

Global Existence of Solutions to the Timoshenko-Fourier System in Critical Besov Spaces

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Abstract

In this work, we are concerned with the Timoshenko-Fourier system in both equal and non-equal wave speeds, which admits a non-symmetric dissipation. Furthermore, the dissipative mechanism of regularity-loss type will occur in the case of non-equal wave speeds. We establish the global-in-time existence of solutions to the Timoshenko-Fourier system in critical Besov spaces with the regularity $s = \frac{3}{2}$.

Keywords

Timoshenko-Fourier System, Regularity-Loss, Critical Besov Spaces

1. Introduction

We are devoted to the following coupled system of the Timoshenko theory of vibrating beams [1] [2] with additional heat conduction effects according to Fourier's law:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - [\sigma(\psi_x)]_x - (\varphi_x - \psi) + \gamma\psi_t + \beta\theta_x = 0, \\ \theta_t - k\theta_{xx} + \beta\psi_{tx} = 0, \end{cases} \quad (1.1)$$

where $t \in (0, +\infty)$ is the time variable and $x \in \mathbb{R}$ is the spatial variable. The function φ and ψ denote the transverse displacement of the beam from equilibrium and the angle of turn of the beam filaments, θ is the temperature difference of the elastic material, and γ , β and k are physical constants dependent on the elastic and thermal properties of the material. For any $\eta > 0$, the smooth

function $\sigma(\eta)$ satisfies $\sigma'(\eta) > 0$, we are devoted to the Cauchy problem for the system (1.1), with initial conditions of

$$(\varphi, \varphi_t, \psi, \psi_t, \theta) \Big|_{t=0} = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)(x), \quad x \in \mathbb{R}.$$

The linearized system of (1.1) is accordingly written as

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \gamma \psi_t + \beta \theta_x = 0, \\ \theta_t - k \theta_{xx} + \beta \psi_{tx} = 0, \end{cases} \quad (1.2)$$

where $a > 0$ represents the speed of sound defined by $a^2 = \sigma'(0)$. $a = 1$ represents the same wave speed and $a \neq 1$ implies different wave speeds for the first two equations corresponding to the Timoshenko-Fourier system.

We investigate the global existence of solutions to (1.3) in critical Besov spaces. Xu and Kawashima [3] have developed a theory of local existence of general symmetric hyperbolic systems in critical Besov spaces, which is seen as a generalization of the basic theory of Kato and Majda [4] [5], and since that the non-symmetric dissipation L has no effect on the local-in-time existence, their results can be directly applied to the system (1.3). Based on the observation for the system (1.1), we argue that the viscous term $k\theta_{xx}$ in (1.1) does not affect the mathematical entropy, allowing us to obtain a global a priori estimate by considering it from the perspective of the hyperbolic energy methods. The specific ideas are detailed in the following proof of Theorem 1.1. According to the references reviewed, very little has been done for the study of the Timoshenko-Fourier system, although many have achieved modest results for studying the fluid dynamic equations in critical Besov spaces, such as the Navier-Stokes equations in [6]-[9], and [10]-[13] for the Euler equation and related models. Due to non-symmetry of Timoshenko-Fourier system, although the first author and third author in [14] have studied general dissipative systems under the assumptions of dissipative entropy and the Shizuta-Kawashima condition in [3] [15], their result cannot be directly applied to the Timoshenko-Fourier system. The conclusions related to the non-symmetry of relaxation matrices have been studied in [11], which provides the theoretical support for our research. Therefore, we first consider the case that (1.3) has the equal wave speed ($a = 1$), and then the non-equal wave speed ($a \neq 1$), of which we can derive the desired prior estimate by using a fundamental fact that the relation between homogeneous and non-homogeneous Chemin-Lerner spaces in Proposition 2.1. Finally, we construct global solutions pertaining to data in the Besov space $B_{2,1}^{3/2}(\mathbb{R})$.

In the case of non-equal waves ($a \neq 1$), there will be regularity loss (a loss of derivatives in the energy estimates), and the system (1.3) admits a weaker dissipation mechanism. To obtain similar result for (1.3) with $a = 1$, we will establish the global-in-time existence: $s_c = \frac{3}{2}$ for the Timoshenko-Fourier system with loss of regularity, which will lead to a significant reduction in the regularity requirement for the initial values compared to [16]. Moreover, compared to the case

$a = 1$, when estimating the dissipation with respect to v , we must utilize Proposition 2.1 for the topological relation between $\|u_x\|_{L^2_T(B_{2,1}^{-1/2})}$ and $\|u_x\|_{L^2_T(B_{2,1}^{-1/2})}$ to obtain the dissipation estimate on v .

This paper is arranged as follows. In Section 2, the related lemmas and propositions are shown. In Section 3, we establish several lemmas of estimating Proposition (3.2), which leads to the proof of Theorem 1.1.

Notations. Throughout, we provide some notations. $f \lesssim g$ denotes $f \leq Cg$, where $C > 0$ is a general constant, the number represented by C in different rows may differ. $C([0, T], X)$ ($C^1([0, T], X)$) denotes the space of continuous (continuously differentiable) functions that take values in $[0, T]$ of the Banach space X . $\|(f, g, h)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X$, where $f, g, h \in X$.

Main Results

Rewrite Equation (1.1) as the following Cauchy problem [3]:

$$\begin{cases} U_t + AU_x + LU = BU_{xx} + G(U)_x, \\ U(x, 0) = U_0(x), \end{cases} \tag{1.3}$$

where the coefficient matrices A , L and B are given by the following formula:

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & -a & 0 & \beta \\ 0 & 0 & 0 & \beta & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix},$$

where $G(U) = (0, 0, 0, g(z), 0)$ with $g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a := O(z^2)$ near $z = 0$.

The main result of our work is stated as follows:

Theorem 1.1. *Suppose $U_0 \in B_{2,1}^{3/2}(\mathbb{R})$. There exists a positive constant δ_0 such that if*

$$\|U_0\|_{B_{2,1}^{3/2}} \leq \delta_0,$$

then the Cauchy problem (1.3) has a unique global classical solution $U \in \tilde{C}^1(\mathbb{R}^+ \times \mathbb{R})$ satisfying

$$U \in \tilde{C}(B_{2,1}^{3/2}(\mathbb{R})) \cap \tilde{C}^1(B_{2,1}^{1/2}(\mathbb{R})).$$

In addition, the following energy inequality holds true

$$\begin{aligned} & \|U\|_{\dot{L}_T^\infty(B_{2,1}^{3/2}(\mathbb{R}))} + \left(\|y\|_{\dot{L}_T^2(B_{2,1}^{3/2})} + \|(v, z_x)\|_{\dot{L}_T^2(B_{2,1}^{1/2})} + \|u_x\|_{\dot{L}_T^2(B_{2,1}^{-1/2})} + \|\theta_x\|_{\dot{L}_T^2(B_{2,1}^{3/2})} \right) \\ & \leq C_0 \|U_0\|_{B_{2,1}^{3/2}(\mathbb{R})}, \end{aligned}$$

where C_0 is some positive constant.

Remark 1.1. Theorem 1.1 exhibits the global-in-time existence of solutions to the Timoshenko-Fourier system in critical Besov spaces with the regularity $s = \frac{3}{2}$, which is the optimal result for the Timoshenko-Fourier system studied so far based on the dimension of the space and the conditions for system stability.

2. Preliminary

In this section, we mainly show the lemmas and propositions that will be used in the following; for proofs of the propositions, see the references [3] [17].

Lemma 2.1. Let $0 < R_1 < R_2$ and $1 \leq a \leq b \leq \infty$.

1) If Suppose $\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R_1\lambda\}$, for any $\alpha \geq 0$, then

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha+n(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a},$$

2) If Suppose $\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \leq |\xi| \leq R_2\lambda\}$, for any $\alpha \in \mathbb{R}$, then

$$\|\Lambda^\alpha f\|_{L^b} \approx \lambda^\alpha \|f\|_{L^a}.$$

Lemma 2.2 Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. then

1) If $s > 0$, then $B_{p,r}^s = L^p \cap \dot{B}_{p,r}^s$.

2) If $\tilde{s} \leq s$, then $B_{p,r}^s \hookrightarrow B_{p,r}^{\tilde{s}}$. This inclusion relation is false for the homogeneous Besov spaces.

3) If $1 \leq r \leq \tilde{r} \leq \infty$, then $\dot{B}_{p,r}^s \hookrightarrow \dot{B}_{p,\tilde{r}}^s$ and $B_{p,r}^s \hookrightarrow B_{p,\tilde{r}}^s$.

4) If $1 \leq p \leq \tilde{p} \leq \infty$, then $\dot{B}_{p,r}^s \hookrightarrow \dot{B}_{\tilde{p},r}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}$ and $B_{p,r}^s \hookrightarrow B_{\tilde{p},r}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}$.

5) If $r \geq \theta$, then $\|f\|_{\dot{L}_T^\theta(B_{p,r}^s)} \leq \|f\|_{L_T^\theta(B_{p,r}^s)}$; If $r \leq \theta$, then $\|f\|_{\dot{L}_T^\theta(B_{p,r}^s)} \geq \|f\|_{L_T^\theta(B_{p,r}^s)}$.

6) $B_{p,1}^{n/p} \hookrightarrow C_0$, $\dot{B}_{p,1}^{n/p} \hookrightarrow C_0$ ($1 \leq p < \infty$).

where C_0 is the spaces of continuous bounded functions that decay at infinity.

Lemma 2.3 Suppose that $\varrho > 0$ and $1 \leq p < 2$. It holds that

$$\|f\|_{\dot{B}_{r,\infty}^{-\varrho}} \lesssim \|f\|_{L^p}.$$

with $\frac{1}{p} - \frac{1}{r} = \frac{\varrho}{n}$. In particular, this holds with $\varrho = \frac{n}{2}, r = 2, p = 1$.

Integral existentiality depends on the connection between homogeneous Chemin-Lerner spaces and non-homogeneous Chemin-Lerner spaces, which we will only briefly illustrate here; see [3] for more detailed proofs.

Proposition 2.1. Let $s \in \mathbb{R}$ and $1 \leq \theta, p, r \leq \infty$, for any $T > 0$

1) It holds that

$$L_T^\theta(L^p) \cap \tilde{L}_T^\theta(\dot{B}_{p,r}^s) \subset \tilde{L}_T^\theta(B_{p,r}^s).$$

2) Moreover, as $s > 0$ and $\theta \geq r$, It holds that

$$L_T^\theta(L^p) \cap \tilde{L}_T^\theta(\dot{B}_{p,r}^s) = \tilde{L}_T^\theta(B_{p,r}^s).$$

Proposition 2.2 Let $s > 0$ and $1 \leq p, r \leq \infty$, then $\dot{B}_{p,r}^s \cap L^\infty$ is an algebra and

$$\|fg\|_{\dot{B}_{p,r}^s} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{B}_{p,r}^s} + \|g\|_{L^\infty} \|f\|_{\dot{B}_{p,r}^s}.$$

Let $s_1, s_2 \leq n/p$ such that $s_1 + s_2 > n \max\left\{0, \frac{2}{p} - 1\right\}$. Then one has

$$\|fg\|_{\dot{B}_{p,1}^{s_1+s_2-n/p}} \lesssim \|f\|_{\dot{B}_{p,1}^{s_1}} \|g\|_{\dot{B}_{p,1}^{s_2}}.$$

Proposition 2.3 Let $1 < p < \infty, 1 \leq \theta \leq \infty$ and $s \in \left(-\frac{n}{p} - 1, \frac{n}{p}\right]$. Then there exists a constant $C > 0$ that depends only on s, n such that

$$\begin{cases} \|[f, \dot{\Delta}_q]g\|_{L^p} \leq Cc_q 2^{-q(s+1)} \|f\|_{\dot{B}_{p,1}^{n+1}} \|g\|_{\dot{B}_{p,1}^s}, \\ \|[f, \dot{\Delta}_q]g\|_{L_T^\theta(L^p)} \leq Cc_q 2^{-q(s+1)} \|f\|_{\tilde{L}_T^\theta(\dot{B}_{p,1}^{n+1})} \|g\|_{\tilde{L}_T^\theta(\dot{B}_{p,1}^s)}. \end{cases}$$

with $\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$, where the commutators $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$

and $\{c_q\}$ denotes a sequence such that $\|c_q\|_{l^1} \leq 1$.

Proposition 2.4. Let $s > 0, 1 \leq p, r, \theta \leq \infty, F \in W_{loc}^{[s]+3, \infty}(I; \mathbb{R})$ with $F(0) = 0, T \in (0, \infty]$ and $f \in \tilde{L}_T^\theta(B_{p,r}^s) \cap L_T^\infty(L^\infty)$, Then there exists a function C that depends only on s, p, r, n and F such that

$$\begin{cases} \|F(f) - F'(0)f\|_{\dot{B}_{p,r}^s} \leq C(\|f\|_{L^\infty}) \|f\|_{\dot{B}_{p,r}^s}^2, \\ \|F(f) - F'(0)f\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} \leq C(\|f\|_{\tilde{L}_T^\theta(L^\infty)}) \|f\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)}^2. \end{cases}$$

Proposition 2.5. The following inequality holds

$$\|fg\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} \lesssim \left(\|f\|_{L_T^{\theta_1}(L^\infty)} \|g\|_{\tilde{L}_T^{\theta_2}(\dot{B}_{p,r}^s)} + \|g\|_{L_T^{\theta_3}(L^\infty)} \|f\|_{\tilde{L}_T^{\theta_4}(\dot{B}_{p,r}^s)} \right).$$

with $s > 0, 1 \leq p \leq \infty, 1 \leq \theta, \theta_1, \theta_2, \theta_3, \theta_4 \leq \infty$ and

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.$$

A direct corollary is that

$$\|fg\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} \lesssim \|f\|_{\tilde{L}_T^{\theta_1}(\dot{B}_{p,r}^s)} \|g\|_{\tilde{L}_T^{\theta_2}(\dot{B}_{p,r}^s)}$$

with $s \geq n/p, \frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

Proposition 2.6. Let $s > 0, 1 \leq p, r, \rho \leq \infty, F \in W_{loc}^{[s]+1, \infty}(I; \mathbb{R})$ with $F(0) = 0, T \in (0, \infty]$ and $v \in \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \cap L_T^\infty(L^\infty)$, then

$$\|F(v)\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \lesssim \left(1 + \|v\|_{L_T^\infty(L^\infty)}\right)^{[s+1]} \|v\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)}.$$

3. The Proof of Theorem 1.1

Xu and Kawashimsa [3] established a local existence theory for general symmetric hyperbolic systems in critical Besov spaces, which is seen as a generalization of the basic theory established by Kato and Majda [4] [5], and it can be applied to Timoshenko-Fourier system (1.3). Precisely,

Proposition 3.1. *Assuming $U_0 \in B_{2,1}^{3/2}$, there exists a time $T_0 > 0$ (dependent only on initial data) such that*

1) (Existence) *The system (1.3) has a unique solution $U(t, x) \in C^1([0, T_0] \times \mathbb{R})$ that satisfying $U \in \tilde{C}_{T_0}^\infty(B_{2,1}^{3/2}) \cap \tilde{C}_{T_0}^1(B_{2,1}^{1/2})$;*

2) (Blow-up Criterion) *If the maximal time $T^* (> T_0)$ existing of such a solution in finite, then*

$$\limsup_{n \rightarrow T^*} \|U(t, \cdot)\|_{B_{2,1}^{3/2}} = \infty,$$

if and only if

$$\int_0^{T^*} \|(U(t, \cdot), \nabla U(t, \cdot))\|_{L^\infty} dt = \infty.$$

Proof. In order to prove Proposition (3.1) (i), it suffices to show that $U \in \tilde{C}_{T_0}^\infty(B_{2,1}^{3/2})$ and $U_t \in \tilde{C}_{T_0}^\infty(B_{2,1}^{1/2})$.

We used the stationary cases of estimates of commutator in [18] (Lemma 2.100, p. 112) to (3.26), we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} N_0 [\dot{\Delta}_q U] + \gamma \|\dot{\Delta}_q y\|_{L^2}^2 + k \|\dot{\Delta}_q \theta_x\|_{L^2}^2 \\ & \lesssim \|y_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2}^2 + \|z_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2} \|\dot{\Delta}_q y\|_{L^2} \\ & \quad + 2^{\frac{q}{2}} \left(\|z_{xx}\|_{L^\infty} \|z\|_{B_{2,1}^{-\frac{1}{2}}} + \|z_x\|_{L^\infty} \|z_{xx}\|_{B_{2,1}^{-\frac{1}{2}}} \right) \|\dot{\Delta}_q y\|_{L^2}. \end{aligned} \tag{3.4}$$

Dividing (3.4) by $(\|\Delta_q U\|_{L^2}^2 + \epsilon)^{\frac{1}{2}}$, we can obtain

$$\begin{aligned} \frac{d}{dt} (\|\Delta_q U\|_{L^2}^2 + \epsilon)^{\frac{1}{2}} + \|\dot{\Delta}_q y\|_{L^2} & \lesssim \|\dot{\Delta}_q \theta_x\|_{L^2} + \|y_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2} + \|z_x\|_{L^\infty} \|\dot{\Delta}_q y\|_{L^2} \\ & \quad + 2^{\frac{q}{2}} \left(\|z_{xx}\|_{L^\infty} \|z\|_{B_{2,1}^{-\frac{1}{2}}} + \|z_x\|_{L^\infty} \|z_{xx}\|_{B_{2,1}^{-\frac{1}{2}}} \right). \end{aligned} \tag{3.5}$$

Integrating (3.5) on the variable $t \in [0, T_0]$, then taking $\epsilon \rightarrow 0$, and using the estimates of commutators and continuity for the composition in the stationary case (see, e.g., [19]), we have

$$\begin{aligned} 2^{-\frac{1}{2}q} \|\Delta_q U\|_{L_t^\infty(L^2)} & \lesssim 2^{-\frac{1}{2}q} \|\Delta_q U_0\|_{L^2} + \int_0^t c_q(\tau) \|(y, \theta_x)\|_{B_{2,1}^{-\frac{1}{2}}}^2 d\tau \\ & \quad + \int_0^t \|(z_{xx}, z_x)\|_{B_{2,1}^{-\frac{1}{2}}} \|z\|_{B_{2,1}^{-\frac{1}{2}}}^{\frac{3}{2}} d\tau, \end{aligned} \tag{3.6}$$

where we used the embedding inequality $(\|z\|_{L^\infty} \lesssim \|z\|_{B_{2,1}^{-\frac{1}{2}}})$, Lemma 2.2 and Young's inequality. Summing up (3.6) on $q \geq -1$ gives

$$\|U\|_{L_t^\infty(B_{2,1}^{3/2})} \lesssim \|U_0\|_{B_{2,1}^{3/2}} + \epsilon \int_0^t \|\nabla U, \Delta U\|_{B_{2,1}^{1/2}}^2 d\tau + C(\epsilon) \int_0^t \|U\|_{B_{2,1}^{3/2}}^2 d\tau. \tag{3.7}$$

Let $\epsilon > 0$ be a small number, using Gronwall's inequality, we get

$$U \in \tilde{C}_{T_0}^\infty(B_{2,1}^{3/2}). \tag{3.8}$$

Furthermore, with proposition (2.5) we can deduce that

$$U_t \in \tilde{C}_{T_0}^\infty(B_{2,1}^{1/2}). \tag{3.9}$$

This completes Proposition (3.1) (i).

From [3], for a symmetric hyperbolic system it is sufficient to establish the blow-up criterion. Such a criterion can be established for the Timoshenko-Fourier system. We consider the symmetric system (1.3) with $LU \equiv 0$ for simplicity, since it is only responsible for the global well-posedness and large time behavior of solutions

$$\partial_t U + A \partial_x U = BU_{xx} + G(U)_x. \tag{3.10}$$

Applying the homogeneous operator $\dot{\Delta}_q$ to (1.3), we infer that $\dot{\Delta}_q U$ satisfies

$$\partial_t \dot{\Delta}_q U + A \dot{\Delta}_q \partial_x U + B \dot{\Delta}_q U_{xx} + \dot{\Delta}_q G(U)_x = 0. \tag{3.11}$$

Perform the inter product with $\dot{\Delta}_q U$ on both sides of (3.11) to get

$$\begin{aligned} & \langle \dot{\Delta}_q U, \dot{\Delta}_q U \rangle_t + \langle A \dot{\Delta}_q U_x, \dot{\Delta}_q U \rangle + \langle B \dot{\Delta}_q U_x, \dot{\Delta}_q U \rangle_x + \langle \dot{\Delta}_q G(U)_x, \dot{\Delta}_q U \rangle \\ & = \langle B \dot{\Delta}_q U_x, \dot{\Delta}_q U_x \rangle, \end{aligned} \tag{3.12}$$

where

$$\langle \dot{\Delta}_q G(U)_x, \dot{\Delta}_q U \rangle \approx \langle [\dot{\Delta}_q, U] U_x, \dot{\Delta}_q U \rangle + \langle U \dot{\Delta}_q U, \dot{\Delta}_q U_x \rangle.$$

By integrating (3.12) with respect to x over \mathbb{R} , we deduce that

$$\begin{aligned} \frac{d}{dt} \|\dot{\Delta}_q U\|_{L^2}^2 & \lesssim \|\dot{\Delta}_q U\|_{L^2} \|\dot{\Delta}_q U_x\|_{L^2} + \|[\dot{\Delta}_q, U] U_x\|_{L^2} \|\dot{\Delta}_q U\|_{L^2} \\ & \quad + \|U\|_{L^\infty} \|\dot{\Delta}_q U\|_{L^2} \|\dot{\Delta}_q U_x\|_{L^2}. \end{aligned} \tag{3.13}$$

Let $\epsilon > 0$ be a small number, dividing (3.13) by $(\|\dot{\Delta}_q U\|_{L^2}^2 + \epsilon)^{\frac{1}{2}}$ gives

$$\begin{aligned} \frac{d}{dt} (\|\dot{\Delta}_q U\|_{L^2}^2 + \epsilon)^{\frac{1}{2}} & \lesssim \|\dot{\Delta}_q U_x\|_{L^2} + \|[\dot{\Delta}_q, U] U_x\|_{L^2} + \|U\|_{L^\infty} \|\dot{\Delta}_q U_x\|_{L^2} \\ & \lesssim c_q(t) 2^{\frac{-q}{2}} \left(\|U_x\|_{L^\infty} \|U\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + \|U_x\|_{L^\infty} \|U_x\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \right) + \|U\|_{L^\infty} \|\dot{\Delta}_q U_x\|_{L^2}, \end{aligned} \tag{3.14}$$

where we used the stationary cases of estimates of commutator in [18] (Lemma 2.100, p. 112) and the sequence $\{c_q(t)\}$ satisfying $\|c_q(t)\|_{l^1} \leq 1$ for all $t \in [0, T_0]$.

Letting $\epsilon \rightarrow 0$ and taking a time integration, we are lead to

$$\|U(t)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \lesssim \|U_0\|_{\dot{B}_{2,1}^{\frac{1}{2}}} + C \int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} \|(U, \nabla U)\|_{\dot{B}_{2,1}^{\frac{1}{2}}} d\tau. \tag{3.15}$$

Taking the L^2 inner production (3.10) with U , it is easy to obtain

$$\|U(t)\|_{L^2} \leq \|U_0\|_{L^2} + C \int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} \|U, \nabla U\|_{L^2} d\tau. \tag{3.16}$$

Adding (3.15) to (3.16), with the aid of Proposition 2.1(1) and Lemma 2.1, we have

$$\begin{aligned} \|U(t)\|_{B_{2,1}^2} &\lesssim \|U_0\|_{B_{2,1}^2} + C \int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} \|U, \nabla U\|_{B_{2,1}^2} d\tau \\ &\lesssim \|U_0\|_{B_{2,1}^2} + C \int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} \|U\|_{B_{2,1}^2} d\tau. \end{aligned} \tag{3.17}$$

The Gronwall’s inequality implies

$$\sup_{t \in [0, T_0]} \|U(t)\|_{B_{2,1}^2} \leq \|U_0\|_{B_{2,1}^2} \exp\left(C \int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} d\tau\right). \tag{3.18}$$

So, the subsequent inequalities will hold true

$$\int_0^{T_0} \|(U, \nabla U)\|_{L^\infty} d\tau \leq C \int_0^{T_0} \|U\|_{B_{2,1}^2} d\tau \leq CT_0 \sup_{t \in [0, T_0]} \|U\|_{B_{2,1}^2}. \tag{3.19}$$

This completes the proof of Proposition 3.1(ii).

Moreover, to prove that the classical solution in Proposition 3.1 is globally defined, we need to construct a priori estimates based on the dissipation mechanism generated by the Timoshenko-Fourier system. To this end, we define energy generalization in terms of $N(T)$ and the corresponding dissipation generalization in terms of $D(T)$:

$$\begin{aligned} N(T) &:= \|U\|_{\tilde{L}_T^2(B_{2,1}^{3/2})}, \\ D(T) &:= \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + \|(v, z_x)\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} + \|u_x\|_{\tilde{L}_T^2(B_{2,1}^{-1/2})} + \|\theta_x\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} \end{aligned}$$

for any time $T > 0$.

In the case of $a = 1$:

Lemma 3.1 (*The dissipation for (y, θ_x)*) *If $U \in \tilde{C}_T(B_{2,1}^{3/2}) \cap \tilde{C}_T^1(B_{2,1}^{1/2})$ is a solution of (1.3) for any $T > 0$, then*

$$E(T) + \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + \|\theta_x\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} \lesssim \|U_0\|_{B_{2,1}^{3/2}} + \sqrt{N(T)}D(T).$$

Proof. We start with the next equations

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - y_x = 0, \\ y_t - [\sigma(z)]_x - v + \gamma y + \beta \theta_x = 0, \\ \theta_t - k\theta_{xx} + \beta y_x = 0. \end{cases} \tag{3.20}$$

The five equations of the above system (3.20) are each multiplied in order by $v, u, \sigma(z) - \sigma(0), y$ and θ . Adding the resulting equations yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (v^2 + u^2 + F(z) + y^2 + \theta^2) - (vu + (\sigma(z) - \sigma(0))y + \beta \theta_y + k\theta_x \theta)_x \\ + \gamma y^2 + k\theta_x^2 = 0, \end{aligned} \tag{3.21}$$

where

$$F(z) = 2 \int_0^z (\sigma(\eta) - \sigma(0)) d\eta.$$

Noting that $F(z)$ is equivalent to z^2 , due to $\sigma'(\eta) > 0$ and the following smallness assumption, then integrating about x yields the basic energy equation

$$\frac{1}{2} \frac{d}{dt} N_0(U) + \gamma \|y\|_{L^2}^2 + k \|\theta_x\|_{L^2}^2 = 0, \tag{3.22}$$

where the energy generalized function $N_0(U)$ is defined as

$$N_0(U) = \|(v, u, y, \theta)\|_{L^2}^2 + \int_{\mathbb{R}} F(z) dx \approx \|U\|_{L^2}^2.$$

By integrating over $t \in [0, T]$ and then taking the square-root of the resultant inequality, we get

$$\|U\|_{L_T^\infty(L^2)} + \sqrt{2\gamma} \|y\|_{L_T^2(L^2)} + \sqrt{2k} \|\theta_x\|_{L_T^2(L^2)} \leq \|U_0\|_{L^2} \tag{3.23}$$

for any of $T > 0$. Next, the dissipation rate of y, θ_x is obtained by frequency localization estimation in homogeneous Chemin-Lerner space. Applying the operator $\dot{\Delta}_q (q \in \mathbb{Z})$ to (3.20), we have

$$\begin{cases} \dot{\Delta}_q v_t - \dot{\Delta}_q u_x + \dot{\Delta}_q y = 0, \\ \dot{\Delta}_q u_t - \dot{\Delta}_q v_x = 0, \\ \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \sigma'(z) \dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma \dot{\Delta}_q y + \beta \dot{\Delta}_q \theta_x = [\dot{\Delta}_q, \sigma'(z)] z_x, \\ \dot{\Delta}_q \theta_t - k \dot{\Delta}_q \theta_{xx} + \beta \dot{\Delta}_q y_x = 0, \end{cases} \tag{3.24}$$

where the commutator is defined as $[f, g] := fg - gf$. Multiplying (3.24) with $\dot{\Delta}_q v, \dot{\Delta}_q u, \sigma'(z) \dot{\Delta}_q z_x, \dot{\Delta}_q y$ and $\dot{\Delta}_q \theta$, respectively, and then adding the resulting equations. We obtain

$$\partial_t \mathcal{H}^l + \partial_x \mathcal{F}^l + \gamma (\dot{\Delta}_q y)^2 + k (\dot{\Delta}_q \theta_x)^2 = \mathcal{R}^l, \tag{3.25}$$

where $z_t = y_x$, and

$$\begin{aligned} \mathcal{H}^l &:= \frac{1}{2} \left\{ (\dot{\Delta}_q v)^2 + (\dot{\Delta}_q u)^2 + \sigma'(z) (\dot{\Delta}_q z)^2 + (\dot{\Delta}_q y)^2 + (\dot{\Delta}_q \theta)^2 \right\}, \\ \mathcal{F}^l &:= \dot{\Delta}_q v \dot{\Delta}_q u + \dot{\Delta}_q y \dot{\Delta}_q \theta + \sigma'(z) (\dot{\Delta}_q z) (\dot{\Delta}_q y) + k \dot{\Delta}_q \theta_x \dot{\Delta}_q \theta, \\ \mathcal{R}^l &:= \frac{1}{2} \sigma'(z)_t (\dot{\Delta}_q z)^2 - \sigma'(z)_x (\dot{\Delta}_q z) (\dot{\Delta}_q y) + \dot{\Delta}_q y [\dot{\Delta}_q, \sigma'(z)] z_x. \end{aligned}$$

Further, integrating over x and with the aid of the Cauchy-Schwarz inequality yield

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} N_0[\dot{\Delta}_q U] + \gamma \|\dot{\Delta}_q y\|_{L^2}^2 + k \|\dot{\Delta}_q \theta_x\|_{L^2}^2 \\ &\lesssim \|\sigma'(z)_t\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2}^2 + \|\sigma'(z)_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2} \|\dot{\Delta}_q y\|_{L^2} + \|[\dot{\Delta}_q, \sigma'(z)] z_x\|_{L^2} \|\dot{\Delta}_q y\|_{L^2}, \end{aligned} \tag{3.26}$$

where

$$N_0[\dot{\Delta}_q U] = \|(\dot{\Delta}_q v, \dot{\Delta}_q u, \dot{\Delta}_q y, \dot{\Delta}_q \theta)\|_{L^2}^2 + \int_{\mathbb{R}} \sigma'(z) |\dot{\Delta}_q z|^2 dx \approx \|\dot{\Delta}_q U\|_{L^2}^2.$$

From the above (1.3) and the following a priori assumption (3.69), we have

$$\|\sigma'(z)\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2}^2 \lesssim \|z_t\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2}^2 \lesssim \|y_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2}^2. \tag{3.27}$$

Similarly,

$$\|\sigma'(z)\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2} \|\dot{\Delta}_q y\|_{L^2} \lesssim \|z_x\|_{L^\infty} \|\dot{\Delta}_q z\|_{L^2} \|\dot{\Delta}_q y\|_{L^2}. \tag{3.28}$$

Combining (3.27)-(3.28) and integrating over $t \in [0, T]$, with the aid of Young's inequality we are led to

$$\begin{aligned} & \sqrt{N_0} [\dot{\Delta}_q U] + \sqrt{2\gamma} \|\dot{\Delta}_q y\|_{L^2_T(L^2)} + \sqrt{2k} \|\dot{\Delta}_q \theta_x\|_{L^2_T(L^2)} \\ & \lesssim \sqrt{N_0} [\dot{\Delta}_q U_0] + \sqrt{\|(y_x, z_x)\|_{L^\infty_T(L^\infty)}} \left(\|\dot{\Delta}_q y\|_{L^2_T(L^2)} + \|\dot{\Delta}_q z\|_{L^2_T(L^2)} \right) \\ & \quad + \sqrt{\|[\dot{\Delta}_q, \sigma'(z)] z_x\|_{L^2_T(L^2)}} \|\dot{\Delta}_q y\|_{L^2_T(L^2)}. \end{aligned} \tag{3.29}$$

By the commutator estimate in Proposition 2.3, it holds

$$\|[\dot{\Delta}_q, \sigma'(z)] z_x\|_{L^2_T(L^2)} \lesssim c_q 2^{\frac{3q}{2}} \|z\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})} \|z_x\|_{L^2_T(\dot{B}^{1/2}_{2,1})}, \tag{3.30}$$

where $\{c_q\}$ denotes a sequence satisfying $\|c_q\|_{l^1} \leq 1$. Therefore, we get

$$\begin{aligned} & 2^{\frac{3q}{2}} \|\dot{\Delta}_q U\|_{L^2_T(L^2)} + \sqrt{2\gamma} 2^{\frac{3q}{2}} \|\dot{\Delta}_q y\|_{L^2_T(L^2)} + \sqrt{2k} 2^{\frac{3q}{2}} \|\dot{\Delta}_q \theta_x\|_{L^2_T(L^2)} \\ & \lesssim \|\dot{\Delta}_q U_0\|_{L^2} + c_q \sqrt{\|(y_x, z_x)\|_{L^\infty_T(\dot{B}^{1/2}_{2,1})}} \left(\|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|z_x\|_{L^2_T(\dot{B}^{1/2}_{2,1})} \right) \\ & \quad + c_q \sqrt{\|z\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}} \left(\|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \|z_x\|_{L^2_T(\dot{B}^{1/2}_{2,1})} \right) \end{aligned} \tag{3.31}$$

Here, we would like to point out that each $\{c_q\}$ may have a different form in the (3.31) equation or the inequality that emerges after, but the bound of $\|c_q\|_{l^1} \leq 1$ is well satisfied. Thus, summing over $q \in \mathbb{Z}$, we have

$$\begin{aligned} & \|U\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \sqrt{2\gamma} \|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \sqrt{2k} \|\theta_x\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \\ & \lesssim \|U_0\|_{(\dot{B}^{3/2}_{2,1})} + \sqrt{\|(y, z)\|_{L^\infty_T(\dot{B}^{3/2}_{2,1})}} \left(\|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|z_x\|_{L^2_T(\dot{B}^{1/2}_{2,1})} \right). \end{aligned} \tag{3.32}$$

Finally, combining (3.23) and (3.32), we conclude from Proposition 2.2 that

$$N(T) + \|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|\theta_x\|_{L^2_T(\dot{B}^{3/2}_{2,1})} \lesssim \|U_0\|_{\dot{B}^{3/2}_{2,1}} + \sqrt{N(T)} D(T). \tag{3.33}$$

Thus, the proof of Lemma 3.1 is completed.

Lemma 3.2. (The dissipation for v) If $U \in \tilde{C}_T(\dot{B}^{3/2}_{2,1}) \cap \tilde{C}^1_T(\dot{B}^{1/2}_{2,1})$ is a solution of (1.3) for any $T > 0$, then

$$\|v\|_{L^2_T(\dot{B}^{1/2}_{2,1})} \lesssim N(T) + \|U_0\|_{\dot{B}^{3/2}_{2,1}} + \|y\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \|\theta_x\|_{L^2_T(\dot{B}^{3/2}_{2,1})} + \sqrt{N(T)} D(T). \tag{3.34}$$

Proof. The system (1.3) can be rewritten in the following form:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - y_x = 0, \\ y_t - z_x - v + \gamma y + \beta \theta_x = g(z)_x, \\ \theta_t - k \theta_{xx} + \beta y_x = 0, \end{cases} \tag{3.35}$$

where the smooth function $g(z)$ is defined as

$$g(z) = \sigma(z) - \sigma(0) - z = O(z^2)$$

and satisfies $g(0) = 0$ and $g'(0) = 0$. Multiplying the first equation in (3.35) by y , the second by z , the third by u , and the fourth by v , respectively, and then adding the resulting equations together give

$$\frac{d}{dt} N_1(U) + \|v\|_{L^2}^2 \leq \|y\|_{L^2}^2 + \gamma \|y\|_{L^2} \|v\|_{L^2} + \beta \|v\|_{L^2} \|\theta_x\|_{L^2} + \|g(z)\|_{L^2} \|v\|_{L^2}, \tag{3.36}$$

where

$$N_1(U) = -\int_{\mathbb{R}} (vy + uz) dx.$$

Using the Young's inequality, we can conclude that

$$\frac{d}{dt} N_1(U) + \frac{1}{2} \|v\|_{L^2}^2 \lesssim \|y\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 + \|z\|_{L^\infty} \|z_x\|_{L^2} \|v\|_{L^2}. \tag{3.37}$$

Integration of (3.37) in $t \in [0, T]$ leads to

$$\begin{aligned} \|v\|_{L^2_t(L^2)}^2 &\lesssim (|N_1(U)| + |N_1(U_0)|) + \|y\|_{L^2_t(L^2)}^2 + \|\theta_x\|_{L^2_t(L^2)}^2 + \|z\|_{L^\infty_t(L^2)} \|z_x\|_{L^2_t(L^2)} \|v\|_{L^2_t(L^2)} \\ &\lesssim N(T)^2 + \|U_0\|_{B^2_{x,t}}^2 + \|y\|_{L^2_t(L^2)}^2 + \|\theta_x\|_{L^2_t(L^2)}^2 + N(T)D(T)^2, \end{aligned} \tag{3.38}$$

for any $T > 0$, here we utilize the embedding property in Lemma 2.2. Then, by Young's inequality again, we arrive at

$$\|v\|_{L^2_t(L^2)} \lesssim N(T) + \|U_0\|_{B^2_{x,t}} + \|y\|_{L^2_t(L^2)} + \|\theta_x\|_{L^2_t(L^2)} + \sqrt{N(T)}D(T). \tag{3.39}$$

Next, Applying the operator $\dot{\Delta}_q$ ($q \in \mathbb{Z}$) to (3.35) yields

$$\begin{cases} \dot{\Delta}_q v_t - \dot{\Delta}_q u_x + \dot{\Delta}_q y = 0, \\ \dot{\Delta}_q u_t - \dot{\Delta}_q v_x = 0, \\ \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma \dot{\Delta}_q y + \beta \dot{\Delta}_q \theta_x = \dot{\Delta}_q g(z)_x, \\ \dot{\Delta}_q \theta_t - k \dot{\Delta}_q \theta_{xx} + \beta \dot{\Delta}_q y_x = 0. \end{cases} \tag{3.40}$$

Multiplying the first equation in (3.40) by $-\dot{\Delta}_q y$, the second by $-\dot{\Delta}_q z$, the third by $-\dot{\Delta}_q u$, and the fourth by $-\dot{\Delta}_q v$, respectively, and then adding the resulting equations. we get

$$-\partial_t \mathcal{H}_1^l + \partial_x \mathcal{F}_1^l + (\dot{\Delta}_q v)^2 - (\dot{\Delta}_q y)^2 = \mathcal{R}_1^l, \tag{3.41}$$

where

$$\begin{aligned} \mathcal{H}_1^l &:= \dot{\Delta}_q v \dot{\Delta}_q y + \dot{\Delta}_q u \dot{\Delta}_q z, \\ \mathcal{F}_1^l &:= \dot{\Delta}_q v \dot{\Delta}_q z + \dot{\Delta}_q u \dot{\Delta}_q y, \\ \mathcal{R}_1^l &:= \gamma \dot{\Delta}_q v \dot{\Delta}_q y + \beta \dot{\Delta}_q v \dot{\Delta}_q \theta_x - \dot{\Delta}_q v \dot{\Delta}_q g(z)_x. \end{aligned}$$

Using the Hölder inequality and Young’s inequality gives

$$\frac{d}{dt} N_1[\dot{\Delta}_q U] + \frac{1}{2} \|\dot{\Delta}_q v\|_{L^2}^2 \lesssim \|\dot{\Delta}_q y\|_{L^2}^2 + \|\dot{\Delta}_q \theta_x\|_{L^2}^2 + \|\dot{\Delta}_q g(z)_x\|_{L^2} \|\dot{\Delta}_q v\|_{L^2}, \quad (3.42)$$

where

$$N_1[\dot{\Delta}_q U] = \int_{\mathbb{R}} (\dot{\Delta}_q v \dot{\Delta}_q y + \dot{\Delta}_q u \dot{\Delta}_q z) dx.$$

By integrating over $t \in [0, T]$, we have

$$\begin{aligned} \|\dot{\Delta}_q v\|_{L^2_T(L^2)}^2 &\lesssim \|\dot{\Delta}_q U\|_{L^2_T(L^2)}^2 + \|\dot{\Delta}_q U_0\|_{L^2}^2 + \|\dot{\Delta}_q y\|_{L^2_T(L^2)}^2 \\ &\quad + \|\dot{\Delta}_q \theta_x\|_{L^2_T(L^2)}^2 + \|\dot{\Delta}_q v\|_{L^2_T(L^2)} \|\dot{\Delta}_q g(z)_x\|_{L^2_T(L^2)}. \end{aligned} \quad (3.43)$$

In addition, Young’s inequality allows us to obtain

$$\begin{aligned} 2^{\frac{q}{2}} \|\dot{\Delta}_q v\|_{L^2_T(L^2)} &\lesssim c_q \|U\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + c_q \|U_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + c_q \|y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\quad + c_q \|\theta_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + c_q \sqrt{\|v\|_{\dot{B}^{\frac{3}{2}}_{2,1}}} \|g(z)_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}}, \end{aligned} \quad (3.44)$$

where the norm of $g(z)_x$ at the right end of (3.44) can be estimated by applying Lemma 2.2 and Proposition 2.4

$$\|g(z)_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \int_0^T \|g(z)\|_{\dot{B}^{\frac{3}{2}}_{2,1}} dt \lesssim \int_0^T \|z\|_{\dot{B}^{\frac{3}{2}}_{2,1}}^2 dt \lesssim \|z_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}}^2. \quad (3.45)$$

Thus, combining (3.44)-(3.45) and summing $q \in \mathbb{Z}$, we get

$$\begin{aligned} \|v\|_{\dot{B}^{\frac{3}{2}}_{2,1}} &\lesssim \|U\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|U_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\quad + \sqrt{\|v\|_{\dot{B}^{\frac{3}{2}}_{2,1}}} \|z_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\theta_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}}. \end{aligned} \quad (3.46)$$

Finally, note that (3.39) and (3.46), by proposition 2.1, we obtain

$$\|v\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim N(T) + \|U_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|\theta_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \sqrt{N(T)} D(T). \quad (3.47)$$

This proves Lemma 3.2.

Lemma 3.3 (The dissipation for z_x) If $U \in \tilde{C}_T(\dot{B}^{\frac{3}{2}}_{2,1}) \cap \tilde{C}_T^1(\dot{B}^{\frac{1}{2}}_{2,1})$ is a solution of (1.3) for any $T > 0$, then

$$\begin{aligned} \|z_x\|_{\dot{B}^{\frac{1}{2}}_{2,1}} &\lesssim N(T) + \|U_0\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|y\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \|v\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \\ &\quad + \|\theta_x\|_{\dot{B}^{\frac{3}{2}}_{2,1}} + \sqrt{N(T)} D(T). \end{aligned} \quad (3.48)$$

Proof. Multiplying the third equation of (3.35) by y_x and the fourth equation by $-z_x$, respectively, and then adding up the resulting equations, we have

$$-\frac{d}{dt} (z_x y) + \partial_x (y z_t) + (z_x)^2 - (y_x)^2 = \mathcal{R}_2^l, \quad (3.49)$$

where

$$\mathcal{R}_2^l := z_x (\gamma y + \beta \theta_x - v) - z_x g(z)_x.$$

Integrating (3.49) over \mathbb{R} , we get

$$\begin{aligned} \|z_x\|_{L^2}^2 &\leq \|y_x\|_{L^2}^2 + \frac{d}{dt} \int_{\mathbb{R}} z_x y dx + \int_{\mathbb{R}} |\mathcal{R}_2'| dx, \\ \int_{\mathbb{R}} |\mathcal{R}_2'| dx &\leq \epsilon \|z_x\|_{L^2}^2 + C_\epsilon \|(v, y, \theta_x)\|_{L^2}^2 + C \|z\|_{L^\infty} \|z_x\|_{L^2}^2. \end{aligned}$$

That is

$$\frac{d}{dt} N_2(U) + \|z_x\|_{L^2}^2 \lesssim \|y_x\|_{L^2}^2 + (\|v\|_{L^2}^2 + \|y\|_{L^2}^2 + \|\theta_x\|_{L^2}^2) + \|z\|_{L^\infty} \|z_x\|_{L^2}^2, \quad (3.50)$$

where

$$N_2(U) := - \int_{\mathbb{R}} z_x y dx.$$

Similar to the process for (3.39), we obtain

$$\begin{aligned} \|z_x\|_{L_T^2(L^2)} &\lesssim N(T) + \|U_0\|_{B_{2,1}^{3/2}} + \|y\|_{L_T^2(B_{2,1}^{3/2})} + \|v\|_{L_T^2(B_{2,1}^{1/2})} \\ &\quad + \|\theta_x\|_{L_T^2(B_{2,1}^{1/2})} + \sqrt{N(T)} D(T). \end{aligned} \quad (3.51)$$

Furthermore, by (3.40), it holds

$$\begin{cases} \dot{\Delta}_q z_t - \dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - \dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma \dot{\Delta}_q y + \beta \dot{\Delta}_q \theta_x = \dot{\Delta}_q g(z)_x. \end{cases} \quad (3.52)$$

The equations (3.52) are multiplied by $\dot{\Delta}_q y_x$ and $-\dot{\Delta}_q z_x$, respectively, and then utilizing the energy estimates on each block, we get

$$\begin{aligned} 2^{\frac{q}{2}} \|\dot{\Delta}_q z_x\|_{L_T^2(L^2)} &\lesssim c_q \left(\|U\|_{L_T^\infty(B_{2,1}^{3/2})} + \|U_0\|_{B_{2,1}^{3/2}} \right) + c_q \|y_x\|_{L_T^2(B_{2,1}^{1/2})} + c_q \epsilon \|z_x\|_{L_T^2(B_{2,1}^{1/2})} \\ &\quad + c_q C_\epsilon \left(\|v\|_{L_T^2(B_{2,1}^{1/2})} + \|y\|_{L_T^2(B_{2,1}^{1/2})} + \|\theta_x\|_{L_T^2(B_{2,1}^{1/2})} \right) \\ &\quad + c_q \sqrt{\|z_x\|_{L_T^\infty(B_{2,1}^{1/2})}} \|g(z)_x\|_{L_T^2(B_{2,1}^{1/2})}^{\frac{1}{2}}. \end{aligned} \quad (3.53)$$

In addition, similar to the processes of (3.45)-(3.46), we arrive at

$$\begin{aligned} \|z_x\|_{L_T^2(B_{2,1}^{1/2})} &\lesssim \|U\|_{L_T^\infty(B_{2,1}^{3/2})} + \|U_0\|_{B_{2,1}^{3/2}} + \|y\|_{L_T^2(B_{2,1}^{3/2})} + \|v\|_{L_T^2(B_{2,1}^{1/2})} \\ &\quad + \|y\|_{L_T^2(B_{2,1}^{1/2})} + \|\theta_x\|_{L_T^2(B_{2,1}^{3/2})} + c_q \sqrt{\|z\|_{L_T^\infty(B_{2,1}^{3/2})}} \|z_x\|_{L_T^2(B_{2,1}^{1/2})}, \end{aligned} \quad (3.54)$$

where choosing $0 < \epsilon \leq \frac{1}{2}$.

Eventually, combining (3.51) and (3.54) gives Lemma 3.3.

Lemma 3.4. (The dissipation for u_x) If $U \in \tilde{C}_T(B_{2,1}^{3/2}) \cap \tilde{C}_T^1(B_{2,1}^{1/2})$ is a solution of (1.3) for any $T > 0$, then

$$\|u_x\|_{L_T^2(B_{2,1}^{-1/2})} \lesssim N(T) + \|U_0\|_{B_{2,1}^{3/2}} + \|v\|_{L_T^2(B_{2,1}^{1/2})} + \|y\|_{L_T^2(B_{2,1}^{3/2})}. \quad (3.55)$$

Proof. Applying the nonhomogeneous operator $\Delta_q (q \geq -1)$ to the first and second equations of (3.35) yields

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0. \end{cases} \quad (3.56)$$

Multiplying (3.56) by $-\Delta_q u_x$ and $\Delta_q v_x$, we get

$$\frac{d}{dt} N_2 [\Delta_q U] + \|\Delta_q u_x\|_{L^2}^2 \leq \|\Delta_q v_x\|_{L^2}^2 + \|\Delta_q u_x\|_{L^2} \|\Delta_q y\|_{L^2}, \tag{3.57}$$

where

$$N_2 [\Delta_q U] := -\int_{\mathbb{R}} \Delta_q v \Delta_q u_x dx.$$

Further, integration of (3.57) with respect to $t \in [0, T]$ leads to

$$\begin{aligned} \|\Delta_q u_x\|_{L^2_t(L^2)}^2 &\leq |N_2 [\Delta_q U]| + N_2 [\Delta_q U_0] + \|\Delta_q v_x\|_{L^2_t(L^2)}^2 \\ &\quad + \|\Delta_q u_x\|_{L^2_t(L^2)} \|\Delta_q y\|_{L^2_t(L^2)}. \end{aligned} \tag{3.58}$$

With the aid of Young's inequality and the embedding property in Lemma 2.2, we obtain

$$\begin{aligned} 2^{-\frac{q}{2}} \|\Delta_q u_x\|_{L^2_t(L^2)} &\lesssim c_q N(T) + c_q \|U_0\|_{B_{2,1}^{3/2}} + c_q \|v\|_{L^2_t(B_{2,1}^{1/2})} \\ &\quad + c_q \sqrt{\|u_x\|_{L^2_t(B_{2,1}^{-1/2})}} \|y\|_{L^2_t(B_{2,1}^{3/2})}. \end{aligned} \tag{3.59}$$

Thus, we finish the proof of Lemma 3.4.

The case of $a \neq 1$: The dissipation with respect to y, θ_x, z_x, u_x is the same as in the case of equal waves ($a = 1$). Here we only need to show the dissipation with respect to v in the case of non-equal waves ($a \neq 1$). It is needed to use proposition 2.1 for the topological relation between $\|u_x\|_{L^2_t(\dot{B}_{2,1}^{-1/2})}$ and $\|u_x\|_{L^2_t(B_{2,1}^{-1/2})}$ to obtain the dissipative estimate of v .

Lemma 3.5. (The dissipation for v) *If $U \in \tilde{C}_T(B_{2,1}^{3/2}) \cap \tilde{C}_T^1(B_{2,1}^{1/2})$ is a solution of (1.3) for any $T > 0$, then*

$$\begin{aligned} \|v\|_{L^2_t(B_{2,1}^{1/2})} &\lesssim N(T) + \|U_0\|_{B_{2,1}^{3/2}} + \varepsilon \|u_x\|_{L^2_t(B_{2,1}^{-1/2})} + (1 + C_\varepsilon) \|y\|_{L^2_t(B_{2,1}^{3/2})} \\ &\quad + C_{\varepsilon_1} \|\theta_x\|_{L^2_t(B_{2,1}^{3/2})} + N(T)D(T), \end{aligned} \tag{3.60}$$

for any $\varepsilon, \varepsilon_1 > 0$, $C(\varepsilon), C(\varepsilon_1)$ is a positive constant here depending on $\varepsilon, \varepsilon_1 > 0$.

Proof. Rewrite the system (1.3) in the following form:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y + \beta \theta_x = g(z)_x, \\ \theta_x - k\theta_{xx} + \beta y_x = 0, \end{cases} \tag{3.61}$$

where the smooth function $g(z)$ is defined by

$$g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$$

satisfying $g(0) = 0$ and $g'(0) = 0$.

First, applying the non-homogeneous frequency localization operator Δ_q ($q \geq -1$) to (3.61) leads to

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0, \\ \Delta_q z_t - a \Delta_q y_x = 0, \\ \Delta_q y_t - a \Delta_q z_x - \Delta_q v + \gamma \Delta_q y + \beta \Delta_q \theta_x = \Delta_q g(z)_x, \\ \Delta_q \theta_x - k \Delta_q \theta_{xx} + \Delta_q \beta y_x = 0. \end{cases} \tag{3.62}$$

Subsequently, multiplying the first equation in (3.62) by $-\Delta_q v$, the second equation by $-a \Delta_q z$, the third equation by $-a \Delta_q u$, and the fourth equation by $-\Delta_q v$, and then adding up the resulting equations, we have

$$-\Delta_t \mathcal{H}_3^l + \partial_x \mathcal{F}_3^l + (\Delta_q v)^2 - (\Delta_q y)^2 = \mathcal{R}_3^l, \tag{3.63}$$

where

$$\begin{aligned} \mathcal{H}_3^l &:= \Delta_q v \Delta_q y + a \Delta_q u \Delta_q z, \\ \mathcal{F}_3^l &:= a \Delta_q v \Delta_q z + a^2 \Delta_q u \Delta_q y, \\ \mathcal{R}_3^l &:= \gamma \Delta_q v \Delta_q y + \beta \Delta_q v \Delta_q \theta_x + (a^2 - 1) \Delta_q u_x \Delta_q y - \Delta_q v \Delta_q g(z)_x. \end{aligned}$$

Integrating (3.63) over $x \in \mathbb{R}$, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \frac{d}{dt} N_3[\Delta_q U] + \frac{1}{2} \|\Delta_q v\|_{L^2}^2 \\ \lesssim \|\Delta_q y\|_{L^2}^2 + \|\Delta_q v\|_{L^2} \|\Delta_q \theta_x\|_{L^2} + |a^2 - 1| \|\Delta_q u_x\|_{L^2} \|\Delta_q y\|_{L^2} + \|\Delta_q v\|_{L^2} \|\Delta_q g(z)_x\|_{L^2}, \end{aligned} \tag{3.64}$$

where

$$N_3[\Delta_q U] := -\int \Delta_q v \Delta_q y + a \Delta_q u \Delta_q z.$$

By integrating over $t \in [0, T]$, we arrive at

$$\begin{aligned} \|\Delta_q v\|_{L_t^2(L^2)}^2 &\lesssim \|\Delta_q U\|_{L_T^\infty(L^2)}^2 + \|\Delta_q U_0\|_{L^2}^2 + \|\Delta_q y\|_{L_T^2(L^2)}^2 + \|\Delta_q v\|_{L_T^2(L^2)} \|\Delta_q \theta_x\|_{L_T^2(L^2)} \\ &\quad + \|\Delta_q y\|_{L_T^2(L^2)} \|\Delta_q u_x\|_{L_T^2(L^2)} + \|\Delta_q g(z)_x\|_{L_T^2(L^2)}, \end{aligned} \tag{3.65}$$

where noting the case of $a \neq 1$. In addition, utilizing the Young's inequality gives

$$\begin{aligned} 2^{\frac{q}{2}} \|\Delta_q v\|_{L_T^2(L^2)} &\lesssim c_q \|U\|_{\tilde{L}_T^\infty(B_{2,1}^{1/2})} + c_q \|U_0\|_{B_{2,1}^{1/2}} + \varepsilon c_q \|u_x\|_{\tilde{L}_T^2(B_{2,1}^{-1/2})} \\ &\quad + c_q (1 + C_\varepsilon) \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + C_{\varepsilon_1} c_q \|\theta_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} + c_q \|g(z)_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})}, \end{aligned} \tag{3.66}$$

where

$$2^q \|\Delta_q v\|_{L_T^2(L^2)} \|\Delta_q \theta_x\|_{L_T^2(L^2)} \leq \varepsilon_1 \|v\|_{\tilde{L}_T^2(B_{2,1}^{1/2})}^2 + C_{\varepsilon_1} \|\theta_x\|_{L_T^2(B_{2,1}^{1/2})}^2,$$

for $\varepsilon, \varepsilon_1 > 0$, where $C_\varepsilon, C_{\varepsilon_1}$ are position constants dependent on $\varepsilon, \varepsilon_1$. Knowing the fact $g'(0) = 0$, it follows from Proposition 2.5 and Proposition 2.6 that

$$\begin{aligned} \|g(z)_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} &= \|g'(z)z_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} \\ &\lesssim \|g'(z) - g'(0)\|_{\tilde{L}_T^\infty(B_{2,1}^{1/2})} \|z_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} \\ &\lesssim \|z\|_{\tilde{L}_T^\infty(B_{2,1}^{1/2})} \|z_x\|_{\tilde{L}_T^2(B_{2,1}^{1/2})}. \end{aligned} \tag{3.67}$$

Combining the above estimations (3.66)-(3.67), and summing over $q \geq -1$, we obtain

$$\begin{aligned} \|v\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} &\lesssim \|U\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + \|U_0\|_{B_{2,1}^{3/2}} + \varepsilon \|u_x\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + (1 + C_\varepsilon) \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} \\ &\quad + C_{\varepsilon_1} \|\theta_x\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + \|z\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} \|z_x\|_{\tilde{L}_T^2(B_{2,1}^{3/2})}. \end{aligned} \quad (3.68)$$

This gives (3.60).

From the above Lemma 3.1-Lemma 3.5, the following Proposition 3.2 holds.

Proposition 3.2. *Suppose that for any $T > 0$, $U \in \tilde{C}_T^1(B_{2,1}^{3/2}) \cap \tilde{C}_T^1(B_{2,1}^{1/2})$ is a solution of (1.3), and there exists $\delta_1 > 0$ such that when*

$$N(T) \leq \delta_1, \quad (3.69)$$

then the following estimate holds

$$N(T) + D(T) \lesssim \|U_0\|_{B_{2,1}^{3/2}} + (N(T) + \sqrt{N(T)})D(T). \quad (3.70)$$

Therefore, the following inequality holds

$$N(T) + D(T) \lesssim \|U_0\|_{B_{2,1}^{3/2}}. \quad (3.71)$$

By using the standard boot-strap argument, the proof of Theorem 1.1 is similar to the process in [20], and we omit the details here for the sake of brevity.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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