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A New Updating Method for the Undamped Vibroacoustic System with No Spill-Over

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Abstract

In this paper, we consider the model updating problem of the undamped vibroacoustic system with no spill-over (MUP-UVA), which is to update the original system to a new system such that some "unstable" eigenvalues are replaced by some newly measured ones. Based on the spectral decomposition of the undamped vibroacoustic system, a necessary and sufficient condition is derived such that the updated system can preserve no spill-over, and a set of parametric solutions of MUP-UVA is characterized. Furthermore, a gradient optimization algorithm for the minimum norm solution of MUP-UVA is proposed and the performance of the algorithm is illustrated by several numerical experiments.

Keywords

Undamped Vibroacoustic System, Model Updating, No Spill-Over

1. Introduction

It is well known that the Finite Element Model (FEM) can be used to predict the dynamic characteristics of the engineering structures. In practice, the structural dynamic mathematical models are usually very large and sparse, and only a small number of natural frequencies (eigenvalues) and model shapes (eigenvectors) can be experimentally measured from a realized actual structure. Comparing with the corresponding ones from the FEM of these structures, there may be inconsistencies between these two sets of eigenpairs, *i.e.*, the FEM may be inaccurate. Therefore, the existing FEM should be updated with minimal changes, so that the updated model can accurately predict the dynamic characteristics of the structure. On the other hand, the method of Model Updating (MUP) can also be used to eliminate resonance. As is known, if frequencies of the system excited are the same as or close to the natural frequencies of a certain order, the vibrations of the system may be significantly amplified and resonance occurs, which may cause very large damage. Passive control [1] and active control [2] are two classical methods to deal with resonance problems. Active control relies on feedback control through actuators and sensors, while passive control, which is also known as model updating, updates the physical parameters of the original system so that some "unstable" natural frequencies are replaced by newly measured or expected ones. For large structures, there are usually only a few natural frequencies that need to be reassigned, due to the limited number and frequency range of measured coordinates. In order to ensure the stability of the structure, the remaining unmeasured natural frequencies and model shapes should be kept unchanged. In other words, it is required to keep the unchanged eigenvalues and their related eigenvectors unaffected by model updating when the system is adjusted and modified, which is called no spill-over property.

In the past decades, various techniques for MUP by the measured eigendata have been discussed, for example, Friswell [3] [4], Kuo [5] [6], Chu [7] [8], and Moreno [9]. The main purpose of MUP is to characterize the coefficient matrices of FEM by some prescribed eigenpairs, and its main challenge is how to preserve the no spill-over property and the structures of system matrices (such as symmetry, positive definiteness and sparsity, and so on) simultaneously. Chu *et al.* [10] investigated the spill-over phenomenon in the MUP. They showed that the MUP with no spill-over is possible in undamped models. Mao [11] considered the MUP with no spill-over where the mass matrix and stiffness matrices are updated simultaneously, and the matrices preserve positive definiteness. For the damped vibration system, Chu [8] provided some sufficient and necessary conditions that the MUP with no spill-over is solvable with mass and stiffness matrices being positive definite. With the spectral decomposition of some matrix polynomials, the solutions of the MUP with no spill-over were characterized for the undamped piezoelectric smart structure system [12] and the $\star$-palindromic quadratic system [13]. Recently, analytical expressions of structure preserving no spill-over updating were determined for some specific structured matrix pencils, including symmetric, Hermitian, $\star$-even and $\star$-odd [14]. They first consider a quadratic inverse eigenvalues problem where the no spill-over property and the $\star$-symmetric structures of coefficient matrices are preserved, simultaneously. Therefore, MUP with no spill-over of some structured vibration systems remains a fundamental challenge in the field.

By the finite element technique, the motion for the structure interacting with the enclosed acoustic medium of the undamped vibroacoustic systems can be expressed as the following differential equation [15]

$$M\ddot{v}(t)+Kv(t)=f(t), \tag{1}$$

where

$$M=\begin{bmatrix} M_A & C_{AS} \\ 0 & M_S \end{bmatrix}, K=\begin{bmatrix} K_A & 0 \\ C_{SA} & K_S \end{bmatrix}, x(t)=\begin{bmatrix} x_A \\ x_S \end{bmatrix} \tag{2}$$

with $n = n_A + n_S$, $M_A, K_A \in \mathbb{R}^{n_A \times n_A}$ and $M_S, K_S \in \mathbb{R}^{n_S \times n_S}$ being symmetric, and $C_{AS} = -C_{SA}^{\mathrm{T}}$. In (1), the finite elements of the structure and the cavity are coupled together, and their dynamic response affects each other, characterized in the acoustic-structural and structural-acoustic coupling matrices C_{AS} and C_{SA}, M_A and M_S are referred to as the acoustic and structural mass matrices, while K_A and K_S are acoustic and structural stiffness matrices. Let $v(t) = e^{\omega t} x$ and $\lambda = \omega^2$, the vibration of the undamped vibroacoustic model (1) can be characterized by eigenvalues and eigenvectors of the following generalized eigenvalue problem

$$P(\lambda) x := (\lambda M + K) x = 0, \tag{3}$$

As is known, if M is nonsingular, then $P(\lambda)$ has n eigenvalues. In this paper, we consider the MUP of the undamped vibroacoustic system with no spill-over, which can be stated as follows:

Problem (**MUP-UVA**): Given an analytical model (M, K), a set of its eigenpairs $\{\lambda_i, x_i\}_{i=1}^{p}$ $(1 \le p < n)$, and a set of measured eigenpairs $\{\mu_i, y_i\}_{i=1}^{p}$, where both λ_i and μ_i are closed under complex conjugate, find $\tilde{M}_A, \tilde{K}_A \in \mathbb{R}^{n_A \times n_A}$ and $\tilde{M}_S, \tilde{K}_S \in \mathbb{R}^{n_S \times n_S}$ such that

- The eigenpairs $\{\lambda_i, x_i\}_{i=1}^{p}$ of original system are replaced by $\{\mu_i, y_i\}_{i=1}^{p}$ in the updated system.
- The remaining $n - p$ unknown eigenpairs of the updated system are the same as those of the original system.
- The acousti–structural coupling matrices C_{AS} and C_{SA} are kept unchanged.

With the assumption that the structural-acoustic coupling matrix being accurately known, Modak [15] provided a direct method of updating an undamped vibroacoustic system, in which the symmetry properties of mass matrices M_A, M_S and stiffness matrices K_A, K_S are preserved. However, they didn't consider the spill-over phenomenon. With the spectral decomposition of a structured quadratic asymmetric pencil, Zhao [16] provided a set of parametric solutions of the MUP with no spill-over for the damped vibroacoustic system. And their method did not need the constraints that the prescribed eigenvectors should span the same subspaces as the original ones. However, these methods cannot be used to solve cases in which the vibroacoustic system has repeated eigenvalues. It is well known that a defective eigenvalue whose geometric multiplicity is less than its algebraic multiplicity, is usually more sensitive to perturbations than a semi-simple eigenvalue. Recently, Guo [17] considered the robust assignment for the repeated poles. Zhao [18] provided a sufficient solvable condition for the MUP with no spill-over for the quadratic asymmetric vibration system. To the best of our knowledge, the MUP with no spill-over for the undamped vibroacoustic system remains open in the following three cases: 1) the system has repeated eigenvalues; 2) the prescribed eigenvectors are chosen arbitrarily; and 3) the sufficient and necessary conditions that the updated system preserves no spill-over.

In this paper, we proposed a new updating method for the undamped vibroacoustic system with no spill-over. The main contributions of this paper are:

1) A sufficient and necessary condition that the updated undamped vibroacoustic system can preserve no spill-over is provided.

2) The parametric solutions of MUP-UVA are characterized by the system matrices and some prescribed eigenpairs without the assumptions that all the eigenvalues of original system and the newly measured eigenvalues are simple.

3) A gradient optimization algorithm for the minimum norm solution of the MUP-UVA is proposed.

2. Sufficient and Necessary Condition for No Spill-Over

Assume that all distinct eigenvalues of $P(\lambda)$ are λ_1, $\bar{\lambda}_1$, $\cdots$, λ_l, $\bar{\lambda}_l$, λ_{2l+1}, $\cdots$, λ_k, where $\lambda_j = \alpha_j + i\beta_j$, $j = 1,\cdots,l$ and $\lambda_j \in \mathbb{R}$, $j = 2l+1,\cdots,k$, each of which has algebraic multiplicity n_j, *i.e.*, $2n_1 + \cdots + 2n_l + n_{(2l+1)} + \cdots + n_k = n_A + n_S \equiv n$. Let

$$X := \left[X_{1R}, X_{1I}, \cdots, X_{lR}, X_{lI}, X_{2l+1}, \cdots, X_k\right] \in \mathbb{R}^{n\times n}, \tag{4}$$

$$J := \mathrm{diag}\left(J_R(\lambda_1), \cdots, J_R(\lambda_l), J\left(\lambda_{(2l+1)}\right), \cdots, J(\lambda_k)\right) \in \mathbb{R}^{n\times n}, \tag{5}$$

where $J_R(\lambda_j) \in \mathbb{R}^{2n_j\times 2n_j}$ and $J(\lambda_j) \in \mathbb{R}^{n_j\times n_j}$ are Jordan canonical forms of λ_j for $j = 1,\cdots,l$ and $j = 2l+1,\cdots,k$, respectively, and X is the matrix of right (generalized) eigenvectors corresponding to J. Clearly, (J, X) is a Jordan pair [19] of $P(\lambda)$ if and only if X is nonsingular and the following matrix equation holds

$$MXJ + KX = 0. \tag{6}$$

Partition X and J as

$$X := [X_1, X_2],\ J = \mathrm{diag}(\Lambda_1, \Lambda_2), \tag{7}$$

where $X_1 \in \mathbb{R}^{n\times p}$, $\Lambda_1 \in \mathbb{R}^{p\times p}$ are the representation of the eigenparis $\{\lambda_j, x_j\}_{j=1}^{p}$ which are to be reassigned. Let $(\Sigma_1, Y) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ be the real representation of the prescribed eigenpairs $\{\mu_j, y_j\}_{j=1}^{p}$.

With the notations above, the MUP-UVA can be mathematically reformulated as follows: given $M, K \in \mathbb{R}^{n\times n}$ with M, K being nonsingular, $\Lambda_1, \Sigma_1 \in \mathbb{R}^{p\times p}$, and $X_1, Y \in \mathbb{R}^{n\times p}$, find $\Delta\mathcal{M} := \mathrm{diag}(\Delta M_A, \Delta M_S)$ and $\Delta\mathcal{K} := \mathrm{diag}(\Delta K_A, \Delta K_S)$ such that

$$(M_A + \Delta M_A)Y_A\Sigma_1 + (K_A + \Delta K_A)Y_A + C_{AS}Y_S\Sigma_1 = 0, \tag{8}$$

$$(M_S + \Delta M_S)Y_S\Sigma_1 + (K_S + \Delta K_S)Y_S + C_{SA}Y_A = 0, \tag{9}$$

$$\tilde{M}X_2\Lambda_2 + \tilde{K}X_2 = 0, \tag{10}$$

where $\tilde{M} = M + \Delta\mathcal{M}$, $\tilde{K} = K + \Delta\mathcal{K}$ and $Y := \begin{bmatrix} Y_A \\ Y_S \end{bmatrix}$ with $Y_A \in \mathbb{R}^{n_A\times p}$ and $Y_S \in \mathbb{R}^{n_S\times p}$.

From (6), we can see that (10) can be equivalently rewritten as

$$\begin{cases} \Delta M_A X_{2A}\Lambda_2 + \Delta K_A X_{2A} = 0, \\ \Delta M_S X_{2S}\Lambda_2 + \Delta K_S X_{2S} = 0, \end{cases} \tag{11}$$

which is the no spill-over property.

Lemma 1. [20] Let X, $J \in \mathbb{R}^{n\times n}$ be defined by (4) and (5), respectively. Partition X as $X = \begin{bmatrix} X_A \\ X_S \end{bmatrix}$, $X_A \in \mathbb{R}^{n_A\times n}$, $X_S \in \mathbb{R}^{n_S\times n}$. There exist nonsingular matrices M and K of the form (2) such that (6) holds, if and only if $\begin{bmatrix} X_A \\ -X_S J \end{bmatrix}$ is nonsingular and there exists a nonsingular matrix $\Gamma \in \mathbb{SR}^{n\times n}$ satisfying

$$\Gamma J^{\mathrm{T}} = J\Gamma, \ X_A \Gamma X_S^{\mathrm{T}} = 0. \tag{12}$$

And in this case, the matrices M and K can be expressed as

$$M_A = \left(X_A \Gamma X_A^{\mathrm{T}}\right)^{-1}, \ M_S = -\left(X_S \Gamma J^{\mathrm{T}} X_s^{\mathrm{T}}\right)^{-1}, \tag{13}$$

$$K_A = -\left(X_A \Gamma J^{-\mathrm{T}} X_A^{\mathrm{T}}\right)^{-1}, \ K_S = \left(X_S \Gamma X_S^{\mathrm{T}}\right)^{-1}, \tag{14}$$

$$C_{AS} = -\left(X_A \Gamma X_A^{\mathrm{T}}\right)^{-1} X_A \Gamma J^{\mathrm{T}} X_S^{\mathrm{T}} \left(X_S \Gamma J^{\mathrm{T}} X_s^{\mathrm{T}}\right)^{-1}, \tag{15}$$

$$C_{SA}^{\mathrm{T}} = -\left(X_A \Gamma J^{-\mathrm{T}} X_A^{\mathrm{T}}\right)^{-1} X_A \Gamma J^{\mathrm{T}} X_S^{\mathrm{T}} \left(X_S \Gamma X_S^{\mathrm{T}}\right)^{-1}, \tag{16}$$

and it holds that $C_{AS} = -C_{SA}^{\mathrm{T}}$.

Theorem 2. Suppose that (J, X) defined by (4) and (5) is a Jordan pair of $P(\lambda)$. Let

$$\Gamma_1 = \left(\begin{bmatrix} X_{1A}^{\mathrm{T}} & -\Lambda_1^{\mathrm{T}} X_{1S}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} M_A & C_{AS} \\ 0 & M_S \end{bmatrix} \begin{bmatrix} X_{1A} \\ X_{1S} \end{bmatrix} \right)^{-1}. \tag{17}$$

Then the MUP-UVA can avoid spill-over, *i.e.*, (11) is satisfied if and only if $\Delta\mathcal{M}$ and $\Delta\mathcal{K}$ jointly satisfy the following matrix equation:

$$[\Delta\mathcal{M}, \Delta\mathcal{K}]\mathcal{H} = 0, \tag{18}$$

where

$$\mathcal{H} = \begin{bmatrix} M_A^{-1} - X_{1A}\Gamma_1 X_{1A}^{\mathrm{T}} & M_A^{-1} C_{AS} M_S^{-1} - X_{1A}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}} \\ -X_{1S}\Gamma_1 X_{1A}^{\mathrm{T}} & -M_S^{-1} - X_{1S}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}} \\ -K_A^{-1} - X_{1A}\Lambda_1^{-1}\Gamma_1 X_{1A}^{\mathrm{T}} & -X_{1A}\Gamma_1 X_{1S}^{\mathrm{T}} \\ K_S^{-1} C_{SA} K_A^{-1} - X_{1S}\Lambda_1^{-1}\Gamma_1 X_{1A}^{\mathrm{T}} & K_S^{-1} - X_{1S}\Gamma_1 X_{1S}^{\mathrm{T}} \end{bmatrix}, \tag{19}$$

with $\mathrm{rank}(\mathcal{H}) = n - p$.

Proof. (Necessity) Since (J, X) is a Jordan pair of $P(\lambda)$, we have

$$MXJ + KX = 0,$$

and $\begin{bmatrix} X_A \\ -X_S J \end{bmatrix}$ is nonsingular. It follows from Lemma 1 that there exists a nonsingular matrix $\Gamma \in \mathbb{SR}^{n\times n}$ such that (12)-(16) hold. Since $\sigma(\Lambda_1) \cap \sigma(\Lambda_2) = \varnothing$, the matrix Γ must be of block diagonal form $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2)$, which satisfy

$$\Lambda_1\Gamma_1 = \Gamma_1\Lambda_1^{\mathrm{T}},\ \Lambda_2\Gamma_2 = \Gamma_2\Lambda_2^{\mathrm{T}}, \tag{20}$$

where Γ_1 is given by (17). Substituting the partitions of X and J given by (7) into (13) and (14), we can obtain that

$$X_{2A}\Gamma_2 X_{2S}^{\mathrm{T}} = -X_{1A}\Gamma_1 X_{1S}^{\mathrm{T}}, \tag{21}$$

$$X_{2A}\Gamma_2 X_{2A}^{\mathrm{T}} = M_A^{-1} - X_{1A}\Gamma_1 X_{1A}^{\mathrm{T}}, \tag{22}$$

$$X_{2S}\Lambda_2\Gamma_2 X_{2S}^{\mathrm{T}} = -M_S^{-1} - X_{1S}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}}, \tag{23}$$

$$X_{2A}\Lambda_2^{-1}\Gamma_2 X_{2A}^{\mathrm{T}} = -K_A^{-1} - X_{1A}\Lambda_1^{-1}\Gamma_1 X_{1A}^{\mathrm{T}}, \tag{24}$$

$$X_{2S}\Gamma_2 X_{2S}^{\mathrm{T}} = K_S^{-1} - X_{1S}\Gamma_1 X_{1S}^{\mathrm{T}}, \tag{25}$$

$$X_{2A}\Lambda_2\Gamma_2 X_{2S}^{\mathrm{T}} = M_A^{-1}C_{AS}M_S^{-1} - X_{1A}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}}, \tag{26}$$

$$X_{2A}\Lambda_2^{-1}\Gamma_2 X_{2S}^{\mathrm{T}} = K_A^{-1}C_{SA}^{\mathrm{T}}K_S^{-1} - X_{1A}\Lambda_1^{-1}\Gamma_1 X_{1S}^{\mathrm{T}}. \tag{27}$$

Pre-multiplying (26) and (21) by ΔM_A and ΔK_A, respectively, we can get

$$\left(\Delta M_A X_{2A}\Lambda_2 + \Delta K_A X_{2A}\right)\Gamma_2 X_{2S}^{\mathrm{T}} := H_1, \tag{28}$$

where $H_1 = \Delta M_A\left(M_A^{-1}C_{AS}M_S^{-1} - X_{1A}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}}\right) - \Delta K_A\left(X_{1A}\Gamma_1 X_{1S}^{\mathrm{T}}\right)$. Pre-multiplying (23) and (25) by ΔM_S and ΔK_S, respectively, we have

$$\left(\Delta M_S X_{2S}\Lambda_2 + \Delta K_S X_{2S}\right)\Gamma_2 X_{2S}^{\mathrm{T}} := H_2, \tag{29}$$

where $H_2 = -\Delta M_S\left(M_S^{-1} + X_{1S}\Lambda_1\Gamma_1 X_{1S}^{\mathrm{T}}\right) + \Delta K_S\left(K_S^{-1} - X_{1S}\Gamma_1 X_{1S}^{\mathrm{T}}\right)$. Similarly, Pre-multiplying (22) and (24) by ΔM_A and ΔK_A, respectively, we can obtain that

$$\left(\Delta M_A X_{2A}\Lambda_2 + \Delta K_A X_{2A}\right)\Lambda_2^{-1}\Gamma_2 X_{2A}^{\mathrm{T}} := H_3, \tag{30}$$

where $H_3 = \Delta M_A\left(M_A^{-1} - X_{1A}\Gamma_1 X_{1A}^{\mathrm{T}}\right) - \Delta K_A\left(K_A^{-1} + X_{1A}\Lambda_1^{-1}\Gamma_1 X_{1A}^{\mathrm{T}}\right)$. Transposing both sides of Equations (21), (27) and pre-multiplying by ΔM_S and ΔK_S, respectively, we can obtain that

$$\left(\Delta M_S X_{2S}\Lambda_2 + \Delta K_S X_{2S}\right)\Lambda_2^{-1}\Gamma_2 X_{2A}^{\mathrm{T}} := H_4, \tag{31}$$

where $H_4 = -\Delta M_S\left(X_{1S}\Gamma_1 X_{1A}^{\mathrm{T}}\right) + \Delta K_S\left(K_S^{-1}C_{SA}K_A^{-1} - X_{1S}\Lambda_1^{-1}\Gamma_1 X_{1A}^{\mathrm{T}}\right)$. It is easy to verify that (28)-(31) can be rewritten as

$$AB = \begin{bmatrix} H_3 & H_1 \\ H_4 & H_2 \end{bmatrix}, \tag{32}$$

where $B = \left[\Lambda_2^{-1}\Gamma_2 X_{2A}^{\mathrm{T}}\Gamma_2 X_{2S}^{\mathrm{T}}\right] \in \mathbb{R}^{(n-p)\times n}$ and

$$A = \begin{bmatrix} \Delta M_A X_{2A}\Lambda_2 + \Delta K_A X_{2A} \\ \Delta M_S X_{2S}\Lambda_2 + \Delta K_S X_{2S} \end{bmatrix} \in \mathbb{R}^{n\times(n-p)}.$$

Obviously, $A = 0$ since (11) holds. It follows from (32) that (18) is satisfied.

(Sufficiency.) Suppose that (18) holds. We can see from (32) that $AB = 0$, and it follows from the Sylvester's rank inequality that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \le n - p. \tag{33}$$

Since $\begin{bmatrix} X_A \\ -X_S\Lambda \end{bmatrix}$, Λ_2 and Γ_2 are nonsingular, it follows from (20) that

$$\operatorname{rank}\left(B^{\mathrm{T}}\right)=\operatorname{rank}\begin{bmatrix}X_{2A}\Gamma_2\Lambda_2^{-\mathrm{T}}\\ X_{2S}\Gamma_2\end{bmatrix}=\operatorname{rank}\begin{bmatrix}X_{2A}\\ X_{2S}\Lambda_2\end{bmatrix}=n-p. \tag{34}$$

It is easy to see from (33) that $A=0$, which implies that (11) holds.

Substituting (21)-(27) into (19) we have

$$\mathcal{H}=\begin{bmatrix}X_2\Lambda_2\\ X_2\end{bmatrix}\Gamma_2\begin{bmatrix}\Lambda_2^{-\mathrm{T}}X_{2A}^{\mathrm{T}} & X_{2S}^{\mathrm{T}}\end{bmatrix}. \tag{35}$$

Since X and $\begin{bmatrix}X_A\\ -X_S\Lambda\end{bmatrix}$ are nonsingular, which implies that the $2n\times(n-p)$ matrix $\begin{bmatrix}X_2\\ X_2\Lambda_2\end{bmatrix}$ and $n\times(n-p)$ matrix $\begin{bmatrix}X_{2A}\Lambda_2^{-1}\\ X_{2S}\end{bmatrix}$ are all of full column rank. We can see from (20) that Γ_2 is nonsingular. It follows from (35) that $\operatorname{rank}(\mathcal{H})=n-p$.

3. Solvability of the MUP-UVA

From the definitions of the matrices $\Delta\mathcal{M}$ and $\Delta\mathcal{K}$, it is easy to verify that (18) is equivalent to the following two equations

$$[\Delta M_A,\Delta K_A]\mathcal{H}_A=0, \tag{36}$$

$$[\Delta M_S,\Delta K_S]\mathcal{H}_S=0, \tag{37}$$

where $\mathcal{H}_A\in\mathbb{R}^{2n_A\times n}$ is composed of the first and third columns of $\mathcal{H}$ and $\mathcal{H}_S\in\mathbb{R}^{2n_S\times n}$ is composed of the second and fourth columns of $\mathcal{H}$. Without loss of generality, we assume that $\operatorname{rank}(\mathcal{H}_A)=s_1$ and $\operatorname{rank}(\mathcal{H}_S)=s_2$. By Theorem 2, we have $s_1+s_2=n-p$. Let the QR decompositions of $\mathcal{H}_A$ and $\mathcal{H}_S$ be given by

$$\mathcal{H}_A=Q\begin{bmatrix}R\\ 0\end{bmatrix},\ \mathcal{H}_S=P\begin{bmatrix}R'\\ 0\end{bmatrix}, \tag{38}$$

where $Q:=[Q_1,Q_2]\in\mathbb{R}^{2n_A\times 2n_A}$ and $P:=[P_1,P_2]\in\mathbb{R}^{2n_S\times 2n_S}$ are orthogonal matrices with $Q_2\in\mathbb{R}^{2n_A\times(2n_A-s_1)}$, $P_2\in\mathbb{R}^{2n_S\times(2n_S-s_2)}$, and $R\in\mathbb{R}^{s_1\times n}$, $R'\in\mathbb{R}^{s_2\times n}$ are of full row rank. Next, we characterize the parametric solutions of the MUP-UVA in terms of Y.

Theorem 3. Let (Λ_1,X_1) be defined by (7), and $(\Sigma_1,Y)\in\mathbb{R}^{p\times p}\times\mathbb{R}^{n\times p}$ be the real representation of the prescribed eigenpairs $\{\mu_j,y_j\}_{j=1}^{p}$. Suppose that $1\le p\le\min\{2n_A-s_1,2n_S-s_2\}$. Partition Q_2 and P_2 as $Q_2=[Q_{21},Q_{22}]$ and $P_2=[P_{21},P_{22}]$, where $Q_{22}\in\mathbb{R}^{2n_A\times p}$ and $P_{22}\in\mathbb{R}^{2n_S\times p}$. If the $p\times p$ matrices

$$Z_A:=Q_{22}^{\mathrm{T}}\begin{bmatrix}Y_A\Sigma_1\\ Y_A\end{bmatrix}, \tag{39}$$

$$Z_S:=P_{22}^{\mathrm{T}}\begin{bmatrix}Y_S\Sigma_1\\ Y_S\end{bmatrix}, \tag{40}$$

are nonsingular, then the $\Delta\mathcal{M}$, $\Delta\mathcal{K}$ defined by

$$[\Delta M_A,\Delta K_A]=-\left(M_AY_A\Sigma_1+K_AY_A+C_{AS}Y_S\Sigma_1\right)Z_A^{-1}Q_{22}^{\mathrm{T}}, \tag{41}$$

$$\left[\Delta M_S, \Delta K_S\right] = -\left(M_S Y_S \Sigma_1 + K_S Y_S + C_{SA} Y_A\right) Z_S^{-1} P_{22}^{\mathrm{T}}, \tag{42}$$

are real and solve the MUP-UVA. And in this case, Y is the matrix of eigenvectors corresponding to Σ_1.

Proof. Let

$$U := \left[U_1, U_2\right] = \left[\Delta M_A, \Delta K_A\right] Q, \tag{43}$$

where $U_1 \in \mathbb{R}^{n_A \times s_1}, U_2 \in \mathbb{R}^{n_A \times (2n_A - s_1)}$. It follows from (36) that

$$\left[\Delta M_A, \Delta K_A\right] \mathcal{H}_A = \left[U_1, U_2\right] \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \tag{44}$$

which indicates that $U_1 = 0$. Furthermore,

$$\left[\Delta M_A, \Delta K_A\right] = \left[U_1, U_2\right] Q^{\mathrm{T}} = U_2 Q_2^{\mathrm{T}}, \tag{45}$$

where U_2 can be further determined that (8) is satisfied. Let $U_2 = \left[U_{21} U_{22}\right]$, where $U_{21} \in \mathbb{R}^{n_A \times (2n_A - s_1 - p)}$, $U_{22} \in \mathbb{R}^{n_A \times p}$. we set $U_{21} = 0$. Form the partition of Q_2, we can see from (45) that

$$\left[\Delta M_A, \Delta K_A\right] = U_{22} Q_{22}^{\mathrm{T}}, \tag{46}$$

Substituting (46) into (8) gives

$$U_{22} Z_A = -\left(M_A Y_A \Sigma_1 + K_A Y_A + C_{AS} Y_S \Sigma_1\right). \tag{47}$$

Since Z_A is nonsingular, it follows from (47) that (41) is satisfied. Similarly, by the QR decomposition of $\mathcal{H}_S$, we can prove that (42) holds.

In practice, we tend to seek the minimum norm solution which can be obtained by the following problem:

$$\min_{Y \in \mathbb{R}^{n \times p}} \mathcal{J} := \mathcal{J}_A + \mathcal{J}_S, \tag{48}$$

where $\mathcal{J}_A = \frac{1}{2}\left(\left\|\Delta M_A\right\|_F^2 + \left\|\Delta K_A\right\|_F^2\right)$ and $\mathcal{J}_S = \frac{1}{2}\left(\left\|\Delta M_S\right\|_F^2 + \left\|\Delta K_S\right\|_F^2\right)$. Clearly, this is an unconstrained optimization problem which can also be solved by the MATLAB function **fminunc**. Next, we provide the gradient formula of $\mathcal{J}$ with respect to Y.

Theorem 4. Suppose that the QR decompositions of $\mathcal{H}_A$ is given by (38). Let $\left[\Delta M_A, \Delta K_A\right] = D Q_{22}^{\mathrm{T}}$, where $D = -\left(M_A Y_A \Sigma_1 + K_A Y_A + C_{AS} Y_S \Sigma_1\right) Z_A^{-1}$ with Z_A being defined by (39). Partition Q_{22} as $Q_{22}^{\mathrm{T}} = \left[Q_{31}, Q_{32}\right]$, where $Q_{31}, Q_{32} \in \mathbb{R}^{p \times n_A}$. Let $V = \left(V_1 - V_2\right)^{\mathrm{T}}$, where

$$V_1 = \left[-\Sigma_1 Z_A^{-1} D^{\mathrm{T}} M_A - Z_A^{-1} D^{\mathrm{T}} K_A, -\Sigma_1 Z_A^{-1} D^{\mathrm{T}} C_{AS}\right], \tag{49}$$

$$V_2 = \left[\Sigma_1 Z_A^{-1} D^{\mathrm{T}} D Q_{31} + Z_A^{-1} D^{\mathrm{T}} D Q_{32}, 0_{p \times n_S}\right]. \tag{50}$$

Then the gradient $\nabla_Y \mathcal{J}_A$ of $\mathcal{J}_A$ with respect to Y is given by

$$\nabla_Y \mathcal{J}_A = V. \tag{51}$$

Proof. From (41), we can obtain that

$$\left[\Delta M_A, \Delta K_A\right] = \left[0_{n_A \times (2n_A - p)}, DQ\right]^{\mathrm{T}}, \tag{52}$$

Recall that Q is an $2n_A \times 2n_A$ orthogonal matrix, then $\mathcal{J}_A$ can be rewritten as

$$\mathcal{J}_A = \frac{1}{2} tr\left(D^{\mathrm{T}} D\right). \tag{53}$$

Now, we establish the gradient $\bigtriangledown_Y \mathcal{J}_A$ of $\mathcal{J}_A$. Taking the differential of both sides of $\mathcal{J}_A$, we can obtain that

$$\Delta\mathcal{J}_A = \frac{1}{2} tr\left(\Delta D^{\mathrm{T}} D\right) + \frac{1}{2} tr\left(D^{\mathrm{T}} \Delta D\right). \tag{54}$$

Again, taking the differential of both sides of

$$DZ_A = -\left(M_A Y_A \Sigma_1 + K_A Y_A + C_{AS} Y_S \Sigma_1\right),$$

we have

$$\Delta DZ_A + D\Delta Z_A = -\left(M_A \Delta Y_A \Sigma_1 + K_A \Delta Y_A + C_{AS} \Delta Y_S\right). \tag{55}$$

Note that Z_A is nonsingular, then we have

$$\Delta D = \left[-\left(M_A \Delta Y_A \Sigma_1 + K_A \Delta Y_A + C_{AS} \Delta Y_S \Sigma_1\right) - D\Delta Z_A\right] Z_A^{-1}. \tag{56}$$

From (56), we have

$$\begin{aligned}
& tr\left(D^{\mathrm{T}} \Delta D\right) \\
&= tr\left(D^{\mathrm{T}}\left[-\left(M_A \Delta Y_A \Sigma_1 + K_A \Delta Y_A + C_{AS} \Delta Y_S \Sigma_1\right) - D\Delta Z_Y\right] Z_Y^{-1}\right) \\
&= tr\left(Z_Y^{-1} D^{\mathrm{T}}\left[-\left(M_A \Delta Y_A \Sigma_1 + K_A \Delta Y_A + C_{AS} \Delta Y_S \Sigma_1\right) - D\Delta Z_Y\right]\right) \\
&= tr\left(\left(-\Sigma_1 Z_Y^{-1} D^T M_A - Z_Y^{-1} D^{\mathrm{T}} K_A\right)\Delta Y_A\right) - tr\left(\Sigma_1 Z_Y^{-1} D^{\mathrm{T}} C_{AS} \Delta Y_S\right) \\
&\quad - tr\left(Z_Y^{-1} D^{\mathrm{T}} D\Delta Z_Y\right) \\
&= tr\left(V_1 \Delta Y\right) - tr\left(Z_Y^{-1} D^{\mathrm{T}} D\Delta Z_Y\right).
\end{aligned} \tag{57}$$

We now show that the second term of (57) can also be expressed in terms of ΔY. Substituting the partition of Q_{22} into Z_Y defined by (39), we have

$$Z_A = Q_{31} Y_A \Sigma_1 + Q_{32} Y_A. \tag{58}$$

It follows that

$$\Delta Z_A = Q_{31} \Delta Y_A \Sigma_1 + Q_{32} \Delta Y_A, \tag{59}$$

which implies that

$$tr\left(Z_A^{-1} D^{\mathrm{T}} D\Delta Z_A\right) = tr\left(Z_A^{-1} D^{\mathrm{T}} D\left(Q_{31} \Delta Y_A \Sigma_1 + Q_{32} \Delta Y_A\right)\right) = tr\left(V_2 \Delta Y\right). \tag{60}$$

From (57) and (60), we can obtain that

$$tr\left(D^{\mathrm{T}} \Delta D\right) = tr\left(\left(V_1 - V_2\right)\Delta Y\right) = tr\left(V^{\mathrm{T}} \Delta Y\right). \tag{61}$$

Similar to the proof of (61), the term $tr\left(D\Delta D^{\mathrm{T}}\right)$ in (54) can be express by ΔY as:

$$tr\left(D\Delta D^{\mathrm{T}}\right) = tr\left(\left(V_1^{\mathrm{T}} - V_2^{\mathrm{T}}\right)\Delta Y^{\mathrm{T}}\right) = tr\left(V\Delta Y^{\mathrm{T}}\right). \tag{62}$$

Substituting (54) and (61) into (62) gives

$$\Delta\mathcal{J}_A = \frac{1}{2}tr\left(V^{\mathrm{T}}\Delta Y\right) + \frac{1}{2}tr\left(V\Delta Y^{\mathrm{T}}\right) = \langle V, \Delta Y\rangle,$$

which implies that the gradient of $\mathcal{J}_A$ is given by (51). □

Similar to the proof of Theorem 4, we can prove the following theorem.

Theorem 5. Suppose that the QR decompositions of $\mathcal{H}_S$ is given by (38). Let $\left[\Delta M_S, \Delta K_S\right] = TP_{22}^{\mathrm{T}}$, where $T = -\left(M_S Y_S \Sigma_1 + K_S Y_S + C_{SA} Y_A\right) Z_S^{-1}$ with Z_S being defined by (40). Partition P_{22} as $P_{22}^{\mathrm{T}} = \left[P_{31}, P_{32}\right]$, where $P_{31}, P_{32} \in \mathbb{R}^{p\times n_S}$. Let $N = \left(N_1 - N_2\right)^{\mathrm{T}}$, where

$$N_1 = \left[-\Sigma_1 Z_S^{-1} T^{\mathrm{T}} M_S - Z_S^{-1} T^{\mathrm{T}} K_S, -\Sigma_1 Z_S^{-1} T^{\mathrm{T}} C_{SA}\right],$$

$$N_2 = \left[\Sigma_1 Z_S^{-1} T^{\mathrm{T}} T P_{31} + Z_S^{-1} T^{\mathrm{T}} T P_{32}, 0_{p\times n_A}\right].$$

Then the gradient $\nabla_Y \mathcal{J}_S$ of $\mathcal{J}_S$ with respect to Y is given by

$$\nabla_Y \mathcal{J}_S = N.$$

4. Numerical Examples

In this section, we give some numerical examples to verify the performance of Algorithm 1. All calculations were carried out using MATLAB R2023b. We compute the relative residuals of the updated system $\left(Res.U_k\right)$ as

Algorithm 1. Finding the minimum solution of the MUP-UVA.

Input:
1. The matrices $M, K \in \mathbb{R}^{n\times n}$ and a set of eigenpairs $\{\lambda_j, x_j\}_{j=1}^{p}$.
2. A set of newly measured eigenvalues $\{\widetilde{\mu_j}\}_{j=1}^{p}$.
3. ε =Termination tolerance.
4. Max_{iter} =Maximum number of iteration.

Output:
The matrices $\Delta\mathcal{M}$ and $\Delta\mathcal{K}$ such that the objective function $\mathcal{J}$ defined in (48) is minimized.
Step 1. Form the matrices Λ_1 and X_1 by (7).
Step 2. Compute the QR decompositions of $\mathcal{H}_A$ and $\mathcal{H}_S$ by (38). **Set** $k = 1$.
Step 3. Randomly generated Y.
Step 4. Compute Z_A and Z_S and the condition numbers cond(Z_A), cond(Z_S). If cond(Z_A) and cond(Z_S) are large, return to **Step 3**.
Step 5. Form matrices D, V_1, V_2, T, N_1, N_2, and compute the gradient $Grad := \nabla_Y\mathcal{J}$ by Theorm 4 and 5.
Step 6. Compute a new Y by gradient-based optimization method (we use the BFGS method [21]).
Step 7. Compute $\Delta\mathcal{M}$ and $\Delta\mathcal{K}$ in (41) and (42) by Y which minimizes the value of $\mathcal{J}$. **Stop**.

$$Res.U = \frac{\left\|\tilde{M}Y\Sigma_1 + \tilde{K}Y\right\|_F}{\left(\left\|\tilde{M}\right\|_F\left\|\Sigma_1\right\|_F + \left\|\tilde{K}\right\|_F\right)\left\|Y\right\|_F}, \tag{63}$$

and the relative residuals of the original system $Res.O_k$ as

$$Res.O = \frac{\left\|MX_2\Lambda_2 + KX_2\right\|_F}{\left(\left\|M\right\|_F\left\|\Lambda_2\right\|_F + \left\|K\right\|_F\right)\left\|X_2\right\|_F}. \tag{64}$$

Example 1. [15] In this example, we consider an undamped vibroacoustic system which is modified from a finite element model of a three-dimensional rectangular-box cavity backed by a flexible plate with $n_A = 336, n_S = 60$. In the raw data, the stiffness matrix K_A is singular, and the magnitude of the elements in the acoustic and structural matrices vary greatly. Those in the acoustic mass matrix M_A is of order $O(10^{-10})$, while those in the structural mass matrix M_S is of orders $O(10^{-5})$ to $O(10^{-2})$. Those in the acoustic stiffness matrix K_A is of order $O(10^{-2})$, while in the structural stiffness matrix K_S is of order $O(10^{5})$. And those in the acoustic-structural coupling matrix C_{AS} is of order $O(10^{-5})$. Therefore, the eigenvalues of $P(\lambda)$ in (3) are of order $O(10^{9})$. So we choose $n_A = 335$ and $n_S = 60$ and balance these matrices by setting

$$\hat{M} := 10^8 M \times T, \ \hat{K} := K \times T,$$

and $\lambda := 10^{-8}\lambda$, where $T = \mathrm{diag}(10^2 I_{n_A}, 10^{-6} I_{n_S})$. After this balancing, the matrices $\hat{M}$ and $\hat{K}$ are nonsingular, and the elements in the matrices are all of analogous order, and the eigenvalues are of normal order. All eigenvalues of the original system $\hat{P}(\lambda) = \lambda\hat{M} + \hat{K}$ are real.

Suppose that we update 8 eigenvalues of original system

$$\lambda_1 = -12.3512, \lambda_2 = -12.1400, \lambda_3 = -11.7411, \lambda_4 = -11.3875,$$

$$\lambda_5 = -11.3389, \lambda_6 = -11.1312, \lambda_7 = -11.1490, \lambda_8 = -11.0803.$$

to the following 4 pairs of complex conjugate eigenvalues

$$\mu_{1,2} = \pm 3.0544i, \ \mu_{3,4} = \pm 6.2118i, \ \mu_{5,6} = \pm 2.7988i,$$

where $\mu_{1,2}$ are of algebraic multiplicity two. We choose Y randomly, and compute the coefficient matrices ΔM_A, ΔM_S, ΔK_A and ΔK_S by Algorithm 3.2 in [16], Theorem 3, **fminunc** and Algorithm 1, respectively. Numerical results are shown in Table 1, which means that all the prescribed eigenvalues are embedded perfectly into the updated system and the remaining eigenpairs of the original system are kept unchanged. We can also see from Table 1 that the errors of updates computed by the Algorithm 1 are smaller, *i.e.*, the proposed algorithm in our paper is more efficient for the minimum solution of the PMP-UVA.

Table 1. Numerical result for Example 1.

Alg	$\|\Delta M_A\|_F$	$\|\Delta M_S\|_F$	$\|\Delta K_A\|_F$	$\|\Delta K_S\|_F$	*Res.U*	*Res.O*	IT.	CT.
Algorithm 3.2 [16]	362.4999	13.7043	2.4e + 03	120.5830	1.9176e−15	6.0511e−14	-	1.1477
Theorem 3	210.7100	6.9986	63.3163	14.9833	2.9799e−17	1.1275e−14	-	0.7024
fminunc	101.6430	3.7118	36.3769	7.3134	2.0297e−17	9.5218e−15	8	30.2630
Algorithm 1	34.8851	3.2175	10.7382	2.06446	1.4559e−17	4.5676e−15	14	9.07322

Example 2. In this example, the matrices $M_A, M_S, K_A, K_S, C_{AS}$ and C_{SA} are the same as in Example 1. We update p (varying from 2 to 38 at increment 4) eigenvalues to μ_j that are generated randomly, and μ_j need not to be simple. We apply **fminunc** and Algorithm 1, respectively, to get the updated systems.

Numerical results are illustrated in Figure 1 and Figure 2. These results are rather close, or at least comparable, for all p. The relative errors of modifications on

ΔM_A, ΔM_S, ΔK_A and ΔK_S are listed in **Figures 3-6**, respectively, which show the norms of the modifications computed by **Algorithm 1** are smaller than those obtained by **fminunc**. Therefore, we can conclude that **Algorithm 1** is more efficient for getting the minimum norm solutions of the MUP-UVA.

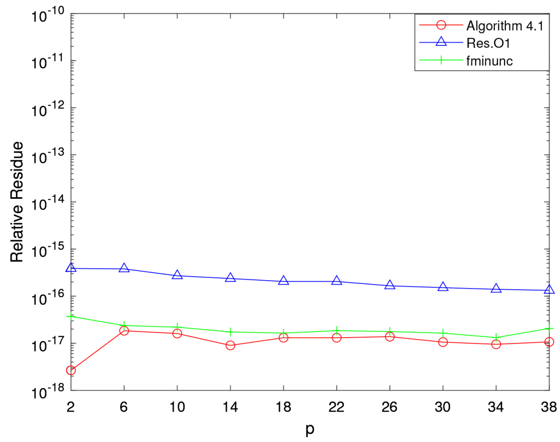

Figure 1. $Res.U$ of updated systems in Example 2.

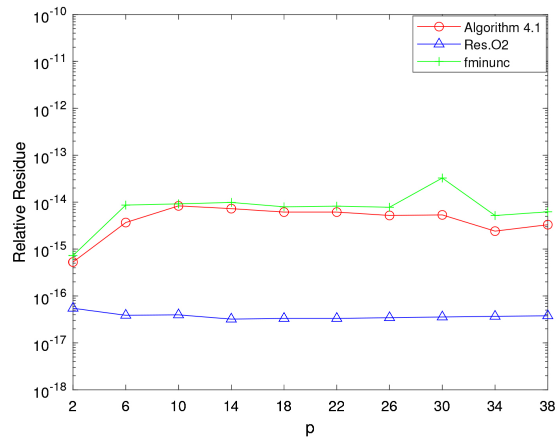

Figure 2. $Res.O$ of updated systems in Example 2.

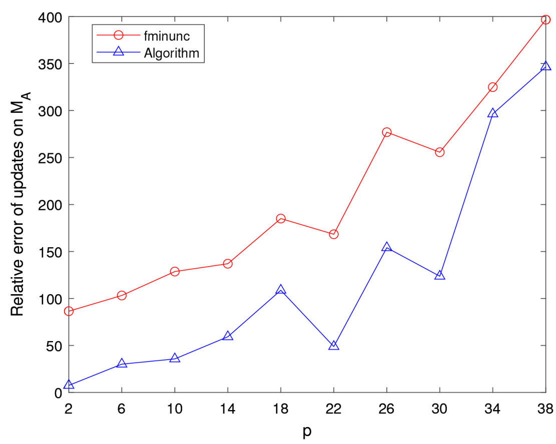

Figure 3. Relative errors of updates on M_A in Example 2.

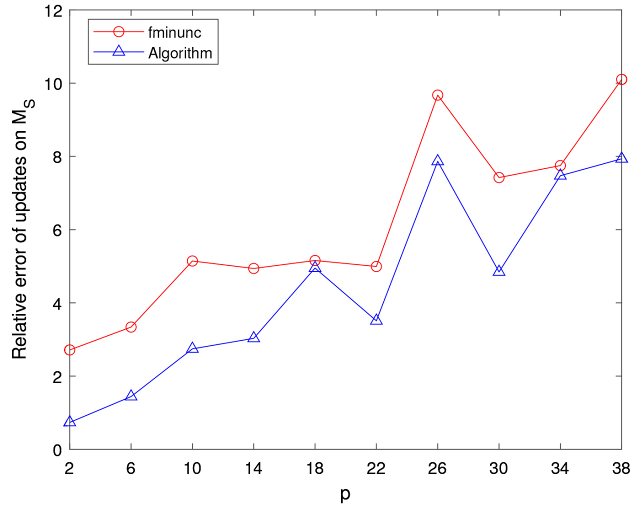

Figure 4. Relative errors of updates on M_S in Example 2.

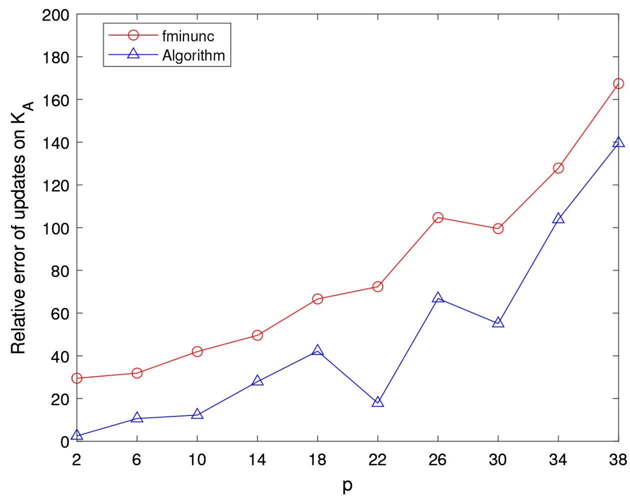

Figure 5. Relative errors of updates on K_A in Example 2.

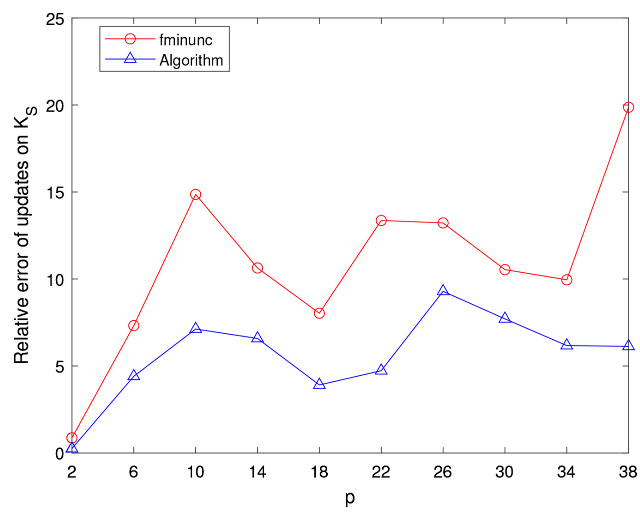

Figure 6. Relative errors of updates on K_S in Example 2.

5. Conclusions and Suggestions

A new model updating method for the undamped vibroacoustic system with no spill-over is provided in this paper. Using the spectral decomposition of a structured asymmetric pencil, a sufficient and necessary condition is derived from which the updated undamped vibroacoustic system can preserve no spill-over. Finally, a gradient-based optimization algorithm for the minimum norm solution of the MUP-UVS is proposed. Numerical examples illustrate the effectiveness the proposed algorithm.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Notations and Assumptions

- $\lambda_1, \lambda_2, \cdots, \lambda_n$ —eigenvalues of the model (M, K).
- $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$ —diagonal matrix of eigenvalues to be reassigned.
- $X_1 = [x_1, \cdots, x_p]$ —matrix of eigenvectors corresponding to Λ_1.
- $\Lambda_2 = \text{diag}(\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_n)$ —diagonal matrix of remaining eigenvalues to be kept unchanged.
- $X_2 = [x_{p+1}, \cdots, x_n]$ —matrix of eigenvector corresponding to Λ_2.
- $\Sigma_1 = \text{diag}(\mu_1, \cdots, \mu_p)$ —diagonal matrix of measured eigenvalues.
- $Y = [y_1, \cdots, y_p]$ —matrix of eigenvectors corresponding to Σ_1.
- $\mathbb{C}^{m \times n}$ —the set of all complex $m \times n$ matrices over $\mathbb{C}$.
- $\mathbb{R}^{m \times n}$ —the set of all real $m \times n$ matrices over $\mathbb{R}$.
- $\mathbb{SR}^{n \times n}$ —the set of all symmetric matrices in $\mathbb{R}^{n \times n}$.
- $tr(A)$ —the trace of matrix A.
- $\|A\|_F$ —Frobenius norm of the matrix A.

In this paper, we will make the following assumptions.

(A1) The matrices M and K are nonsingular.

(A2) All the eigenvalues of Λ_1, Λ_2 and Σ_1 are nonzero.

(A3) $\sigma(\Lambda_1) \cap \sigma(\Lambda_2) = \varnothing$, $\sigma(\Lambda_1) \cap \sigma(\Sigma_1) = \varnothing$.