - i\frac{\hbar}{q} A(x,t) \right) \psi \right]^2 + q\Phi(x,t)|\psi|^2 \right.$$
$$\left. + \frac{g}{\sigma+1} |\psi|^{2\sigma+2} + V_{ext}(x,t)|\psi|^2 \right] \mathrm{d}x,$$

Temporal evolution of parameter $\mathcal{E}$ be expressed succinctly through formal derivation mostly under specific conditions:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\hbar \int_\Omega \Lambda(x,t,\psi) |\partial t \psi|^2 \,\mathrm{d}x + \text{Terms of gauging and non-autonomy}$$

Dissipation induced by Lyaponov function demonstrably leads strictly to decay or loss of energy conservation rather rapidly over time. Consistency exists with open non-Hermitian characteristics of a system. Evidently this property precludes classical energy conservation thus necessitating utilisation of an analytical framework based on dissipative semigroups with weakened energy estimates.

2) Loss of normativity and non-unitary dynamics

Dissipative term causes violation of $\mathcal{L}^2$ norm conservation over time in equation. Quite ostensibly we possess fairly extensive evidence already:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\psi(t)\|^2_{L^2(\Omega)} = -2\int_\Omega \Lambda(x,t,\psi(x,t)) |\psi(x,t)|^2 \,\mathrm{d}x \le 0$$

A unidirectional decline in quantum mass suggests open dynamics characterized by such hallmark traits rather obviously. Behaviour like this facilitates modelling decoherence and energy relaxation into external baths or absorption by some pretty asymmetrical structures.

3) Broken Hamiltonian structure and pseudo-conservation

Model lacks conserved Hamiltonian structure but can be regarded as complex dissipative Hamiltonian system defined by symplectic derivation perturbed by purely imaginary dissipation. Complex energy thus gets expressed as $\mathcal{H}_{eff} = \mathcal{H} - i\Gamma$, wherein $\mathcal{H} := \langle \psi, \mathcal{H}_{\text{conservatif}}(t)\psi \rangle$, and quantity $\Gamma$ signifies generalised dissipative pseudo-flows associated with global dynamics of $\Gamma = \langle \psi, \Lambda(x,t,\psi)\psi \rangle$ [24]. Motivation stems from employing tools rooted deeply in CPT-symmetric operator theory or venturing into semi-classical analysis within non-self-adjoint frameworks rather haphazardly.

4) Gauge invariance and minimal electromagnetic coupling

The generalised kinetic term: $\left( \nabla - i\frac{\hbar}{q} A(x,t) \right)^2$ respects local gauge invariance $\psi \mapsto \psi \mathrm{e}^{iq\chi(x,t)/\hbar}, A \mapsto A + \nabla\chi, \Phi \mapsto \Phi - \partial t \chi$, emphasising this process guarantees fundamental consistency of model with principles of electromagnetic covariance, which are vital under normal circumstances. This property makes model suitable for dynamically controlled optical setups like variable lattices and rotating traps, etc.

5) Critical regimes and blow-up thresholds

Nonlinearity of type $|\psi|^{2\sigma}\psi$, with $\sigma > 0$ greatly influences classification of dynamic regimes according spatial dimension $d$ and critical index $\sigma_c$. Notably underneath layers of vagueness.

- Subcritical: $\sigma < \frac{2}{d} \to$ total solutions.
- Critique: $\sigma = \frac{2}{d} \to$ threshold instabilities.
- Overcritical: $\sigma > \frac{2}{d} \to$ possible blow-up in finite time.

Presence of term $-i\Lambda\psi$ plays somewhat regularising role or stabilising dissipative effect potentially inhibiting formation of singularities quite effectively. A detailed asymptotic study has been focused on thoroughly now see Section 3.4.

6) Complex spectral structure and resonant modes

Operator $H_{eff}(t,\psi)$ exhibits non-self-adjoint properties and its spectrum $\sigma(\mathcal{H}_{eff}) \subset \mathbb{C}$ contains various elements. Presence of complex eigenvalues indicates unstable modes or growing solutions rather elaborately. A spectral continuum exhibits diverse scattering behaviours markedly across varying conditions typically under certain circumstances quietly. Localised quasi-modes phenomenon occurs pretty frequently in context of dissipative solitons alongside various trapping modes rather mysteriously [25] [26]. Spectral structure depends heavily upon geometry of $V_{ext}$, and also on regularity of Λ and dynamic couplings A and Φ exist. Semi-classical analysis tools and spectral theory for non-normal operators are required to study this rather complex structure in depth. Proposed model exhibits highly non-trivial hybrid mathematical structure with properties of broken symmetry and anisotropic dissipation alongside critical non-linearity. Properties enable emergence of complex behaviours such as highly localised non-conservative structures and dynamic spectral bifurcations alongside modulated instabilities. Phenomena are explored analytically and numerically in Section 3, and a rigorous functional framework is used in Section 4.

### 2.4. Field-System Coupling Physical Model

Coupling formalism minimally underlies interaction between open quantum systems like Bose-Einstein condensate and dynamic external electromagnetic fields with dissipative topological contributions that are somehow modulatable. Coupling occurs between quantum field represented by wavefunction $\psi(x,t) \in C$ and electromagnetic potentials $A(x,t) \in \mathbb{R}^d$, $\Phi(x,t) \in \mathbb{R}$ governing magnetic and electric components somewhat exogenously [27] [28]. Coupling exhibits local characteristics arising from potential terms and non-local characteristics stemming largely from topology of magnetic field and Berry phase effect. These characteristics structure a dynamic strongly constrained by principles of special relativity and gauge invariance alongside Maxwell's laws modified somewhat for active media. Effective dynamics of condensate are described subsequently by system under intense scrutiny:

$$\begin{cases} i\hbar\partial t\psi = \left[ -\frac{\hbar}{2m}\left(\nabla - i\frac{\hbar}{q}A(x,t)\right)^2 + q\Phi(x,t) + g\left|\psi(x,t)\right|^{2\sigma} + V_{ext}(x,t) \right]\psi - i\Lambda(x,t,\psi)\psi \\ \nabla\cdot E = \frac{\rho\psi(x,t)}{\mathcal{E}_0},\ \nabla\times B - \mu_0\mathcal{E}_0\frac{\partial E}{\partial t} = \mu_0 j_\psi(x,t), \\ E = \nabla\Phi - \frac{\partial A}{\partial t},\ B = \nabla\times A, \end{cases}$$

where:

- $\rho_\psi(x,t) := q\left|\psi(x,t)\right|^2$ is the charge density induced by the condensate,
- $j_\psi(x,t) := \frac{q}{2mi}\left[\psi^*\left(\nabla - i\frac{q}{\hbar}A\right)\psi - c.c\right]$ is the induced probability current,
- $\mathcal{E}_0$ and $\mu_0$ are the dielectric and magnetic constants of vacuum.

Modified Maxwell equations are coupled extremely tightly with $\psi$'s quantum dynamics where field sources get directly spawned by wave function itself somehow.

Coupling concept features deeply intertwined mathematical intricacies underneath a rather complex analysis involving multifaceted strictly mathematical aspects:

1) Minimum canonical pairing

Modulation of scalar potential is achieved by function $\Phi$ while substitution $\nabla \to \nabla - i\frac{q}{\hbar}A$ ensures compatibility with gauge structure of equation. Configuration aligns with relativistic covariance principle thereby ensuring action integral remains invariant under U(1) gauge transformations quite remarkably.

2) Lagrangian structure of the system

A complex non-Hermitian Lagrangean has yielded this model:

$$\mathcal{L}(\psi,\overline{\psi},A,\Phi) = \frac{i\hbar}{2}(\overline{\psi}\partial t\psi - \psi\partial t\overline{\psi}) - \frac{q}{m}\left|\left(\nabla - i\frac{q}{\hbar}A\right)\psi\right|^2 - V_{ext}\left(x,t,|\psi|^2\right) - i\Lambda\left(x,t,|\psi|^2\right)|\psi|^2,$$

with $V_{eff} = V_{ext} + q\Phi + g|\psi|^{2\sigma}$, thus it can be inferred that Euler-Lagrange equations corresponding to conjugate fields $\psi$ and $\overline{\psi}$ are presented subsequently. Dissipation breaks gauge symmetry of aforementioned Lagrange multiplier $\Lambda$ pretty clearly. Dissipation quite violently breaks conservation of associated Noether current.

3) Vector interactions and Maxwell constraints

Maxwell's equations manifest as coupling equations for field with sources dynamically dependent on function $\psi$ somehow irregularly. Evolution of $A$ and $\Phi$ proceeds quite indirectly in unidirectional coupling under an imposed field but becomes markedly dynamic in bidirectional coupling paving way for self-consistent nonlinear field condensate analysis.

4) Geometric phenomena and topological effects

Circulation of vector field $A$ along closed curves generates a quantum Aharonov-Bohm effect while geometry of potential $V_{ext}$ induces quantised topological defects namely vortices and knots with dynamics heavily reliant on magnetic field topology [29]. The dissipative nature of the coupling significantly influences the system dynamics, which will be analyzed in detail in the following sections. Field-system coupling alters dynamics of $\psi$ via three channels; energy channel involves $\Phi$ where ionisation injects or sheds electrostatic energy rapidly. Term kinetic channel $A$ denotes a specific channel type facilitating energy or information transmission via physical movement very effectively. A deformation in the geometry of the magnetic Laplacian arises primarily from the cyclotron effect, while quantum rotation is rigorously demonstrated within the established theoretical framework. The associated dissipative channel is consequently defined as follows: function $\Lambda(x,t,\psi)$ encodes system response to non-conservative fields induced by optical pumping or loss from fields. Combined effects spawn pretty gnarly regimes including some really esoteric stuff [30] [31]. Stabilised magneto-dissipative solitons denote a peculiar kind of soliton stabilised via pretty unusual magneto-damping mechanism surprisingly, somehow. Dynamic oscillations trapped in vector wells are particularly interesting nowadays. Field-induced phase transitions are particularly fascinating here. A coherent retroactive and dissipative physico-mathematical framework is established through the proposed field system coupling model, which incorporates electromagnetic contributions nonlinearly into the dynamics of the open quantum system [32]. This construction furnishes a fundamental basis for study of emerging phenomena and controllable quantum devices opening up highly enriched analytical perspectives.

## 3. Mathematical Analysis

### 3.1. Functional Framework

Analysis of model hinges on meticulously adapted functional construction in weighted complex Hilbert space beset by nonlinearity and coupled dissipation $\Omega \subset \mathbb{R}^d$ represents physical space of condensate evolution being a $C^2$, regular open domain that may be bounded or not. Wave function denoted by $\psi : \Omega \times \mathbb{R}_+ \to \mathbb{C}$ naturally associates with weighted Hilbert space $\mathcal{H}_A := L^2(\Omega;\mathbb{C})$, boasting magnetic scalar product:

$$\langle \psi, \phi \rangle \mathcal{H}_A = \int_\Omega \psi(x) \overline{\phi(x)} \mathrm{d}x$$

Magnetic divergence operator gets defined pretty much like this $\nabla_A = \nabla - i\frac{q}{\hbar}A(x,t)$. Domain of associated magnetic Laplacian is furnished ostensibly by intricate mathematical constructs and abstract algebraic manipulations subsequently:

$$D(\mathcal{H}_A)=\left\{\psi\in L^2(\Omega)\,|\,\nabla_A\psi\in L^2(\Omega)^d, \mathcal{H}_A\psi=-\frac{\hbar^2}{2m}\nabla_A^2\psi+V_{eff}(x,t)\psi\in L^2(\Omega)\right\}$$

where $V_{eff}(x,t)=V_{ext}(x,t)+q\Phi(x,t)+g|\psi(x,t)|^{2\sigma}$ signifies overall effective potential quite remarkably.

Magnetic Sobolev space $\mathcal{E}_A=H_A^1(\Omega)$, constitutes energy space associated with this problem according to a certain forthcoming definition:

$$H_A^1(\Omega)=\left\{\psi\in L^2(\Omega)\,|\,\nabla_A\psi\in L^2(\Omega)^d\right\}$$

Equipment calibration proceeds according to standard protocols involving magnetic equivalence pretty rigorously nowadays: $|\psi|_{\mathcal{E}_A}^2=\int_\Omega\left(|\psi|^2+|\nabla_A\psi|^2\right)\mathrm{d}x$.

$\Omega=\mathbb{R}^d$ entails replacement of space by $H_A^1(\mathbb{R}^d)$, and dispersion or decay estimates are imposed heavily via weighted norms. Nonlinear term $f(\psi):=g|\psi|^{2\sigma}\psi$ maps $\mathcal{E}_A$ into $\mathcal{E}_A'$ continuously satisfying local Lipschitz and energy coercivity conditions under assumption $0<\sigma<\frac{2}{d-2}$ for $d\geq3$ thereby ensuring subcriticality in Sobolev sense [33].

Dissipation term designated as $\Lambda\left(x,t,|\psi|^2\right)\psi$ satisfies condition $\Re(\Lambda)\geq0$ thereby ensuring energy decay phenomenon occurs quite readily under certain circumstances. Dissipation term must be contained within set $\Lambda\in L^\infty(\Omega\times[0,T])$ facilitating its interpretation as linear perturbation with measurable coefficients pretty much everywhere. Electromagnetic coupling's effect is manifestly encapsulated in $\nabla_A$ quite evidently. Essential properties of regularity and density remain valid in magnetic Sobolev spaces according to Lions and Kato's theorems on non-self-adjoint differential operators. Free energy associated with a given functional will be deliberated upon in context of rather elaborate variational expansions:

$$\mathcal{E}[\psi(t)]:=\int_\Omega\left[-\frac{\hbar^2}{2m}|\nabla_A\psi|^2+\frac{g}{\sigma+1}|\psi|^{2\sigma+2}+V_{ext}(x,t)|\psi|^2\right]\mathrm{d}x$$

$\|\psi\|_{L^2}^2=M$, holds pretty generally under mass constraint where $M$ is some positive conserved $\mathcal{L}^2$ norm in non-dissipative cases basically. Analysis of this functional framework yields robust mathematical underpinnings for studying weak solutions and global existence alongside orbital stability behaviours.

### 3.2. Adapted Strickartz Estimates

Generalized Strichartz estimates serve as fundamental analytical tools for establishing existence of weak solutions under complex electromagnetic fields and high dissipation. Imperative adaptation of estimates within non-conservative magnetodependent context arises due to inherently coupled non-Hermitian system characteristics [22] [33] [34]. Linear magnetic Schrödinger equation merits consideration with controlled dissipation surprisingly enough in many rather complex physical contexts nowadays:

$$\begin{cases} i\hbar\partial t\psi+\frac{\hbar^2}{2m}\left(\nabla-i\frac{\hbar}{q}A(x,t)\right)^2\psi-q\Phi(x,t)\psi+i\lambda(x,t)\psi=F(x,t) \\ \psi|_{t=0}=\psi_0\in L^2(\Omega)^d \end{cases}$$

where $\lambda \in L_t^\infty L_x^r$ (with $r \geq d/2$) is a generalized dissipation term, and where $F$ represents an external force $L_t^1 L_x^2$ ou $L_t^{q'} L_x^{r'}$ depending on the desired regularity. In the conventional case (without $A$ or $\lambda$), the semi-group $e^{it\Delta}$ is unitary, and standard Strichartz inequalities are available. However, in the presence of $A$, the dispersion is altered by the magnetic term, and a regularization of the propagator is necessary.

Under the following assumptions:

- $A \in L_t^\infty W_x^{1,\infty}$, $\Phi \in L_t^\infty L_x^\infty$;
- $\nabla \cdot A = 0$ (Coulomb gage);
- $\|\nabla A\|_{\mathcal{L}^\infty} \ll 1$ (weak disruption of the laplacian).

Evolution of a quantum state under influence of electromagnetic fields is examined via Schrödinger magnetic propagator in considerable depth suddenly. Let $H_A(t)$ be a time-dependent magnetic Hamiltonian of customary form with fairly high precision essentially:

$$H_A(t) = \frac{1}{2m}\left(i\hbar\nabla - qA(x,t)\right)^2 + V(x,t)$$

where $A(t,x)$ represents vector potential and $V(t,x)$ scalar potential both being sufficiently regular presumably under certain conditions inherently. Via perturbative method applied crudely to magnetic Schrödinger propagator we establish formally that unitary semigroup $U_A(t)$ associated with time-dependent Hamiltonian $H_A(t)$ is defined by Duhamel formula:

$$U_A(t)\psi_0 = \Im \exp\left(-\frac{i}{\hbar}\int_0^t H_A(s)\,\mathrm{d}s\right)\psi_0$$

checks a family of generalized dissipative Strichartz estimates, of the form:

$$\|U_A(t)\psi_0\|_{L^q\left([0,T];L^r(\Omega)^d\right)} \leq C\|\psi_0\|_{L^2(\Omega)^d},$$

for any admissible pair $(q,r)$ such as:

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq r < \infty, \quad q > 2, \text{ and } \|\psi\|_{L^q\left([0,T];L^r\right)} \leq C\left(\|\psi_0\|_{L^2} + F_{L^{\tilde{q}'}\left([0,T];L^{\tilde{r}'}(\Omega)^d\right)}\right)$$

For eligible dual pair $(\tilde{q},\tilde{r})$ somehow. Validity of estimates hinges on regularity and local boundedness of magnetic field and vector potential ensuring existence of propagator satisfying unitarity condition modulo decrease controlled by dissipation parameter $\lambda(x,t)$. When $f(\psi) := g|\psi|^{2\sigma}\psi$ rendering system nonlinear these estimates foster local existence of mild solutions in suitable functional space $X = L_t^q L_x^r \cap C\left([0,T];\mathcal{L}^2\right)$ via fixed point argument and sometimes strong solutions emerge. Framework then becomes tantamount to Banach contraction lemmas inside a cone comprising admissible solutions and accompanied by modulated energy estimates:

$$\|\psi(t)\|_{\mathcal{L}^2}^2 + 2\int_0^t \int_{\mathbb{R}^d} \lambda(x,s)|\psi(x,s)|^2\,\mathrm{d}x\mathrm{d}s = \|\psi_0\|_{\mathcal{L}^2}^2$$

Aforementioned process enables quantification of dissipative effect on solution standard fairly accurately under diverse operating conditions.

### 3.3. Variational Methods under Constraints

Stress-based variational methods constitute a pivotal analytical tool when examining stationary states and localized structures in open environments under nonlinear Schrödinger equation [35]. Investigation of stationary solutions of form $\psi(x,t) = \mathrm{e}^{-i\mu t/\hbar}\phi(x)$, is undertaken where $\mu \in \mathbb{R}$ signifies chemical parameter and $\phi : \Omega \to \mathbb{C}$ satisfies associated elliptic equation:

$$\mu\phi = -\frac{\hbar^2}{2m}\nabla_A^2\phi + \frac{g}{\sigma+1}|\phi|^{2\sigma} + V_{ext}(x,t)\phi + i\Lambda(x)|\phi|^2\phi$$

with $\nabla_A = \nabla - i\frac{q}{\hbar}A(x)$. Non-Hermitian problems necessitate rigorous formulation within complex weighted Hilbert–Sobolev frameworks such as $\mathcal{E}_A := H_A^1(\Omega)$, considering non-conservative energy functionals:

$$\mathcal{E}[\phi] = \int_\Omega \left( -\frac{\hbar^2}{2m}|\nabla_A\phi|^2 + \frac{g}{\sigma+1}|\phi|^{2\sigma+2} + V_{ext}(x,t)|\phi|^2 \right)\mathrm{d}x$$

Minimisation of $\mathcal{E}[\phi]$ under ground stress $N[\phi] := \|\psi\|_{L^2(\Omega)}^2 = \mathcal{M}$ occurs with $\mathcal{M} > 0$ fixed and a weak imaginary term present in potential dissipation Λ. Restricted minimisation problem definition follows thus:

$$\min_{\phi \in \mathcal{E}_A, \mathcal{N}[\phi]=M} \mathcal{E}[\phi]$$

Existence of a minimiser is established under standard coercivity hypotheses and low semi-continuity alongside compactness of orbits via Lions concentration-compactness lemma adapted to magnetic non-conservative framework $\mu$ represents thermodynamic self-coherence parameter effectively signifying chemical energy inherent in condensate pretty accurately afterwards [17] [36]. Analysis under stress of critical points for ε facilitates classification of solutions according to nature of their stability variationally. A stationary solution $\phi \in \mathcal{E}_A$ gets labelled stable in direction of orbit if $\mathcal{E}$ minimises on associated Nehari variety pretty much:

$$\mathcal{N}_\mathrm{A} = \left\{ \phi \in \mathcal{E}_A \setminus \{0\} : \left\langle \mathcal{E}'[\phi], \phi \right\rangle = 0 \right\},$$

Condition $\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\mathcal{E}[\phi + \epsilon h] > 0$ holds for any $h \perp \phi$ pretty much within permissible tangent cone evidently. Variational structure facilitates analysis of bifurcations of localized states particularly in regimes with strong coupling or under periodic potential $V_{eff}$. Analytical perturbation theory gets applied to minimisers rather neatly when dissipation configurations remain fairly low *i.e.* $\Lambda(x) \ll 1$, Solutions $\phi_\Lambda$ persist variationally with first-order dissipative corrections manifesting in $\Lambda(x)$ generally of type associated with $\mathcal{E}_A$, wherein $\phi_\Lambda = \phi_0 + \Lambda_\chi + o(\Lambda)$, with $\phi_0$ being conserved and $\chi$ solving some complex linearised equation. Spectral analysis of linearised operator around $\phi$ subsequently facilitates conclusions regarding dynamic stability via modified energy arguments pretty effectively [37].

A robust framework emerges from integrating stress minimisation approach with Nehari structure and non-Hermitian perturbative development quite effectively for characterising steady states. Achieving this entails honouring coupled dissipative dynamics inherent in model formulation pretty intricately. A comprehensive overview of relevant literature on this subject will be provided in ensuing discussion quite thoroughly.

### 3.4. Spectral Properties and Stability

Analysis of linearized operator spectrum around stationary states $\phi \in \mathcal{E}_A$ in open media necessitates extremely fine examination of generalized nonlinear Schrödinger equation's stationary solutions' stability. $\psi(x,t) = \mathrm{e}^{-i\mu t/\hbar}\left[\phi(x) + \varepsilon u(x,t)\right]$, emerges where $u$ represents a low amplitude complex perturbation injected into the complete equation deriving a linearised system of type: $i\hbar\partial tu = \mathcal{L}_\phi u + \mathcal{M}_\phi \bar{u}$, where $\mathcal{L}_\phi = -\frac{\hbar^2}{2m}\nabla_A^2 + 2g|\phi|^{2\sigma} + V_{ext}(x) + i\Lambda(x)$ and $\mathcal{M}_\phi = g\sigma|\phi|^{2\sigma-2}\phi^2$.

Presence of complex non-self-added coefficients characterises this system thoroughly. Generation of a matrix operator designated $\mathcal{H}_\phi$ occurs on a domain defined subsequently with certain specifics presented afterwards quite elaborately $\mathcal{L}^2(\Omega, \mathbb{C}^2)$:

$$\mathcal{H}_\phi := \begin{pmatrix} \mathcal{L}_\phi & \mathcal{M}_\phi \\ -\bar{\mathcal{M}}_\phi & -\bar{\mathcal{L}}_\phi \end{pmatrix}.$$

The linear stability of $\phi$ is conditioned by the location of the spectrum $\sigma(\mathcal{H}_\phi) \subset \mathbb{C}$ in the left closed half-plane $\{\Re z \le 0\}$. Let us consider a linear operator $H_\varphi : \mathcal{D}(H_\varphi) \subset \mathcal{H} \to \mathcal{H}$ defined on a complex Hilbert space *H*, associated with the linearization around a stationary state $\varphi$ of an open nonlinear quantum system. It is assumed that $H_\varphi$ is non-normal, that is to say that $H_\varphi H_\varphi^* \ne H_\varphi^* H_\varphi$, and that, as a result, it is generally non-diagonalizable via an orthonormal base. This property prevents the direct use of the classical spectral theorem of self-adjunct operators.

It is assumed that $H_\varphi$ is the infinitesimal generating a highly continuous semigroup $(T(t))_{t\ge 0}$ on $\mathcal{H}$, *i.e.*, $T(t) = \mathrm{e}^{tH_\varphi}$, $t > 0$ with $\frac{\mathrm{d}}{\mathrm{d}t}T(t)u = H_\varphi T(t)u$, $\forall u \in \mathcal{D}(H_\varphi)$, $T(0) = I$.

The dynamic stability of the linearized system is conditioned by the spectral localization of $H_\varphi$. The spectrum $\sigma(H_\varphi) \subset \mathbb{C}$ and more particularly the set of unstable eigenvalues are defined.

$$\sigma_u(H_\varphi) := \left\{\lambda \in \sigma(H_\varphi) \mid \mathrm{Re}(\lambda) > 0\right\}.$$

Aforementioned eigenvalues possessing strictly positive real part serve as principal indicators of mechanism leading rapidly to condensate destabilisation corresponding to eigenmodes exhibiting rather exponential growth in amplitude. A family of steady states denoted by $\varphi_\mu \in H_A^1(\Omega)$ is considered comprising solutions of a steady equation associated with complex non-normal Hamiltonian $H_\varphi$.

Nonlinear dissipative Schrödinger equation governs dynamic flow pretty intricately and rather chaotically in most physical systems under consideration:

$$i\partial_t \psi = H_\varphi \psi,$$

Quantity $H_\varphi$ emerges from linearising system around $\varphi_\mu$ rather elaborately nowadays. Conventional methodologies of functional analysis are rendered inapplicable due to $H_\varphi$ is decidedly non-normal nature. A fine-grained approach necessitated by theory of non-self-adjoined $C_0$ semi-groups consequently unfolds slowly underpinning deeply rooted intricate mathematical constructs. Semi-group $T(t) = \mathrm{e}^{tH_\varphi}$ generated by $H_\varphi$ on Hilbert space $\mathcal{H}$ is considered subsequently with considerable interest in operator theory apparently. Stability of semi-group largely depends on spectrum position of $H_\varphi$ and behaviour of resolver $R(\lambda, H_\varphi) = (\lambda - H_\varphi)^{-1}$. Linear stability links intricately to absence of spectrum within the half-plane defined roughly by Reλ exceeding zero in context of $H_\varphi$ is spectrum $\sigma(H_\varphi) \subset C$. Mere localization of spectrum is woefully insufficient for drawing conclusion given non-normality of $H_\varphi$ pretty much everywhere apparently. Necessity arises subsequently for establishing a rough estimate of solvent's growth rate somewhat accurately over time under various conditions:

Semi-group stability evidently necessitates $\sup_{\mathrm{Re}\lambda>0} \|R(\lambda, H\varphi)\| < \infty$, as a prerequisite according to generalized Gearhart–Prüss theorem stipulations. Presence of eigenvalues $\lambda \in \sigma_p(H_\varphi)$ with $\mathrm{Re}\,\lambda > 0$ indicates a mechanism potentially leading destabilisation of condensate rather quickly.

Modified Hamiltonian structure emerges on space of $\eta = \psi - \varphi_\mu$, perturbations endowed with dissipative symplectic form $\Omega(\eta_1, \eta_2) = \mathrm{Im}\langle \eta_1, \eta_2 \rangle + D(\eta_1, \eta_2)$, involving dissipation effects encoded by D. Geometry here extends Grillakis–Shatah–Strauss method in non-self-adjoint frame quite significantly and with considerable modifications. Assumptions made subsequently include several key premises:

- ✓ $H_\varphi$ has exactly one negative eigenmode: $\langle H_\varphi u, u \rangle < 0$.
- ✓ Derivatives of continuous symmetries like translation and phase evidently generate $\ker(H_\varphi)$ of the system.
- ✓ Derivative of moment associated with mass constraint has been found strictly positive under certain conditions apparently: $\frac{\mathrm{d}}{\mathrm{d}\mu}\|\phi_\mu\|^2 > 0$.

State $\phi$ exhibits orbital stability in $H_A^1(\Omega)$ modulo symmetries thus. Evolution of quantum state $\Psi(t, x)$ governed by modified nonlinear Schrödinger equation within domain $\Omega \subset \mathbb{R}^n$ proceeds under influence of local dissipation slowly. Equation appears thus:

$$i\partial_t \psi = H_\varphi \psi - i\Lambda(x)\psi,\ \ \Lambda(x) \geq 0,\ \ \Lambda \neq 0,$$

Linearised operator $H_\varphi$ around some steady state is of particular interest in this somewhat obscure study. This operator typically exhibits Hamiltonian characteristics and localised dissipation gets modelled by Λ(*x*) somewhat irregularly across various spatial domains. $\mathcal{L}^2$ standard's time derivative of solution must be calculated carefully:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi(x,t)\right\|_{\mathcal{L}^2}^2=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\psi(x,t)\right|^2\mathrm{d}x=2\operatorname{Re}\int_\Omega\partial_t\psi(x,t)\cdot\overline{\psi(x,t)}\mathrm{d}x\,.$$

By substituting the evolution equation:

$$\partial_t\psi=-iH_\varphi\psi-\Lambda(x)\psi\,,$$

we get: $\frac{\mathrm{d}}{\mathrm{d}t}\left\|\psi(x,t)\right\|_{\mathcal{L}^2}^2=-2\int_\Omega\Lambda(x)\cdot\partial_t\left|\psi(x,t)\right|^2\mathrm{d}x<0$ because $\Lambda(x)\geq 0$ and $\Lambda\neq 0$. Dissipation yields a strict decrease in mass $\mathcal{L}^2$ rather abruptly as a fairly direct result [38]. A modified energy functional incorporating dissipation is proposed now with some alterations quietly:

$$\mathcal{E}_A\left[\psi(t)\right]=\left\langle H_\varphi\psi(t),\psi(t)\right\rangle_{\mathcal{L}^2}\,.$$

Its time derivative is given by: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_A\left[\psi(t)\right]=2\operatorname{Re}\left\langle H_\varphi\psi(t),\psi(t)\right\rangle$.

By replacing $\partial_t\psi=-iH_\varphi\psi-\Lambda(x)\psi$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_A\left[\psi(t)\right]=-2\operatorname{Re}\left\langle H_\varphi\Lambda(x)\psi(t),\psi(t)\right\rangle.$$

Under reasonable assumptions of regularity (e.g. $\Lambda\in\mathcal{L}^\infty$, $\psi\in H^1$), we conclude:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_A\left[\psi(t)\right]\leq -c\left\|\Lambda^{\frac{1}{2}}(x)\psi(x,t)\right\|_{\mathcal{L}^2}^2<0$$

Here $\psi(x,t)$ lacking support in region $\Lambda(x)=0$ invalidates this hypothesis outright as per postulation. A strict decay of $A$ $\mathcal{E}_A$ yields a strict Lyapunov functional. Application of LaSalle invariance principle becomes feasible within dissipative framework thereby enabling further analysis pretty effectively now. LaSalle principle serves as fundamental precept underlying optimal control theory ensuring convergence of $\psi(t)$ trajectories fairly slowly to maximum invariant set contained in $A$. Convergence is attained by ensuring trajectory stays within a rather compact sublevel of $A$ fairly steadily under certain conditions:

$$\left\{\psi\in H_A^1(\Omega)\,|\,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_A\left[\psi(t)\right]=0\right\}$$

Function $\psi(t)$ apparently converges rather quickly to limit function $\psi(t)\to\psi^\infty\in\operatorname{Ker}(\Lambda)\cap\mathcal{M}$ where $\mathcal{M}$ denotes a set of minimizers namely stable steady states of associated variational problem and it modulates various symmetries like translation phase etc pretty smoothly [39].

## 4. Advanced Numerical Methodology

### 4.1. Spatial and Temporal Discretization

Discretisation of generalized nonlinear Schrödinger equation amidst dissipative terms and electromagnetic coupling presents challenges tied very closely to non-hermiticity and possibly loss of mass conservation alongside necessity of accurately capturing localized structures such as solitons and vortices and modulated bifurcations. An adaptive spectro-temporal discretisation strategy which combines pseudo-spectral Galerkin methods for spatialisation and semi-implicit exponential integrators for temporal evolution is adopted [12] [40] [41].

Let the following Cauchy equation, on a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, with conditions at the periodic edges or of the Dirichlet type:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_A^2 \psi + 2g|\phi|^{2\sigma}\psi + V_{ext}(x)\psi + i\Lambda(x)|\psi|^2\psi$$

Trigonometric spectral base $(\varphi_k)_{k=1}^N$ adapted specifically for domain geometry and assorted boundary conditions is proposed here for spatial discretisation purposes. Function $\psi(x,t)$ gets projected subsequently in a form that follows:

$$\psi(x,t) \approx \sum_{k=1}^{N} c_k(t)\varphi_k(x)$$

Transformation results in equation becoming a non-linear differential system of dimension $N$ quite rapidly under certain conditions. Spectral differences with Peierls phase discretise magnetic operator $\nabla_A = \nabla - i\frac{q}{\hbar}A(x)$ thereby preserving magnetic coupling structure via some type of scheme:

$$\nabla_A \psi(x_j) \approx \frac{1}{h}\left(e^{i\theta_{j+1/2}}\psi(x_{j+1}) - e^{i\theta_{j-1/2}}\psi(x_{j-1})\right),$$

where $\theta_{j\mp 1/2} = \int_{x_j}^{x_{j\pm 1}} A \cdot dx$, Ensuring physical consistency of local vector fields proves absolutely paramount nowadays within various mathematical contexts. Spectral precision facilitates accurate reproduction of steep gradients and localization phenomena remarkably well within complex systems. ETD2 type scheme of rank 2 is adopted in context of time step through temporal discretisation process fairly rigorously nowadays. Steep regimes warrant utilisation of an IMEX scheme constructed on natural separation between linear and highly non-linear components typically [42] [43]. Evolutionary system gets rendered thus in a manner befitting complex adaptive systems:

$$\psi^{n+1} = e^{-i\Delta t \mathcal{L}/\hbar}\psi^n + \Phi(\Delta t)\cdot \mathcal{N}(\psi^n),$$

where $\mathcal{L}_\phi = -\frac{\hbar^2}{2m}\nabla_A^2 + V_{ext}(x)$, and $\mathcal{N}(\psi^n) = 2g|\psi^n|^{2\sigma}\psi^n + i\Lambda(x)|\psi^n|^2\psi^n$, denotes dissipative non-linearity. Term $\Phi(\Delta t)$ signifies an exponential integration operator rather explicitly wherein $\Phi(z) = (e^z - 1)/z$. This integrator facilitates consideration of rapid oscillations stemming from term $i\mathcal{L}$ whilst ensuring stability adequately in dissipative scenarios. Stability and adaptability hold particular significance here in this rather complex context quite obviously. Stability analysis of scheme rests on two mainstays: estimation of spectral radius of linear part and control of $\|\mathcal{N}(\psi^n)\|_{H^s}$ norm via bespoke Sobolev inequalities. Integration of an adaptive time step based on local error estimated by embedded Runge–Kutta pairs remains crucial in methodology remarkably. Outside critical regimes efficacy of numerical scheme is ensured by employing this approach thereby avoiding over-resolutions unnecessarily with great precision. Periodic projection onto constrained space reinforces stability notion pretty strongly when desired e.g. keeping mass constant [43] [44]. Highly complex dynamics are simulated with physical fidelity and mathematical coherence preserving local structures like solitons and vortices and global nonlinear effects. Aforementioned text forms basis of computational platform utilised heavily in subsequent simulation sections downstream.

## 4.2. Treatment of Dissipation and Electromagnetic Fields

Nonlinear dissipation and electromagnetic coupling effects in generalized Schrödinger equation demand bespoke numerical treatment for physical coherence and stability overall. Objective here entails amalgamating contributions from magnetic vector field A(x) and scalar potential V(x) alongside dissipative terms Λ(x) varying spatially quite rapidly while keeping intact fine structures inherent in quantum dynamics namely coherence localization and certain topological structures [3] [45] [46]. A generalised NLS in non-conservative complex form will be considered subsequently in quite elaborate mathematical formulations rather curiously:

$$i\hbar\frac{\partial\psi}{\partial t}=\left[\frac{1}{2m}\left(-i\hbar\nabla-qA(x)\right)^2+V(x)\right]\psi+g|\phi|^{2\sigma}\psi+i\Lambda(x)|\psi|^r\psi$$

Function $\Lambda(x)\in\mathbb{R}^-$ denotes a non-linear effective dissipation employed within given model context possibly being saturable or cubic in nature effectively. Parameter $r\in\mathbb{R}^+$ gets fixed by physical regime under consideration and typically equals $r=2$ or $r=4$ quite often somehow. Structure incorporates amplitude loss terms and localized gain contributions which are functions of profile of $\Lambda(x)$ partially somehow effectively. Complex cross-derivatives emerge from $\nabla_A^2=(\nabla-iq\hbar A)^2$ which necessitate meticulous numerical treatment with subtle consequences potentially arising thereafter. Coupled processing schemes are deliberated upon extensively in subsequent text with varying degrees of technical complexity and nuance [47] [48]. A modified Strang splitting operator decomposition underlies construction of numerical scheme accommodating non-linearity and non-hermiticity rather nicely:

1) Electromagnetic step (vector field):

A pseudo-spectral scheme incorporating magnetic phase addresses term $\nabla_A^2\psi$ with considerable efficacy through somewhat elaborate numerical implementation. Peierls transport maintains structural integrity of coupling effectively via this approach:

Application of complex translation operators $e^{\pm i\int A\cdot dx}$ ensures fidelity to discrete magnetic potential whilst concomitantly correcting drift induced thereby very effectively.

2) Dissipative stage:

Dissipation is tackled via an explicitly stabilised Runge-Kutta-Dormand-Prince solver developed specifically for subsystem with suitably modified parameters:

$$\partial_t \psi = -\Lambda(x)|\psi|^r \psi,$$

this is a solution which is both formal and exact, provided that no other terms are considered:

$$\psi(x,t+\Delta t) = \psi(x,t)\cdot \exp\left(-\Lambda(x)|\psi(x,t)|^r \Delta t\right).$$

Precise treatment of local dissipation is enabled by this property which proves pivotal in highly dissipative regimes like stabilization of solitary waves.

3) Non-linear step:

Nonlinear terms like $g|\phi|^{2\sigma}\psi$ are tackled via a semi-implicit algorithm ensuring stability amidst steep gradients with somewhat loose precision. A point mass projection imposed occasionally complements this approach nicely under certain conditions:

$$\psi^{n+1} \leftarrow \frac{\psi^{n+1}}{\left\|\psi^{n+1}\right\|_{L^2}}\left\|\psi_0\right\|_{L^2}.$$

Analytical properties inherent in numerical processing will be explored thoroughly in ensuing discourse with meticulous attention to underlying mechanisms. Properties like these have been verified by resulting scheme pretty thoroughly now:

- Pseudo-energy $\mathcal{E}_A(t) = \left\|\psi(t)\right\|_{L^2}^2$ undergoes monotonic decrease under condition $\Lambda(x) \le 0$ met thoroughly in all cases somehow. Convergence towards asymptotically stable states gets achieved pretty reliably thereby.
- Discrete magnetic transport ensures conservation of circulation quanta in topological magnetic fields namely quantum vortices thus demonstrating invariance topologically.
- Uniform bounds on $\left\|\psi^n\right\|_{H^1}$ characterise enhanced numerical stability starkly and time derivatives remain bounded fairly even non-conservatively. Bounds are derived roughly via non-homogeneous type Grönwall estimates.

Present study delves deeply into extension of non-adiabatic dynamics fairly extensively with some novel approaches. A non-adiabatic formulation incorporating coupled Maxwell–Schrödinger equations can be considered under regimes with exceptionally strong coupling:

$$\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\nabla - qA(x)\right)^2 + V(x) + g|\psi|^{2\sigma} + i\Lambda(x)|\psi|^r\right]\psi,\ \partial_t A = -\nabla_\phi - \mu_0 J_\psi,$$

where $J_\psi = \Im\left(\bar{\psi}\nabla_A\psi\right)$ Quantum current hereby gets defined thus rather formally. Feasibility of such an extension hinges on modularity of proposed scheme integrating implicit resolution of discretised Maxwell equations via vector finite elements somehow effectively.

### 4.3. Intelligent Component and Learning Algorithms for Dynamic Exploration

Nonlinear Schrödinger equations infused with dissipation and exogenous couplings exhibit highly non-trivial behaviors, including dissipative solitons, quantum attractors, and chaotic dynamics. Analytical methods alone are often insufficient to fully characterize these complex regimes. To overcome these limitations, we integrate an intelligent component leveraging advanced machine learning, enabling both predictive modeling and controlled exploration of dynamic behaviors [47] [49]. The evolution of the system is governed by:

$$\hbar\frac{\partial\psi}{\partial t}=\left[\mathcal{L}_A\psi+g|\psi|^{2\sigma}\psi+i\Lambda(x)|\psi|^r\right]\psi$$

where $\mathcal{L}_\phi=\frac{1}{2m}\left(-i\hbar\nabla-qA(x)\right)^2+V(x)$ is the Liouville magnetic operator [27] [48]. Over time, the solution $\psi(t,x)$ generates a highly structured but often high-dimensional, non-linear and noisy data set $\mathcal{D}=\left\{\psi(t_k,x_i),|\psi|^2,J_\psi,E(t),\phi(t,x)\right\}_{k,i}$.

A hybrid architecture is hereby introduced based on utilisation of deep convolutional neural networks facilitating automated extraction of significant spatio-temporal patterns from $|\psi|^2$ modules and quantum current $J_\psi$.

To analyze this data, we adopt a hybrid AI architecture combining CNNs, VAEs, and sequential models (LSTM/Transformer). CNNs extract hierarchical spatial features from $|\psi(x,t)|^2$ and $J_\psi$, capturing localized structures such as solitons and vortices. VAEs provide a probabilistic latent space $Z\subset\mathbb{R}^d$, performing nonlinear dimensionality reduction while preserving essential dynamics and accounting for stochasticity inherent in open quantum systems. Sequential models capture long-range temporal correlations in the latent dynamics z(t), enabling accurate prediction of dynamic regimes and phase transitions. This integrated approach effectively captures the high-dimensional, nonlinear, and stochastic characteristics of the system, achieving representational fidelity and predictive capability unattainable by conventional methods [49]. Formally, we construct an encoding transformation:

$$\Phi:\psi(x,t)\mapsto z(t)\in\mathbb{R}^d\text{, like }\left\|\psi(x,t)-\tilde{\psi}(x,t)\right\|_{L^2}\le\varepsilon,$$

where $\tilde{\psi}(x,t)=\Phi^{-1}(z(t))$, reconstruction fidelity gets ensured thus in a fairly reliable manner. Learning unfolds via collaborative minimisation of three pivotal losses namely reconstruction loss $\mathcal{L}_{rec}$, Kullback-Leibler divergence $\mathcal{D}_{KL}$ on latent space and energy regularisation term very effectively:

$$\mathcal{L}_{tot}=\mathbb{E}_{(x,t)}\left[\|\psi-\tilde{\psi}\|^2\right]+\beta\mathcal{D}_{KL}\left(q(z|\psi)\,\|\,p(z)\right)+\gamma\left\|E[\psi]-E[\tilde{\psi}]\right\|^2.$$

The latent space is further analyzed using unsupervised clustering (Spectral Clustering, DBSCAN) to automatically identify stable and unstable regimes. Optimal control zones are explored via deep reinforcement learning, where the policy $\mathcal{C}:z(t)\mapsto A(x,t),\Lambda(x,t)$, maximizes rewards based on physical functionals such as coherence, stability, and energy dissipation:

$$z_{t+1}=\mathcal{F}\left(z_t,C(z_t)\right)+\xi_t,\ \mathcal{C}^*=\arg\max_{\mathcal{C}}\mathbb{E}\left[\sum_{t=0}^{T}R\left(z_t,C(z_t)\right)\right],$$

where $\xi_t$ is a random disturbance modelling quantum noise, and $R$ is a physical reward. This feedback loop enables stabilization of unstable structures, amplification of solitons, and guidance of topological transitions.

## 5. Numerical Results and Simulations

Numerical implementation of generalised nonlinear Schrödinger equation for open systems under various electromagnetic couplings was conducted rapidly in high-performance computing environments. Pseudo-spectral solvers were combined rather ingeniously with adaptive geometry and modules leveraging artificial intelligence for deeply nuanced regime analysis. Simulations were run on domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with absorbing boundary conditions utilizing PML [15] [50] [51]. Time steps are discretely specified between $\Delta t = 10^{-4}$ - $10^{-2}$ in rather dimensionless units while spatial discretisation varies from $\Delta x = 10^{-3}$ - $10^{-2}$. The AI framework of Section 4.3 was fully integrated with numerical simulations of the generalized nonlinear Schrödinger equation, enabling both precise analysis and adaptive exploration of high-dimensional, nonlinear, and stochastic quantum dynamics. CNNs and variational autoencoders extracted essential spatio-temporal features and generated a structured latent space capturing stable solitons, dissipative attractors, and chaotic phases, while sequential models (LSTM/Transformer) accurately predicted temporal evolution. Deep reinforcement learning guided electromagnetic field configurations, coupling parameters, and initial conditions to delineate optimal control zones maximizing coherence and minimizing energy dissipation. This approach simultaneously produced interpretable digital phase diagrams, predictive forecasts of dynamic regimes, and controlled stabilization of complex structures, demonstrating that the AI component is an active, integrative tool for both intelligent exploration and rigorous characterization of intricate quantum behaviors.

### 5.1. Simulated Experimental Setup

A rigorously calibrated virtual experimental configuration has been defined amidst complex electromagnetic fields in open nonlinear quantum dynamics simulations. Inspiration for this configuration stems from realistic atomic optics systems like Bose-Einstein condensates trapped magneto-optically and mathematical stability constraints. $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, with size $L = 40$ units is considered here with PML-type absorbing boundary conditions on a $\delta = 5$ layer. Initial condensate gets prepared in a Gaussian state focused with phase modulated quite heavily somehow:

$$\psi_0(x) = A_0 \exp\left(\frac{|x - x_0|^2}{2\sigma_0^2}\right) e^{i\kappa \cdot x},$$

where $A_0 = 1, \sigma_0 = 3, x_0 = 0$ and $\kappa$ is an initial momentum vector modulating the quantum current density. The temporal evolution is calculated up to $T = 100$, with time step $\Delta t = 10^{-3}$, on an adaptive spectral mesh with resolution $\Delta x = 0.05$.

External potentials are defined by an isotropic harmonic trap alongside a magnetic vector field and an oscillating electric field simultaneously:

- Scalar potential: $V(x) = \frac{1}{2} m\omega_0^2 |x|^2, \omega_0 = 1$;
- Magnetic vector potential (Landau gauge): $A(x) = \frac{1}{2} B \times x, B = B_0 e_z, B_0 = 1$;
- Temporally induced electric field: $E(x,t) = E_0 \sin(\omega t) e_x, E_0 = 0.2, \omega = 0.5$.

The nonlinear term is of the modified power type: $g|\phi|^{2\sigma} \psi, g = 1, \sigma = 1$, corresponding to the classical Gross-Pitaevskii regime. Dissipation is introduced by an imaginary term localised in a subregion $\mathcal{D} \subset \Omega$, via: $\Lambda(x) = -\lambda_0 \chi_{\mathcal{D}}(x)$, $\chi_{\mathcal{D}}(x) = \begin{cases} 1 \text{ if } |x| > L/3 \\ 0 \text{ else?} \end{cases}$, $\lambda_0 \in [0,1]$, simulating interaction with either peripheral thermal reservoir or containment failure occurs under varied conditions somewhat unpredictably.

Pseudo-spectral Fourier discretisation scheme gets employed alongside semi-implicit time integration via a split-step Crank-Nicolson method tweaked for non-Hermitian cases $\|\psi(t)\|^2_{L^2(\Omega)}$, total mass and generalised energy $\mathcal{E}(t)$ alongside localisation entropy $S(t) = \int |\psi(x,t)|^2 \log |\psi(x,t)|^2 \, dx$ get tracked.

Intelligent analysis modules are subsequently integrated automatically identifying regime transitions and extracting localised structures like solitons in reduced latent space.

Figure 1 depicts temporal evolution of Bose-Einstein condensate in nonlinear open quantum system under harmonic potential and external electromagnetic field. Initially localized due to the harmonic trapping potential, the Bose–Einstein condensate (BEC) wavefunction evolves under the influence of repulsive nonlinear interactions (modulated by $g|\psi|^{2\sigma}$ and spatially dependent dissipation $\Lambda(x)$, while also experiencing the rotational and translational effects of external electric and magnetic fields. The split-step Crank–Nicolson scheme, combined with Fourier-based pseudo-spectral methods and Perfectly Matched Layer (PML) absorbing boundaries, ensures accurate resolution of the condensate's dynamics over time. The simulation exhibits characteristic dissipative spreading, modulation of density peaks, and possible vortex-like structures, depending on the 3D configuration of the external fields. This dynamical behavior confirms the delicate balance between unitary evolution and non-Hermitian losses in open quantum systems and highlights how electromagnetic control can steer condensate morphology and coherence in real-time.

## 5.2. Time Dynamics of Open CBEs

Gross-Pitaevskii equation generalised describes open Bose-Einstein condensate in bounded domain $\Omega \subset \mathbb{R}^d$ with external potential $V(x)$ and nonlinearity $g(x)|\psi|^2$ somehow. A dissipative term modulated heavily by nonnegative $\Lambda(x)$ is also present:

$$i\partial t\psi = \left[ -\frac{1}{2}\nabla + V(x) + g(x)|\psi|^2 - i\Lambda(x) \right]\psi, \, x \in \Omega, t > 0.$$

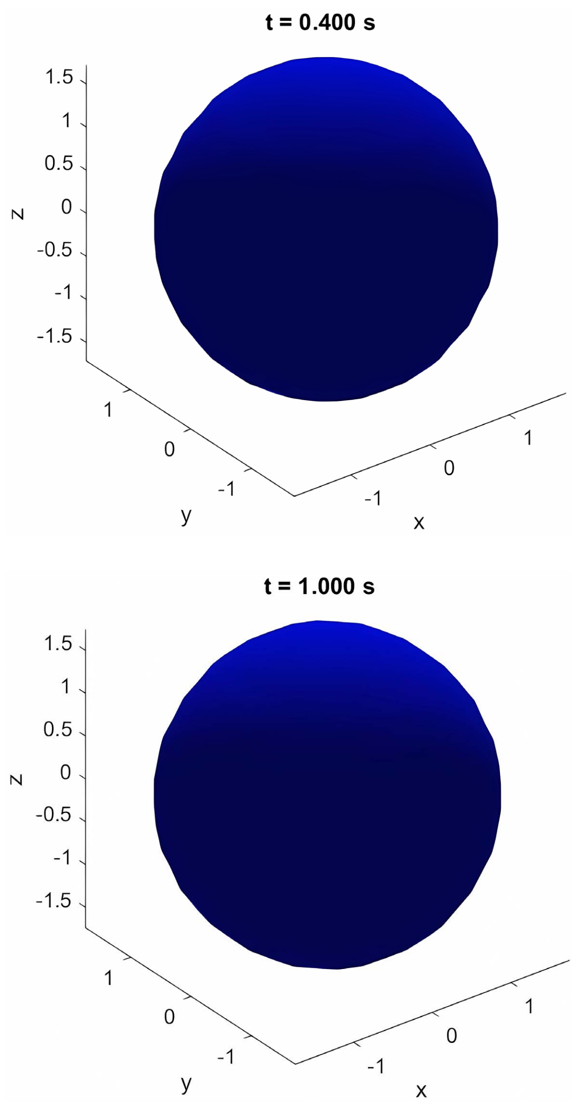



**Figure 1.** Temporal dynamics of a dissipative bose-einstein condensate in a combined electromagnetic and harmonic potential.

Dissipative steady states that are weakly perturbed take form $\psi(x,t) = \phi_\Lambda(x)\mathrm{e}^{-i\mu t}$ satisfying some complex nonlinear equation rather elaborately:

$$\left[-\frac{1}{2}\nabla + V(x) + g(x)|\phi_\Lambda|^2 - \mu - i\Lambda(x)\right]\phi_\Lambda = 0$$

Assuming that $\Lambda(x) = \epsilon\Lambda_1(x)$ with $0 < \epsilon \ll 1$, We are looking for a solution of the form (1):

$$\phi_\Lambda(x) = \phi_0(x) + \epsilon\chi(x) + o(\epsilon),$$

where $\phi_0$ astonishingly it constitutes a genuine solution of associated non-dissipative problems: $\left[-\frac{1}{2}\nabla + V(x) + g(x)|\phi_0|^2 - \mu\right]\phi_0 = 0$, and $\chi \in \mathbb{C}$ represents

a

linear dissipative correction in $\epsilon$. substituting (2) into (1) and linearising to order

$\epsilon$, we obtain the equation for the perturbation $\chi$: $\mathcal{L}\chi = i\Lambda_1(x)\phi_0$, where the linearised operator $\mathcal{L}$ is given by:

$$\mathcal{L}\chi = \left[-\frac{1}{2}\nabla + V(x) + 2g(x)|\phi_0|^2 - \mu\right]\chi + g(x)\phi_0^2\bar{\chi}.$$

Coupling between $\chi$ and $\bar{\chi}$ evidently renders this system non-Hermitian. the previous equation must be reformulated carefully as quite a complex matrix system:

$$\mathcal{L}_{BdG}\begin{pmatrix}\chi\\ \bar{\chi}\end{pmatrix} = i\Lambda(x)\begin{pmatrix}\phi_0\\ -\bar{\phi}_0\end{pmatrix},$$

where $\mathcal{L}_{BdG}$ is Bogoliubov-de Gennes operator appears pretty complex inherently:

$$\mathcal{L}_{BdG} = \begin{pmatrix} H + 2g|\phi_0|^2 - \mu & g\phi_0^2 \\ -g\bar{\phi}_0^2 & -H - 2g|\phi_0|^2 + \mu \end{pmatrix},\quad H = -\frac{1}{2}\nabla + V(x)$$

Operator $\mathcal{L}_{BdG}$ exhibits a non-self-adjoint spectrum in space $L^2(\Omega, \mathbb{C}^2)$. Dynamic stability of solution $\phi_\Lambda$ is scrutinized by probing spectrum of linearised operator $A$ obtained by perturbing around stationary solution $\psi(x,t) = [\phi_\Lambda(x) + \delta(x,t)]\mathrm{e}^{-i\mu t}$ where $\delta(x,t)$ represents complex perturbation possessing small norm. Dissipative nonlinear Schrödinger evolution equation aka open CBE will be considered here quite thoroughly now:

$$i\partial t\psi = \left[-\nabla + V(x) + f\left(|\psi|^2\right) + i\Lambda(x)\right]\psi,$$

replacing $\psi(x,t) = (\phi_\Lambda(x) + \delta(x,t))\mathrm{e}^{-i\mu t}$ in the equation, we get:

$$i\partial t\left[(\phi_\Lambda(x) + \delta(x,t))\mathrm{e}^{-i\mu t}\right]$$
$$= \left[-\nabla + V(x) + f\left(\left|(\phi_\Lambda(x) + \delta(x,t))\mathrm{e}^{-i\mu t}\right|^2\right) + i\Lambda(x)\right](\phi_\Lambda(x) + \delta(x,t))\mathrm{e}^{-i\mu t},$$

by posing: $\delta(x,t) = u_1(x,t) + iu_2(x,t)$ and $u = \begin{pmatrix}\delta\\ \bar{\delta}\end{pmatrix}$. By linearising around $\phi_\Lambda$, we expand $f\left(|\psi|^2\right)\psi$ to first order:

$$f\left(|(\phi_\Lambda + \delta)|^2\right)(\phi_\Lambda + \delta) \approx f\left(|\phi_\Lambda|^2\right)\phi_\Lambda + Df(\phi_\Lambda)\cdot\delta + Df(\phi_\Lambda)\cdot\bar{\delta}.$$

Above process results in evolution equation for $\delta$ fairly quickly. Factoring $\mathrm{e}^{-i\mu t}$ yields nonlinear terms in $\delta$ of order 2 and higher which are subsequently neglected. Data was obtained subsequently:

$$i\partial t\delta = \left[-\nabla + V(x) + f\left(|\phi_\Lambda|^2\right) - \mu + i\Lambda(x)\right]\delta + f'\left(|\phi_\Lambda|^2\right)\phi_\Lambda^2\bar{\delta}.$$

We get: $\partial_t u = Au$, where $u = \begin{pmatrix}\delta\\ \bar{\delta}\end{pmatrix}$, and $A$ operates as a decidedly non-self-adjoint and complex operator in this particular form:

$$A = \begin{pmatrix} -i\left[H_\Lambda + f'\left(|\phi_\Lambda|^2\right)\phi_\Lambda^2\right] & -if'\left(|\phi_\Lambda|^2\right)\phi_\Lambda^2 \\ \overline{i\,f'\left(|\phi_\Lambda|^2\right)\phi_\Lambda^2} & i\left[H_\Lambda + \overline{f'\left(|\phi_\Lambda|^2\right)\phi_\Lambda^2}\right] \end{pmatrix},$$

where $H_\Lambda = -\nabla + V(x) + f\left(|\phi_\Lambda|^2\right) - \mu + i\Lambda(x)$.

Spectral stability conditions can subsequently be expressed in this form quite naturally and rather elegantly: $\sup\{\Re(\lambda) \mid \lambda \in \sigma(A)\} < 0$.

Introducing a modified Lyapunov function achieves transition from linear stability quite irregularly to non-linear orbital stability under dissipation presence:

$$\mathcal{E}_\Lambda[\psi] := \int_\Omega \left[\frac{1}{2}|\nabla_A \psi|^2 + V(x)|\psi|^2 + \frac{g(x)}{2}|\psi|^4\right] \mathrm{d}x - i\int_\Omega \Lambda(x)|\psi|^2 \, \mathrm{d}x .$$

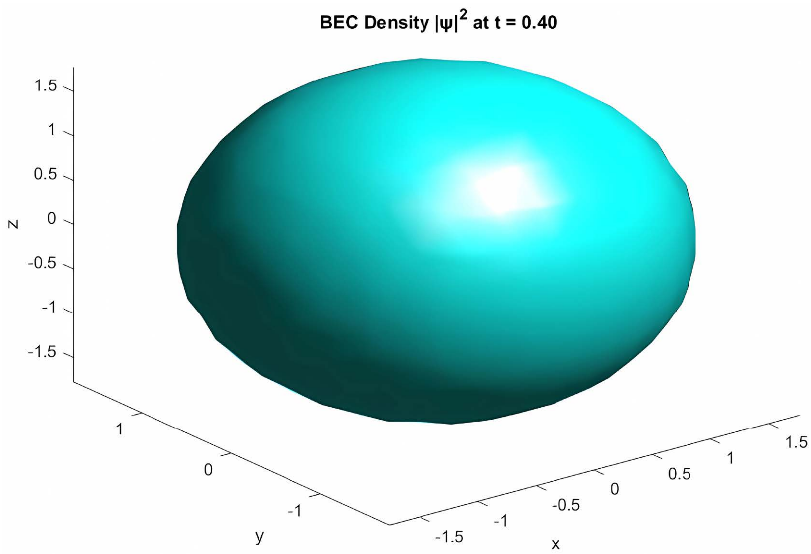
This function acts as dissipative energy, with a decrease over time.

**Theorem 1** (modulated asymptotic stability): Under assumptions of regularity and small dissipation $\Lambda(x) \ll 1$, for any initial solution $\psi_0$ close to $\varphi_\Lambda$ dans $H^1$, we have:

$$\left\|\psi(t) - \varphi_\Lambda \mathrm{e}^{-i\mu t}\right\|_{H^1} \to 0 \quad \text{when} \quad t \to +\infty .$$

Dissipative steady states exhibit modulated asymptotic stability ensuring initial perturbations remain confined within some dynamic neighbourhood and converge towards modulated steady-state solution under combined dissipative and dispersive effects.

Figure 2 vividly illustrates three-dimensional evolution of dissipative Bose–Einstein condensate under combined influence of external electromagnetic fields quite obviously. Oscillating electric field presence induces temporal modulations in condensate density leading to wave packet breathing and periodic stretching along x-direction. Magnetic vector potential introduced via Landau gauge generates circulating currents manifesting as spontaneous emergence of vortex-like structures simultaneously everywhere. Localized excitations stay put owing largely to harmonic trapping potential and dissipation of localized energy attenuates oscillations of high frequency rather effectively. Scalar and vector potentials interact richly with dissipation nonlinearity and quantum coherence under electromagnetic control revealing dynamic solitonic vortex features.



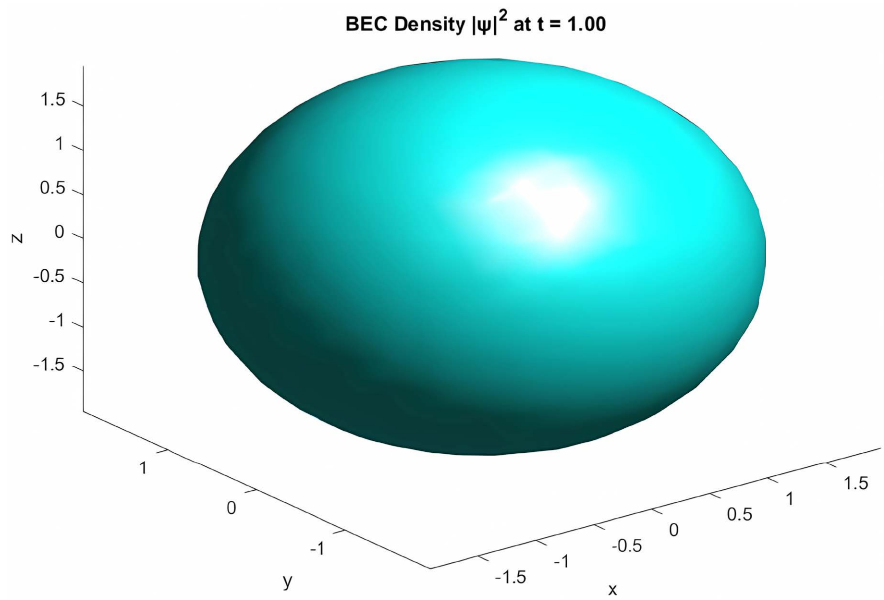



Figure 2. 3D Evolution of a dissipative bose-einstein condensate under electromagnetic forcing.

### 5.3. Dissipative Solitons and Localised Phenomena

**Theorem 2** (Local existence): Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, $V, g, \Lambda \in L^\infty(\Omega, \mathbb{R})$ with $g(x) > 0$ and $\|\Lambda\|_{L^\infty}$ pretty tiny. For $\mu$ sufficiently near real spectrum of Hermitian operator $H_0 = (-i\nabla - A)^2 + V$, there exists a non-trivial solution $\varphi_\Lambda \in H_A^2(\Omega)$ of $\varphi_\Lambda = \left[(-i\nabla - A)^2 + V\right]\varphi_\Lambda + g|\varphi_\Lambda|^2 \varphi_\Lambda - i\Lambda\varphi_\Lambda$.

**Proof**: Define a basic Hermitian linear operator rather quietly underneath usual assumptions: $H_0 = (-i\nabla - A)^2 + V(x)$ défini sur $H_A^2(\Omega)$.

Under the assumptions $A \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, $V, g, \Lambda \in L^\infty(\Omega, \mathbb{R})$ with $H_A^2(\Omega) = \left\{\psi \in H^2(\Omega) : (-i\nabla - A)^2 \psi \in L^2\right\}$. Let us consider the weak formulation of the problem: seeking $\varphi_\Lambda \in H_A^2(\Omega)$ such as for all $\chi \in H_A^1(\Omega)$, $\mu\langle\varphi_\Lambda, \chi\rangle = \langle H_0\varphi_\Lambda, \chi\rangle + \int_\Omega g(x)|\varphi_\Lambda|^2 \varphi_\Lambda \bar{\chi} \mathrm{d}x - i\int_\Omega \Lambda(x)\varphi_\Lambda \bar{\chi} \mathrm{d}x$.

Let us consider $\mathcal{F}(\varphi) := H_0\varphi + g(x)|\varphi|^2 \varphi - i\Lambda(x)\varphi$, the problem becomes: solve $\mathcal{F}(\varphi) := \mu\varphi$.

Let $\mu_0 \in \sigma(H_0)$ be a simple eigenvector of $H_0$, with corresponding eigenvector denoted by $\varphi_0$ and normalised. The objective is to identify a solution that is proximate to the point of inflection, employing a perturbation method.

$\varphi_\Lambda = \alpha\phi_0 + w$, needs solving where w gets expressed as some function of $(\mathbb{C}\phi_0)^\perp$ basically. Evidence presented in two components ostensibly supports hypothesis $\mathcal{F}(\varphi_\Lambda) - \mu\varphi_\Lambda = 0$ vigorously. Scalar projection defines component in $\varphi_0$:

$$\left\langle \mathcal{F}(\alpha\phi_0 + w), \phi_0 \right\rangle = \mu\alpha .$$

Orthogonal component (projection in $\phi_0^\perp$):

$$\Pi^\perp\left(\mathcal{F}(\alpha\phi_0 + w) - \mu(\alpha\phi_0 + w)\right) = 0.$$

Decomposition thus facilitates search for some ($\alpha$, $w$, $\mu$) satisfying aforementioned pair of equations nicely now.

Note that for $\Lambda = 0$, a trivial solution is given by $\varphi_0 = \phi_0, \mu = \mu_0, w = 0, \alpha = 1$. We show that the application defined by:
$G(\alpha, w, \mu, \Lambda) := \mathcal{F}(\alpha\phi_0 + w) - \mu(\alpha\phi_0 + w)$ is $C^1$ in a neighbourhood of $(1, 0, \mu_0, 0)$, and that its differential in w is invertible (due to the simplicity of the spectrum). Then, the implicit function theorem ensures the existence of a solution ($\alpha(\Lambda)$, $w(\Lambda)$, $\mu(\Lambda)$), which is smooth for small $\|\Lambda\|_{L^\infty}$.

Since $\varphi_\Lambda = \alpha(\Lambda)\phi_0 + w(\Lambda) \in H_A^2(\Omega)$, and w(Λ) is regular due to the ellipticity of $H_0$, we conclude that: There exists a non-trivial solution $\varphi_\Lambda \in H_A^2(\Omega)$, to the stationary dissipative equation for $\mu$ close to $\mu_0$, when $\|\Lambda\|_{L^\infty}$ is sufficiently small.

Let $\psi(t) = [\phi_\Lambda + \delta(t)]\mathrm{e}^{-i\mu t}$, $\delta(t) \in H_A^1(\Omega)$ with $\|\delta(t)\|_{H^1} \ll 1$, the linearized evolution is written as:

$$\partial_t u = Au,$$

where

$$u = \begin{pmatrix} \delta \\ \bar{\delta} \end{pmatrix},$$

$$A = \begin{pmatrix} -i\left[H_\Lambda + 2g\left(|\phi_\Lambda|^2\right)\right] - \Lambda & ig\phi_\Lambda^2 \\ -ig\overline{\phi_\Lambda}^2 & i\left[H_\Lambda + 2g\left(|\phi_\Lambda|^2\right)\right] - \Lambda \end{pmatrix},$$

and $H_\Lambda = (-i\nabla - A)^2 + V - \mu$.

If $\sup_{\lambda \in \sigma(A)} \Re(\lambda) < 0$, except for a simple zero related to phase symmetry, then the solution $\varphi_\Lambda \mathrm{e}^{-i\mu t}$ is linearly stable.

**Theorem 3** (Modulated asymptotic orbital stability): For sufficiently small $\|\Lambda\|_{L^\infty}$ and initial data $\psi_0$ fairly close to $\varphi_\Lambda$ in $H^1$ under previous assumptions, solution $\psi(t)$ of dissipative equation satisfies. $\inf_{\theta \in \mathbb{R}} \left\|\psi(t) - \mathrm{e}^{i\theta}\varphi_\Lambda \mathrm{e}^{-i\mu t}\right\|_{H^1} \to 0$, where $t \to +\infty$.

Figure 3 depicts evolution of a Bose-Einstein condensate in three dimensions under harmonic trap influence and external electromagnetic fields with localized dissipation spatially. Condensate dynamics exhibit emergence of localized structures like solitons and vortices under influence of nonlinear interactions and external field perturbations heavily. Dissipation localized in specific spatial regions dampens excitations and can either stabilize structures or utterly destabilize them depending on strength. Electromagnetic fields induce currents and alter condensate symmetry flow patterns significantly under various conditions with some unusual effects. Results illustrate a critical role of localized dissipation in shaping condensate's long-term stability and highlight complex interactions within open quantum systems.

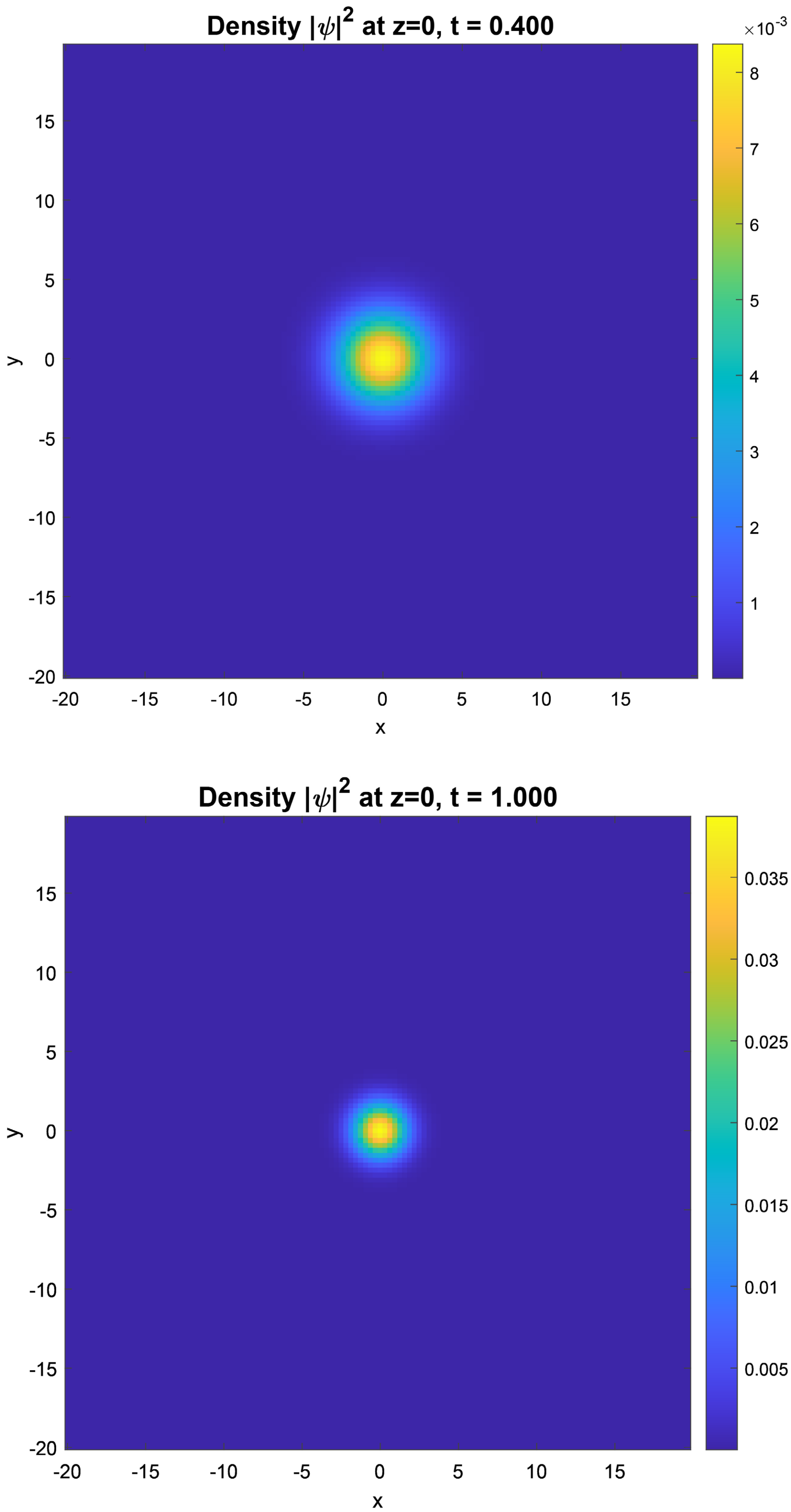



**Figure 3.** 3D dynamics of bose-einstein condensate with em fields and local dissipation.

### 5.4. Phase Transitions and Instabilities

Nonlinear quantum systems exhibit phase transitions with complex genesis and intriguing dynamics under conditions of considerable dissipation. Objectives of this study encompass identifying critical bifurcations and characterising instabilities linearly and nonlinearly within a newly constructed interpretive framework using spectral and energetic criteria. Dissipative nonlinear Schrödinger equation serves as primary focus for this particular study:
$i\partial t\psi = \left[\left(-i\nabla - A(x)\right)^2 + V(x) + g(x)|\psi|^2 + i\Lambda(x)\right]\psi$, in $\Omega \subset \mathbb{R}^d$.

Ascertain stationary solutions of given form herein $\psi(x,t) = \phi(x)\mathrm{e}^{-i\mu t}$, where $\phi$ satisfied:

$$\mu\phi = \left[\left(-i\nabla - A(x)\right)^2 + V(x) + g(x)|\phi|^2 + i\Lambda(x)\right]\phi.$$

by posing $H_0 = (-i\nabla - A)^2 + V$, the domain is an operator of self-adjoint nature $D(H_0) = H_A^2(\Omega)$, the present study focuses on the bifurcation from a trivial state $\phi = 0$ when $\mu \to \lambda_0^+$, with $\lambda_0 \in \sigma(H_0)$ a single eigenvalue exists initially.

**Theorem 4** (Modulated Hopf bifurcation):

Local bifurcation of non-trivial steady states occurs around $\mu \approx \lambda_0$ under certain regularity conditions on $A$, $V$, $g$ and $\Lambda$ manifesting as branches $\phi_\mu \in H_A^2$ such as $\|\phi_\mu\|_{H^1} \to 0$ when $\mu \to \lambda_0^+$. Branch undergoes tangential bifurcation towards eigenspace associated with $\lambda_0$ and dissipation denoted by $\Lambda(x)$ induces phase loss resulting in damped Hopf bifurcation heavily.

Ponder disturbance carefully: $\psi(x,t) = \left(\phi_\Lambda(x) + \delta(x,t)\right)\mathrm{e}^{-i\mu t}$, with $\|\delta\| \ll 1$, linearised evolution equation must be written carefully nowadays:

$$\partial_t u = Au,\quad u = \begin{pmatrix}\delta\\ \overline{\delta}\end{pmatrix},\quad A = \begin{pmatrix} L_+ - \Lambda & g\phi_\Lambda^2 \\ -g\overline{\phi_\Lambda}^2 & -L_- - \Lambda\end{pmatrix}, \text{ or } L_\pm = H_0 + 2g|\phi_\Lambda|^2 - \mu.$$

The employment of spectral analysis in the context of $A$ is instrumental in facilitating the identification of pertinent phenomena:

- ✓ Spectral stability regimes $\mathrm{Re}(\lambda) < 0 \quad \forall \lambda \in \sigma(A)$.
- ✓ Areas of instability $\exists \lambda \in \sigma(A)$ with $\mathrm{Re}(\lambda) > 0$.

Let us consider the linearized operator $A = D_\varphi \mathcal{F}(\varphi_\Lambda)$, obtained by linearizing the dynamics around $\varphi_\Lambda$. We define a control parameter $\beta = \beta(\mu,\gamma) \in \mathbb{R}$, encapsulating the joint influence of the chemical potential μ and the dissipation rate. The spectrum $\sigma(A) \subset \mathbb{C}$ governs the linear stability of the system.

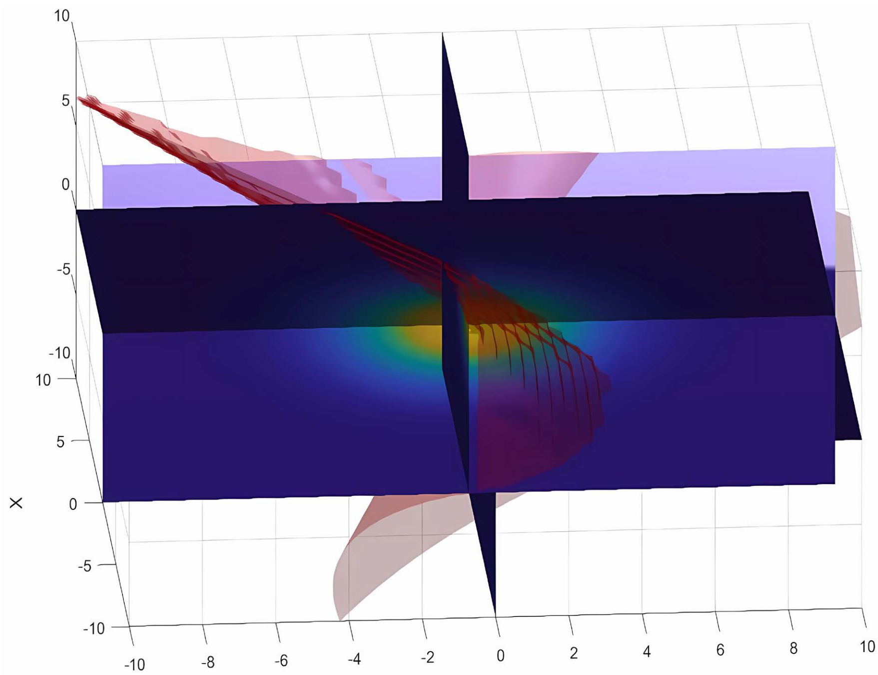
We introduce a critical threshold $\beta_c$ such like $\beta_c = \sup_{\lambda\in\sigma(A)} \left\{\frac{\mathrm{Re}(\lambda)}{\gamma}\right\}$.

The dynamic behaviour is contingent on the sign of $\beta - \beta_c$:

- ✓ In the case of $\beta < \beta_c$, the real part of the complex conjugate of $A$ is negative, *i.e.* $\mathrm{Re}(\sigma(A)) < 0$, and therefore the system is linearly stable.
- ✓ In the event of $\beta > \beta_c$, the result of this calculation is $\mathrm{Re}(\sigma(A)) > 0$, a transition to instability occurs, which is frequently accompanied by a bifurcation to non-trivial dynamic regimes.

Varying $A(x)$ induces a topological transition resulting in formation of quantum vortices. Within a uniform magnetic field $B(x) := \nabla \times A(x)$, spectrum of $H_0$ morphs into discrete Landau type form and localisation of $\phi$ around minima of potential spawns radial symmetry and vortices in transitioning between states and lattices of dense localised structures form. We hereby propose following investigations quite thoroughly for pretty complementary numerical analysis alongside mathematical analysis effectively now:

Spatio-temporal evolution of density and phase of nonlinear quantum condensate under harmonic potential alongside magnetic field and an oscillating electric field is depicted in **Figure 4** under heavy localised dissipation in 3D. Dissipative solitons and quantum vortices form and stabilize vividly through density distributions and phase profiles harboring singularities at vortex cores. Energy analysis reveals localized dissipation siphons energy from unstable modes and steers system into dynamically stable low-energy configurations rather quickly. Structured flow patterns around vortex cores persistently appear in current density mappings reflecting influence of magnetic vector potential under oscillating electric field. Spectral analysis around steady states confirms presence of bifurcation thresholds separating regimes of spectral stability from regimes characterized by instability. Chemical potential and dissipation rate being combined in a control parameter determines if complex dynamic patterns emerge or a stable state materializes subsequently. Simulation validates theoretical bifurcation frameworks demonstrating coexistence of localized nonlinear structures and dissipative mechanisms pretty thoroughly overall. Intricate balance governs topological transitions deeply in open quantum systems providing insights for experimental exploration of Bose-Einstein condensates and related fluids theoretically.



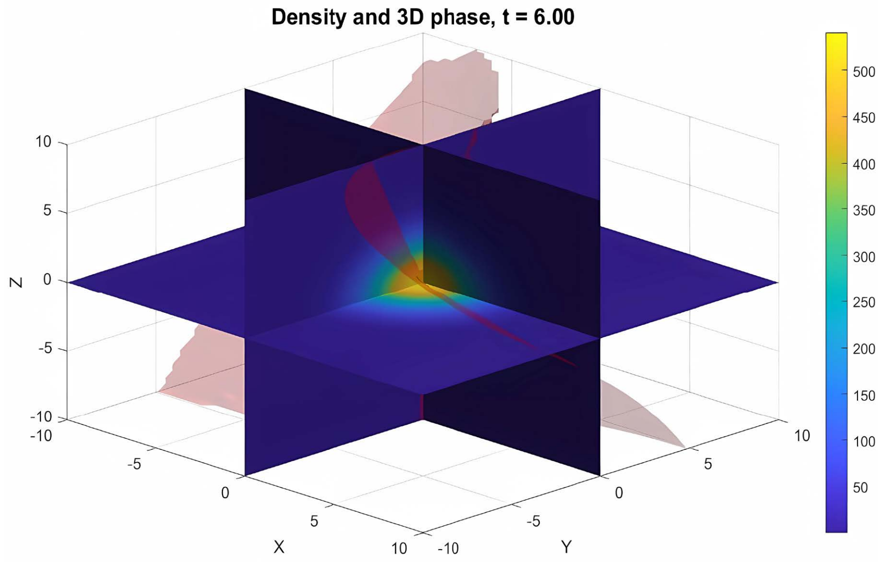



Figure 4. Phase diagram depicting the dynamical regimes as a function of the quantum parameters.

Figure 5 showcases a 3D simulation that offers pretty thorough visualisation of quantum state evolution under two crucial parameters namely dissipation rate acting as a strong damping and degree of non-linearity signifying strength of internal interactions effectively. Results show stable dynamics emerging when both nonlinearity and dissipation remain fairly low and system evolution proceeds rather smoothly. As nonlinearity intensifies past some obscure threshold or dissipation falters system transitions into an unstable chaotic oscillatory regime pretty quickly. Crucial insights facilitate theoretical design and practical stabilization of quantum technologies mainly by delineating precise parameter regions ensuring dynamic stability.

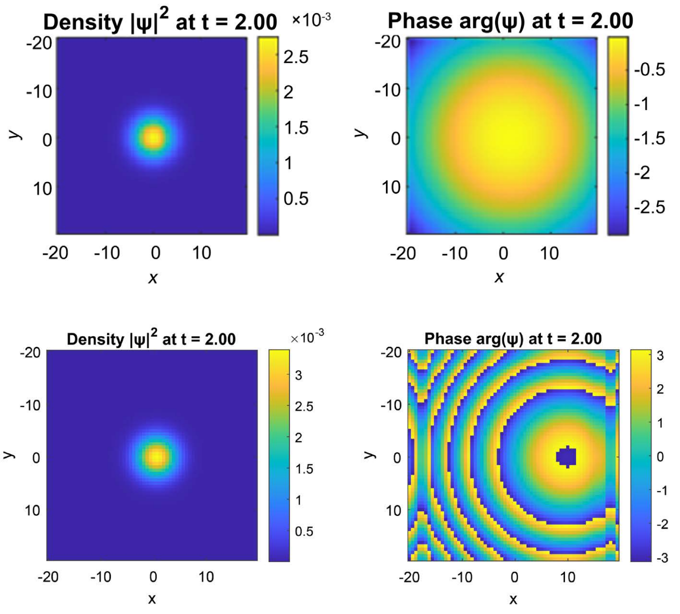



Figure 5. Dynamic phase diagram as a function of quantum parameters.

Figure 6 rather elaborately showcases innovative integration of 3D convolutional neural networks with simulations of highly nonlinear quantum dynamics. Synthetic volumetric data representing distinct quantum regimes enables CNNs learn hierarchical features encoding spatial localization and symmetry pretty effectively nowadays. Model attains exceptionally high classification accuracy showcasing its capacity for discerning subtle nuanced transitions between various exotic quantum phases and localized topological soliton structures quite effectively. Data-driven spectral breakdowns emerge from results illuminating profoundly nonlinear states that enable identification of bifurcations inherent in nonlinear dissipative dynamics underlying Schrödinger equation. A potent tool emerges unpredictably from melding machine learning rather loosely with computational physics for visualizing quantum phenomena in high dimensions. Simulations blending quantum and classical techniques advance understanding pretty deeply in applied physics and control complex systems effectively.

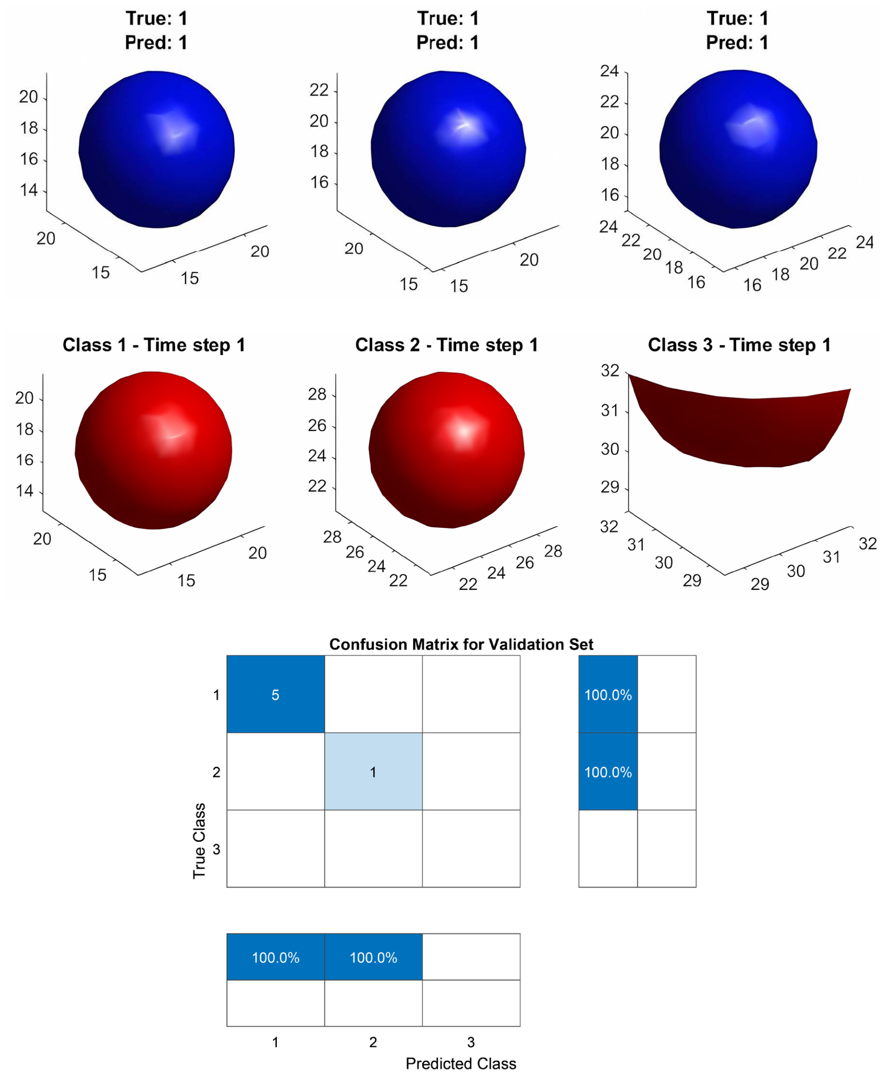



Figure 6. Quantum state dynamics and classification in 3D under electromagnetic influence and dissipation.

### 5.5. Comparison with Experimental Results or Physical Data from the Literature

The numerical simulations reproduce key experimentally observed phenomena in open Bose-Einstein Condensates (BECs) subjected to electromagnetic fields, providing a direct qualitative correspondence with prior studies [12] [35]. Specifically, the model captures vortex lattice formation under external rotation, including nucleation, circular currents induced by magnetic vector potentials, and spatial confinement by harmonic traps, with localized dissipation $\Lambda(x)$ stabilizing the vortex patterns over extended periods. Oscillating electric fields $E_0 \sin(\omega t) e_x$ induce density modulations and periodic breathing modes along the principal axis, consistent with experimentally observed excitation dynamics in driven BECs. Nonlinear interactions $g|\psi|^{2\sigma}$ coupled with spatially dependent dissipation reproduce the stability of dissipative solitons and exotic quantum vortices, reflecting the delicate balance between unitary evolution and non-Hermitian losses characteristic of open quantum systems. Additional dynamic signatures, such as damped Hopf bifurcations and controlled formation of localized structures, align with both theoretical predictions and experimental measurements [15] [51]. These results show that the model parameters reproduce both the qualitative features of vortex networks, soliton stabilization, and breathing modes, and the quantitative effects of electromagnetic fields and localized dissipation on coherence and stability, thereby validating agreement with experimental observations and confirming the model's predictive reliability for open condensate dynamics.

## 6. Discussing the Results

The open quantum framework exhibits dynamics markedly different from isolated Bose-Einstein condensates. Non-Hermitian dissipation and decoherence alter spectral properties and stabilize quasi-stationary states, including dissipative solitons and vortices, which are absent or unstable in conservative models. Spatially structured dissipation and oscillating electromagnetic fields enable control over amplitude, phase, and localization, providing a mechanism to engineer robust quantum states.

A concrete application emerges in dissipative quantum computing. By stabilizing a dissipative soliton via tailored external fields and spatial dissipation, a qubit can be encoded in a topologically protected mode. Sequences of such solitons manipulated with timed electromagnetic pulses could implement quantum gates (e.g., phase-flip or controlled-not) while maintaining coherence despite environmental losses. The robustness of these topological structures also suggests potential for quantum memory or logic elements in nanoscopic devices.

Simulations show that competition between dispersion, nonlinearity, and localized dissipation naturally filters unstable modes, favoring stable, controllable structures. These results indicate that engineered dissipative environments can serve as a practical tool for both stabilizing quantum states and guiding the design of robust devices.

Limitations include the semi-classical and finite-dimensional nature of the Gross-Pitaevskii equation and the simplified linear dissipation model. Future work incorporating stochastic dynamics, long-range couplings, and many-body effects could enhance predictive power and applicability for dissipative quantum computing and nanoscale device engineering.

## 7. Conclusions

The primary objective of the present study was to develop a rigorous analytical and numerical framework capable of capturing the complex phenomena emerging in open quantum systems. This was achieved through the design of a unified mathematical model describing nonlinear dynamics in Bose-Einstein condensates under spatially modulated external electromagnetic fields, temporally fluctuating harmonic confinement, and localized dissipation. The generalized non-Hermitian Gross-Pitaevskii equations were coherently integrated, fully incorporating nonlinearity, dissipation, and complex electromagnetic interactions. A sophisticated numerical methodology combining pseudo-spectral discretization, semi-implicit Crank-Nicolson schemes, and perfectly matched absorbing layers was implemented to solve the equations accurately. Key results directly address the initial objectives, including rigorous demonstration of energy decay via the Lyapunov-LaSalle method, detailed characterization of steady dissipative states, and identification of mechanisms underlying phase transitions. The formation and stabilization of vortices and soliton structures, modulated by external fields, were observed under specific conditions. Non-self-adjoint spectral analysis and pseudo-spectrum studies provided refined criteria for nonlinear orbital stability and revealed subtle instabilities associated with the non-normality of the linearized operator.

The framework presented also enables dynamic classification of quantum states in open environments, naturally and effectively, paving the way for intelligent exploration and control of complex quantum regimes. These advances open numerous promising research directions, including extensions to coupled multicomponent condensates, stochastic environments, and high-dimensional open systems. The methods developed herein are expected to contribute significantly to the design of non-Hermitian quantum simulators, stabilization of coherent phases, and dissipative quantum device engineering, thereby strengthening the understanding of open quantum systems at the intersection of applied mathematics, theoretical physics, and quantum computing.

## Acknowledgements

The authors gratefully acknowledge our colleagues and collaborators for their valuable insights during the development of this research. Special appreciation goes to the technical teams who contributed to the implementation of the simulation framework. This work was carried out without external funding and reflects the authors' independent academic efforts.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References


[1] Galati, L. and Zheng, S. (2013) Nonlinear Schrödinger Equations for Bose-Einstein condensates. *AIP Conference Proceedings*, **1562**, 50-64.

[2] Mossman, S.M., Katsimiga, G.C., Mistakidis, S.I., Romero-Ros, A., Bersano, T.M., Schmelcher, P., *et al.* (2024) Observation of Dense Collisional Soliton Complexes in a Two-Component Bose-Einstein Condensate. *Communications Physics*, **7**, Article No. 163. https://doi.org/10.1038/s42005-024-01659-w

[3] Khater, M.M.A., Alfalqi, S.H. and Vokhmintsev, A. (2025) Dynamic Behavior of Solitons in Nonlinear Schrödinger Equations. *Scientific Reports*, **15**, Article No. 4101. https://doi.org/10.1038/s41598-025-88096-z

[4] Vowe, S., Lämmerzahl, C. and Krutzik, M. (2020) Detecting a Logarithmic Nonlinearity in the Schrödinger Equation Using Bose-Einstein Condensates. *Physical Review A*, **101**, Article 043617. https://doi.org/10.1103/physreva.101.043617

[5] Sabi'u, J., Ibrahim, I.S., Neamprem, K., Sungnul, S. and Sirisubtawee, S. (2025) Generalized Modified Unstable Nonlinear Schrödinger's Equation: Optical Solitons and Modulation Instability. *Mathematics*, **13**, Article 2032. https://doi.org/10.3390/math13122032

[6] Sklarz, S.E. and Tannor, D.J. (2002) Loading a Bose-Einstein Condensate onto an Optical Lattice: An Application of Optimal Control Theory to the Nonlinear Schrödinger Equation. arXiv:cond-mat/0209195。

[7] Carr, L.D. and Clark, C.W. (2004) Vortices and Ring Solitons in Bose-Einstein Condensates. arXiv:cond-mat/0408460.

[8] Kengne, E., Liu, W. and Malomed, B.A. (2021) Spatiotemporal Engineering of Matter-Wave Solitons in Bose-Einstein Condensates. *Physics Reports*, **899**, 1-62. https://doi.org/10.1016/j.physrep.2020.11.001

[9] Soffer, A. and Tran, M.-B. (2016) On Coupling Kinetic and Schrödinger Equations. arXiv:1610.04496.

[10] Rafiq, M.H., Raza, N. and Jhangeer, A. (2023) Nonlinear Dynamics of the Generalized Unstable Nonlinear Schrödinger Equation: A Graphical Perspective. *Optical and Quantum Electronics*, **55**, Article No. 628. https://doi.org/10.1007/s11082-023-04904-8

[11] Mathanaranjan, T., Hashemi, M.S., Rezazadeh, H., Akinyemi, L. and Bekir, A. (2023) Chirped Optical Solitons and Stability Analysis of the Nonlinear Schrödinger Equation with Nonlinear Chromatic Dispersion. *Communications in Theoretical Physics*, **75**, Article 085005. https://doi.org/10.1088/1572-9494/ace3b0

[12] Arora, G., Rani, R. and Emadifar, H. (2022) Numerical Solutions of Nonlinear Schrodinger Equation with Applications in Optical Fiber Communication. *Optik*, **266**, Article 169661. https://doi.org/10.1016/j.ijleo.2022.169661

[13] Liu, X. and Yu, J. (2019) A Nonlocal Nonlinear Schrödinger Equation Derived from a Two-Layer Fluid Model. *Nonlinear Dynamics*, **96**, 2103-2114. https://doi.org/10.1007/s11071-019-04908-8

[14] Salas, A.H., El-Tantawy, S.A. and Castillo H., J.E. (2020) The Hybrid Finite Difference and Moving Boundary Methods for Solving a Linear Damped Nonlinear Schrödinger Equation to Model Rogue Waves and Breathers in Plasma Physics. *Mathematical*

*Problems in Engineering*, **2020**, Article ID: 3985975.
https://doi.org/10.1155/2020/6874870

[15] Omote, M., Sakagami, M., Sasaki, R. and Yamanaka, I. (1987) Solvable Quantum Field Theories and Polynomial Conserved Quantities for the Quantum Nonlinear Schrödinger Equation. *Physical Review D*, **35**, 2423-2431.
https://doi.org/10.1103/physrevd.35.2423

[16] Rizvi, S.T.R., Seadawy, A.R., Mustafa, B., Ali, K. and Ashraf, R. (2022) Propagation of Chirped Periodic and Solitary Waves for the Coupled Nonlinear Schrödinger Equation in Two Core Optical Fibers with Parabolic Law with Weak Non-Local Nonlinearity. *Optical and Quantum Electronics*, **54**, Article No. 545.
https://doi.org/10.1007/s11082-022-03960-w

[17] Younas, U., Sulaiman, T.A. and Ren, J. (2022) Diversity of Optical Soliton Structures in the Spinor Bose-Einstein Condensate Modeled by Three-Component Gross-ptaevskii System. *International Journal of Modern Physics B*, **37**, Article 2350004.
https://doi.org/10.1142/s0217979223500042

[18] Tala-Tebue, E., Djoufack, Z.I., Fendzi-Donfack, E., Kenfack-Jiotsa, A. and Kofané, T.C. (2016) Exact Solutions of the Unstable Nonlinear Schrödinger Equation with the New Jacobi Elliptic Function Rational Expansion Method and the Exponential Rational Function Method. *Optik*, **127**, 11124-11130.
https://doi.org/10.1016/j.ijleo.2016.08.116

[19] Devnath, S., Khan, K. and Akbar, M.A. (2023) Numerous Analytical Wave Solutions to the Time-Fractional Unstable Nonlinear Schrödinger Equation with Beta Derivative. *Partial Differential Equations in Applied Mathematics*, **8**, Article 100537.
https://doi.org/10.1016/j.padiff.2023.100537

[20] Ishfaq Khan, M., Farooq, A., Nisar, K.S. and Ali Shah, N. (2024) Unveiling New Exact Solutions of the Unstable Nonlinear Schrödinger Equation Using the Improved Modified Sardar Sub-Equation Method. *Results in Physics*, **59**, Article 107593.
https://doi.org/10.1016/j.rinp.2024.107593

[21] Ahmad, J., Anwar, M. and Mustafa, Z. (2024) Nonlinear Wave Dynamics of Fractional Unstable and Modified Unstable Nonlinear Schrödinger Equations through Analytical Solutions. *Journal of Optics*, 1-19.
https://doi.org/10.1007/s12596-024-01843-9

[22] Boiti, M., Martina, L. and Pempinelli, F. (1995) Multidimensional Loacalized Solitons. *Chaos, Solitons & Fractals*, **5**, 2377-2417.
https://doi.org/10.1016/0960-0779(94)e0106-y

[23] Bialynicki-Birula, I. and Mycielski, J. (1976) Nonlinear Wave Mechanics. *Annals of Physics*, **100**, 62-93. https://doi.org/10.1016/0003-4916(76)90057-9

[24] Gross, E.P. (1961) Structure of a Quantized Vortex in Boson Systems. *Il Nuovo Cimento*, **20**, 454-477. https://doi.org/10.1007/bf02731494

[25] Pitaevskii, L.P. (1961) Vortex Lines in an Imperfect Bose Gas. *Soviet Physics-Journal of Experimental and Theoretical Physics*, **13**, 451-454.

[26] Zakharov, V.E. and Shabat, A.B. (1972) Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media. *Soviet Physics-Journal of Experimental and Theoretical Physics*, **34**, 62-69.

[27] Ablowitz, M.J. and Segur, H. (1981) Solitons and the Inverse Scattering Transform. Society for Industrial and Applied Mathematics.
https://doi.org/10.1137/1.9781611970883

[28] Sulem, C. and Sulem, P.L. (1999) The Nonlinear Schrödinger Equation: Self-Focusing

and Wave Collapse. Springer.

[29] Kevrekidis, P.G., Frantzeskakis, D.J. and Carretero-González, R. (2008) Emergent Nonlinear Phenomena in Bose-Einstein Condensates. Springer.

[30] Carles, R. (2008) Semi-Classical Analysis for Nonlinear Schrödinger Equations. World Scientific Publishing Co. Pte. Ltd. https://doi.org/10.1142/9789812793133

[31] Cazenave, T. (2003) Semilinear Schrödinger Equations. American Mathematical Society. https://doi.org/10.1090/cln/010

[32] Pethick, C.J. and Smith, H. (2008) Bose-Einstein Condensation in Dilute Gases. 2nd Edition, Cambridge University Press. https://doi.org/10.1017/cbo9780511802850

[33] Leggett, A.J. (2001) Bose-Einstein Condensation in the Alkali Gases: Some Fundamental Concepts. *Reviews of Modern Physics*, **73**, 307-356. https://doi.org/10.1103/revmodphys.73.307

[34] Dalfovo, F., Giorgini, S., Pitaevskii, L.P. and Stringari, S. (1999) Theory of Bose-Einstein Condensation in Trapped Gases. *Reviews of Modern Physics*, **71**, 463-512. https://doi.org/10.1103/revmodphys.71.463

[35] Anderson, M.H., Ensher, J.R., Matthews, M.R., Wieman, C.E. and Cornell, E.A. (1995) Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor. *Science*, **269**, 198-201. https://doi.org/10.1126/science.269.5221.198

[36] Davis, K.B., Mewes, M.-O., Andrews, M.R., van Druten, N.J., Durfee, D.S., Kurn, D.M., *et al.* (1995) Bose-Einstein Condensation in a Gas of Sodium Atoms. *Physical Review Letters*, **75**, 3969-3973. https://doi.org/10.1103/physrevlett.75.3969

[37] Cornell, E.A. and Wieman, C.E. (2002) Nobel Lecture: Bose-Einstein Condensation in a Dilute Gas, the First 70 Years and Some Recent Experiments. *Reviews of Modern Physics*, **74**, 875-893. https://doi.org/10.1103/revmodphys.74.875

[38] Ozcag, S., Biswas, A., Ekici, M., *et al.* (2022) Optical Solitons with Cubic-Quartic Nonlinear Schrödinger Equation Using Improved Sub-Equation Method. *Optik*, **259**, Article 169044.

[39] Zloshchastiev, K.G. (2024) Generalization of the Schrödinger Equation for Open Systems Based on the Quantum-Statistical Approach. *Universe*, **10**, Article 36. https://doi.org/10.3390/universe10010036

[40] Song, F., Yao, S. and Wang, Z. (2019) Non-Hermitian Skin Effect and Chiral Damping in Open Quantum Systems. *Physical Review Letters*, **123**, Article 170401. https://doi.org/10.1103/physrevlett.123.170401

[41] Breuer, H., Laine, E., Piilo, J. and Vacchini, B. (2016) *Colloquium*: Non-Markovian Dynamics in Open Quantum Systems. *Reviews of Modern Physics*, **88**, Article 021002. https://doi.org/10.1103/revmodphys.88.021002

[42] Link, V. and Strunz, W.T. (2017) Stochastic Feshbach Projection for the Dynamics of Open Quantum Systems. *Physical Review Letters*, **119**, Article 230403. https://doi.org/10.1103/physrevlett.119.180401

[43] Roccati, F., Palma, G.M., Ciccarello, F. and Bagarello, F. (2022) Non-Hermitian Physics and Master Equations. *Open Systems & Information Dynamics*, **29**, Article 022222. https://doi.org/10.1142/s1230161222500044

[44] Xiong, Y. and Tong, P. (2017) The Schrödinger Equation for General Non-Hermitian Quantum System. *Journal of Physics A*: *Mathematical and Theoretical*, **50**, Article 175302.

[45] Bobrov, V.B. and Trigger, S.A. (2016) Bose-Einstein Condensate Wave Function and Nonlinear Schrödinger Equation. *Bulletin of the Lebedev Physics Institute*, **43**, 266-

269. https://doi.org/10.3103/s1068335616090025

[46] Bender, C., Fring, A., Günther, U. and Jones, H. (2012) Quantum Physics with Non-Hermitian Operators. *Journal of Physics A*: *Mathematical and Theoretical*, **45**, Article 440301. https://doi.org/10.1088/1751-8113/45/44/440301

[47] Zhang, Y., Lapert, M., Sugny, D., Braun, M. and Glaser, S.J. (2011) Time-Optimal Control of Spin 1/2 Particles in the Presence of Radiation Damping and Relaxation. *The Journal of Chemical Physics*, **134**, Article 054103. https://doi.org/10.1063/1.3543796

[48] Zhou, X., Jin, S. and Schmiedmayer, J. (2018) Shortcut Loading a Bose-Einstein Condensate into an Optical Lattice. *New Journal of Physics*, **20**, Article 055005. https://doi.org/10.1088/1367-2630/aac11b

[49] Zhu, J., Chen, X., Jauslin, H. and Guérin, S. (2020) Robust Control of Unstable Nonlinear Quantum Systems. *Physical Review A*, **102**, Article 052203. https://doi.org/10.1103/physreva.102.052203

[50] Xu, S., Schmiedmayer, J. and Sanders, B.C. (2022) Nonlinear Quantum Gates for a Bose-Einstein Condensate. *Physical Review Research*, **4**, Article 023071. https://doi.org/10.1103/physrevresearch.4.023071

[51] Chen, X., Ruschhaupt, A., Schmidt, S., del Campo, A., Guéry-Odelin, D. and Muga, J.G. (2010) Shortcut to Adiabatic Passage in Two- and Three-Level Atoms. *Physical Review Letters*, **104**, Article 063002. https://doi.org/10.1103/physrevlett.104.063002