Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(8), 2700-2711
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Proving a Special Case of the Coxeter-Hadwiger Conjecture

István Lénárt

Faculty of Science, Eötvös Loránd University, Budapest, Hungary
Email: ilenart@cs.elte.hu, h12572len@ella.hu

How to cite this paper: Lénárt, I. (2025) Proving a Special Case of the Coxeter-Hadwiger Conjecture. *Journal of Applied Mathematics and Physics*, **13**, 2700-2711.
https://doi.org/10.4236/jamp.2025.138154

Received: July 14, 2025
Accepted: August 22, 2025
Published: August 25, 2025

Abstract

An orthoscheme or Pythagorean simplex is a solid in n-dimensional Euclidean space whose faces are right triangles. In 1956, Hadwiger asked whether an n-dimensional general (not necessarily Pythagorean) simplex can always be decomposed into a finite number of Pythagorean simplexes. Tschirpke proved in 1994 that this division is always possible in 5D space. Coxeter proved that a 3D Pythagorean simplex can be split into three smaller ones. In a 2024 paper, I generalized Coxeter's trisection to prove that the dissection of an n-dimensional Pythagorean simplex into n pieces of the same type is possible if each leg of the original solid is equal to the unit distance. In the present paper, I extend this proof to an n-dimensional Pythagorean simplex with legs of arbitrary measure. This means the proof of the Hadwiger conjecture in the special case of a Pythagorean simplex.

Keywords

Pythagorean Simplex, Coxeter Partition of a Pythagorean Simplex in n-Dimensional Space

1. Pythagorean Simplexes in n-Dimensional Euclidean Spaces

This article is an extension of [1]. I try to make it understandable in itself, but I ask the Reader to consult the previous article for preliminary details, which I do not wish to repeat in full here.

1.1. Definition of a Pythagorean Simplex

The literature on n-dimensional geometry usually treats the general triangle as the 2D simplex and the general tetrahedron as the 3D simplex. Since the mid-19th century, however, researchers have recognized the importance of the 2D right triangle and the 3D quadrirectangular tetrahedron as possible alternatives to the definition of the simplex in the respective dimensions [2].

Definition. Consider an ordered chain of straight line segments $a_1, a_2, \cdots, a_n$ in an *n*-dimensional Euclidean space where the starting point of segment $a_i\,(i > 1)$ coincides with the endpoint of segment a_{i-1}, and segment a_i is perpendicular to the (*i* − 1)-dimensional space formed by the preceding segments $a_1, a_2, \cdots, a_{i-1}$. Tschirpke calls this series of segments a totally orthogonal edge path [3].

Define segments $a_1, a_2, \cdots, a_n$ the **legs** (catheti) of the solid, and segment a_{n+1} the **hypotenuse** connecting the endpoint of segment a_n with the starting point of segment a_1.

The extension of the Pythagorean Theorem applies for this *n*-dimensional shape, namely, for the hypotenuse, $a_{n+1}^2 = a_1^2 + a_2^2 + \cdots + a_n^2 = \sum_1^n a_i^2$, and for the volume, $V = \frac{1}{n!} a_1 a_2 a_3 \cdots a_n = \frac{\prod_1^n a_i}{n!}$.

The *n*-dimensional hypotenuse formula is a direct generalization of the 2D Pythagorean Theorem.

The volume formula can be deduced from the Cayley-Menger determinant which uses the edges of a tetrahedron to calculate its volume. I demonstrate the method for a 3D Pythagorean tetrahedron each face of which is a right triangle (**Figure 1**). Let us denote the length of the three legs as a, b, c, then the Cayley-Menger determinant gives the following result:

$$\left|D_{3,CM}\right| = \frac{2^3 (3!)^2}{(-1)^{3+1}} V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & a^2 & 0 & b^2 & b^2+c^2 \\ 1 & a^2+b^2 & b^2 & 0 & c^2 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & c^2 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & a^2 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & 0 & b^2 & b^2+c^2 \\ 1 & b^2 & 0 & c^2 \\ 1 & b^2+c^2 & c^2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & a^2 & b^2 & b^2+c^2 \\ 1 & a^2+b^2 & 0 & c^2 \\ 1 & a^2+b^2+c^2 & c^2 & 0 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 0 & a^2 & a^2+b^2+c^2 \\ 1 & a^2 & 0 & b^2+c^2 \\ 1 & a^2+b^2 & b^2 & c^2 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & a^2 & a^2+b^2 \\ 1 & a^2 & 0 & b^2 \\ 1 & a^2+b^2 & b^2 & 0 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & c^2 \end{vmatrix}$$

$$= 4 \cdot a^2 \cdot b^2 c^2 + 4 \cdot a^2 \cdot b^2 c^2 + 0 + 0 = 2^3 \cdot a^2 \cdot b^2 \cdot c^2$$

$$V = \frac{1}{3!} abc$$

This formula can be directly generalized using the Cayley-Menger determinant in higher dimensions. For example, a 4D simplex of this type with legs a, b, c, d has volume $V = \frac{1}{4!} abcd$.

These hypotenuse and volume formulas play a key role in the following proofs.

The n-dimensional shape has been called by various names, such as a pyramid, an ortoscheme, Orthogonalsimplex, or in the 3D case, a quadrirectangular tetrahedron.

I propose the name **a Pythagorean n-simplex** for the n-dimensional shape with n legs and one hypotenuse.

In this paper, I shorten this name to a **Pyth simplex**, as for example, a Pyth triangle (right triangle), Pyth tetrahedron (quadrirectangular tetrahedron), Pyth pentachoron, etc.

1.2. Representing Multidimensional Objects

Several methods have been developed to visualize multidimensional objects on the flat sheet (see for example [3] or [4]).

The traditional right angle symbols seemed confusing to me in the drawings, so I chose a different marking method. A right angle is denoted here as an arc in the angle region terminating with two dots on the corresponding sides, regardless of the apparent length of the arc or the apparent position of the lines in the drawing. Figure 1 shows the picture of a 3D model of a Pyth tetrahedron, while Figure 2 displays the representation of the same model in this article.

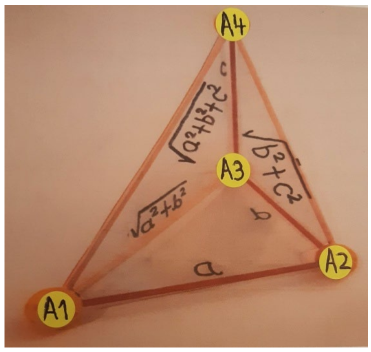

Figure 1. A Pythagorean tetrahedron, every face of which is a right triangle.

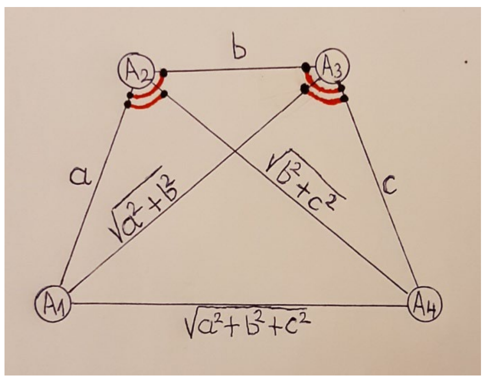

Figure 2. A Pythagorean tetrahedron displayed in this article (see explanation of marking below).

The points of intersection that really exist in the n-dimensional solid are only those which are marked with letters/numbers in the figure (as in graph theory). Lines that intersect in the flat figure without marking the point of intersection are skew lines, as A_1A_3 and A_2A_4 in Figure 2.

To better understand the essence of the geometric structure, the actual length of the segment and the measure of the angle are distorted in most of the drawings. Therefore, I ask the Reader to rely on the notation and legend of the lines and angles, rather than the apparent position of the visual image.

1.3. Distribution of Right Angles in a Pyth *n*-Simplex

As seen in Figure 2 or Figure 3, if we specify the length of the legs and the location of right angles in the simplex, the length of all other sides can be calculated. The recursive formula for the number of right triangles in an *n*-dimensional Pyth simplex is: $s_n = s_{n-1} + \frac{(n-1)n}{2}$, ($n \geq 2$, $s_1 = 0$).

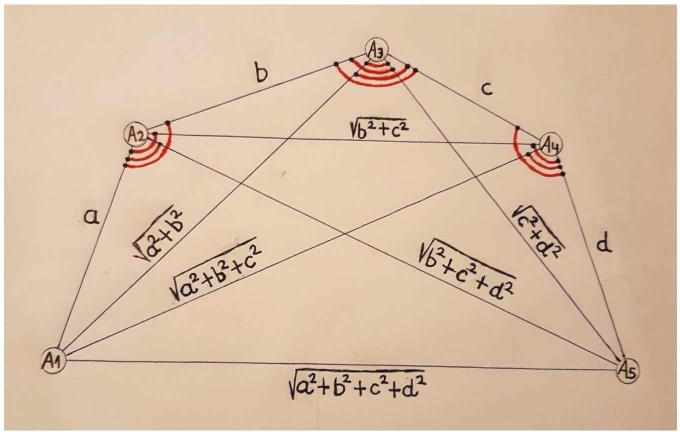

Figure 3. Pyth simplex in 4D.

2. The Coxeter Partition as a Tool to Prove a Special Case of the Hadwiger Conjecture

Coxeter proved that a 3D Pythagorean simplex (Pyth tetrahedron) can be cut into three smaller 3D Pyth simplexes [5].

Hadwiger proposed the conjecture in 1956 that any *n*-dimensional general simplex can be decomposed into a finite number of Pyth simplexes of the same dimension [6]. Tschirpke proved this conjecture to 5D in 1994 [4].

In what follows, I use the generalization of the Coxeter trisection to prove a special case of the Hadwiger conjecture, namely, the partition of an *n*-dimensional Pyth simplex into *n* pieces of smaller Pyth simplexes of the same dimension.

It is of course possible that other types of partitions also exist, and the conjecture can be proven through other partitions.

2.1. The Coxeter Partition in 2D Space: A 2D Simplex Partitioned in Two 2D Simplexes

Consider a Pyth triangle $(A_1A_2A_3)$. Drop a perpendicular from vertex A_2 to side (A_1A_3), mark the foot B_1, and get two smaller triangles $(A_2B_1A_1)$ and $(A_2B_1A_3)$ (Figure 4).

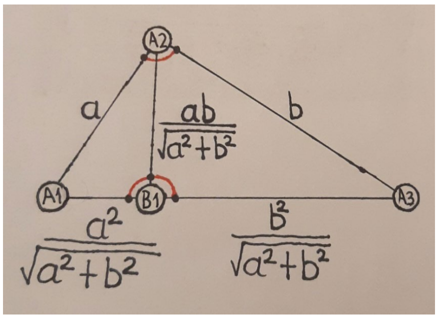

Figure 4. Coxeter partition in 2D.

The volumes of the two smaller Pyth 2D simplexes (Pyth triangles) add up to the volume of the original Pyth triangle:

$$V_1\left(A_2B_1A_1\right)+V_2\left(A_2B_1A_3\right)=\frac{1}{2!}\frac{a^2}{\sqrt{a^2+b^2}}\frac{ab}{\sqrt{a^2+b^2}}+\frac{1}{2!}\frac{b^2}{\sqrt{a^2+b^2}}\frac{ab}{\sqrt{a^2+b^2}}$$
$$=\frac{1}{2!}ab\frac{a^2+b^2}{a^2+b^2}=\frac{1}{2!}ab=V\left(A_2A_3A_1\right)$$

This result completes the proof of the Coxeter partition for a Pyth simplex in 2D space.

Remark: In what follows, I will make frequent use of the fact that the non-trivial altitude of a 2D right triangle with legs a,b is equal to $\frac{ab}{\sqrt{a^2+b^2}}$, and the hypotenuse is divided by the foot into two segments $\frac{a^2}{\sqrt{a^2+b^2}}$ and $\frac{b^2}{\sqrt{a^2+b^2}}$ respectively.

2.2. The Coxeter Partition in 3D Space: A 3D Simplex Partitioned in Three 3D Simplexes

Consider a Pyth tetrahedron $\left(A_1A_2A_3A_4\right)$. Drop a perpendicular from vertex A_2 to edge $\left(A_1A_3\right)$, mark the foot B_1. Erect a perpendicular from B_1 to edge $\left(A_1A_4\right)$, mark the foot B_2. Connect B_2 and A_2, and get a tetrahedron $\left(A_2B_1B_2A_1\right)$. Connect B_1 and A_4, and get two further tetrahedra $\left(A_2B_1B_2A_4\right)$ and $\left(A_2B_1A_3A_4\right)$ (**Figure 5** and **Figure 6**).

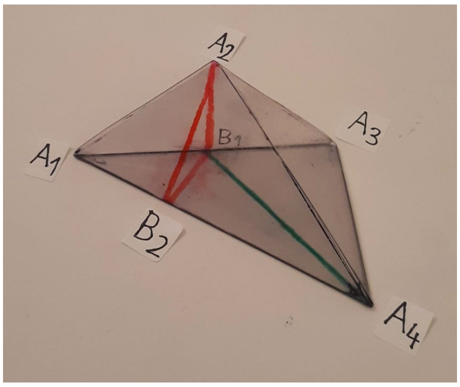

Figure 5. A 3D model of the Coxeter partition of a 3D Pyth tetrahedron.

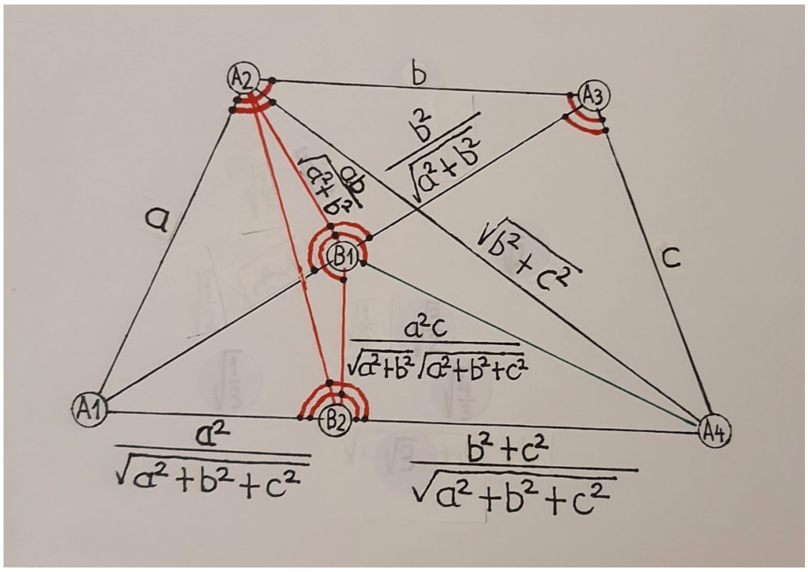

Figure 6. Flat sheet representation of the Coxeter partition of a 3D Pyth tetrahedron.

The triangle $(A_2B_1B_2)$ is a right triangle with the right angle at vertex (B_1), because the plane $(A_1A_3A_4)$ is perpendicular to the plane $(A_1A_2A_4)$ in the original Pyth tetrahedron $(A_1A_2A_3A_4)$. Therefore, the line (A_2B_1) in the plane $(A_1A_2A_4)$ is perpendicular to the lines $(B_1B_2),(B_1A_4)$ in the plane $(A_1A_3A_4)$. This gives that the three smaller tetrahedra $(A_2B_1B_2A_1),(A_2B_1B_2A_4),(A_2B_1A_3A_4)$ are also Pyth 3D simplexes, because each face is a right triangle.

There are 12 right triangles in **Figure 6**, namely,

$$(A_1A_2A_3),(A_2A_3A_4),(A_1A_2A_4),(A_1A_3A_4),(A_1B_1A_2),(A_2B_1A_3),$$
$$(A_2B_1B_2),(A_2B_1A_4),(A_1B_2A_2),(A_1B_2B_1),(A_2B_2A_4),(B_1B_2A_4),$$

with the right angle vertex at the middle in each triplet.

Let $(A_1A_2)=a,(A_2A_3)=b,(A_3A_4)=c,(A_1A_4)=\sqrt{a^2+b^2+c^2}$,

then $(A_2B_1)=\dfrac{ab}{\sqrt{a^2+b^2}}$, because (A_2B_1) is the non-trivial altitude of right triangle $(A_1A_2A_3)$;

$(A_2B_2)=\dfrac{a\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$, because (A_2B_2) is the non-trivial altitude of right triangle $(A_1A_2A_4)$;

$$(B_1B_2)^2=(A_2B_2)^2-(A_2B_1)^2=\frac{a^2(b^2+c^2)}{a^2+b^2+c^2}-\frac{a^2b^2}{a^2+b^2}=\frac{a^4c^2}{(a^2+b^2)(a^2+b^2+c^2)},$$

because (B_1B_2) is a leg of right triangle $(A_2B_1B_2)$;

$(A_1B_2)^2=(A_1A_2)^2-(A_2B_2)^2=a^2-\dfrac{a^2(b^2+c^2)}{a^2+b^2+c^2}=\dfrac{a^4}{a^2+b^2+c^2}$, because (A_1B_2) is a leg of right triangle $(A_1B_2A_2)$;

$$(B_2A_4)=(A_1A_4)-(A_1B_2)=\sqrt{a^2+b^2+c^2}-\frac{a^2}{\sqrt{a^2+b^2+c^2}}=\frac{b^2+c^2}{\sqrt{a^2+b^2+c^2}}.$$

The volumes of the three smaller Pyth 3D simplexes (Pyth tetrahedra) add up to the volume of the original Pyth simplex:

$$
\begin{aligned}
& V_1\left(A_2B_1B_2A_1\right)+V_2\left(A_2B_1B_2A_4\right)+V_3\left(A_2B_1A_3A_4\right) \\
& =\frac{1}{3!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2}{\sqrt{a^2+b^2+c^2}} \\
& \quad+\frac{1}{3!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{b^2+c^2}{\sqrt{a^2+b^2+c^2}} \\
& \quad+\frac{1}{3!}\frac{ab}{\sqrt{a^2+b^2}}\frac{b^2}{\sqrt{a^2+b^2}}c \\
& =\frac{1}{3!}\frac{a^5bc}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)}+\frac{1}{3!}\frac{a^3bc\left(b^2+c^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)}+\frac{1}{3!}\frac{ab^3c}{a^2+b^2} \\
& =\frac{1}{3!}abc\frac{b^2\left(a^2+b^2+c^2\right)+a^4+\left(b^2+c^2\right)a^2}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)} \\
& =\frac{1}{3!}abc\frac{a^4+a^2\left(b^2+c^2\right)+b^2\left(a^2+b^2+c^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)} \\
& =\frac{1}{3!}abc\frac{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)}=\frac{1}{3!}abc=V\left(A_2A_3A_4A_1\right)
\end{aligned}
$$

This result completes the proof of the Coxeter partition for a Pyth simplex in 3D space.

2.3. The Coxeter Partition in 4D Space: A 4D Simplex Partitioned in Four 4D Simplexes

For clarity, Figure 7 only indicates the right angles that are necessary for the proof.

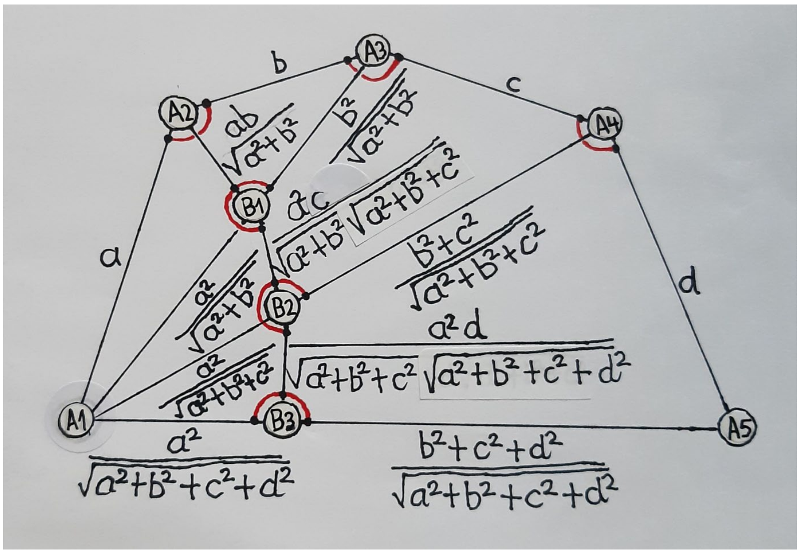

Figure 7. Coxeter partition in 4D.

The volumes of the four smaller Pyth 4D simplexes (Pyth pentachorons) add up to the volume of the original Pyth 4D simplex:

$$V_1\left(A_2B_1B_2B_3A_1\right)+V_2\left(A_2B_1B_2B_3A_5\right)+V_3\left(A_2B_1B_2A_4A_5\right)+V_4\left(A_2B_1A_3A_4A_5\right)$$
$$=\frac{1}{4!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2d}{\sqrt{a^2+b^2+c^2}}\frac{a^2}{\sqrt{a^2+b^2+c^2+d^2}}$$
$$+\frac{1}{4!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2d}{\sqrt{a^2+b^2+c^2}}\frac{b^2+c^2+d^2}{\sqrt{a^2+b^2+c^2+d^2}}$$
$$+\frac{1}{4!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{b^2+c^2}{\sqrt{a^2+b^2+c^2}}d+\frac{1}{4!}\frac{ab}{\sqrt{a^2+b^2}}\frac{b^2}{\sqrt{a^2+b^2}}cd$$
$$=\frac{1}{4!}abcd\frac{a^6+a^4\left(b^2+c^2+d^2\right)+a^2\left(b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)+b^2\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}$$
$$=\frac{1}{4!}abcd\frac{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}=\frac{1}{4!}abcd=V\left(A_2A_3A_4A_5A_1\right)$$

This result completes the proof of the Coxeter partition for a Pyth simplex in 4D space.

2.4. The Coxeter Partition in 5D Space: A 5D Simplex Partitioned in Five 5D Simplexes

For clarity, Figure 8 only indicates the right angles that are necessary for the proof.

The volumes of the five smaller Pyth 5D simplexes add up to the volume of the original 5D Pyth simplex:

$$V_1\left(A_2B_1B_2B_3B_4A_1\right)+V_2\left(A_2B_1B_2B_3B_4A_6\right)+V_3\left(A_2B_1B_2B_3A_5A_6\right)$$
$$+V_4\left(A_2B_1B_2A_4A_5A_6\right)+V_5\left(A_2B_1A_3A_4A_5A_6\right)$$
$$=\frac{1}{5!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2d}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2+d^2}}$$
$$\times\frac{a^2e}{\sqrt{a^2+b^2+c^2+d^2}\sqrt{a^2+b^2+c^2+d^2+e^2}}\frac{a^2}{\sqrt{a^2+b^2+c^2+d^2+e^2}}$$
$$+\frac{1}{5!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2d}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2+d^2}}$$
$$\times\frac{a^2e}{\sqrt{a^2+b^2+c^2+d^2}\sqrt{a^2+b^2+c^2+d^2+e^2}}\frac{b^2+c^2+d^2+e^2}{\sqrt{a^2+b^2+c^2+d^2+e^2}}$$
$$+\frac{1}{5!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{a^2d}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2+d^2}}$$
$$\times\frac{b^2+c^2+d^2}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2+d^2}}e$$
$$+\frac{1}{5!}\frac{ab}{\sqrt{a^2+b^2}}\frac{a^2c}{\sqrt{a^2+b^2}\sqrt{a^2+b^2+c^2}}\frac{b^2+c^2}{\sqrt{a^2+b^2+c^2}}de$$
$$+\frac{1}{5!}\frac{ab}{\sqrt{a^2+b^2}}\frac{b^2}{\sqrt{a^2+b^2}}cde$$
$$=\frac{1}{5!}abcde\frac{A+B}{C}$$

$$\boxed{\begin{aligned} A &= a^8 + a^6\left(b^2+c^2+d^2+e^2\right) + a^4\left(b^2+c^2+d^2\right)\left(a^2+b^2+c^2+d^2+e^2\right) \\ &\quad + a^2\left(b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2+c^2+d^2+e^2\right) \\ B &= b^2\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2+c^2+d^2+e^2\right) \\ C &= \left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)\left(a^2+b^2+c^2+d^2+e^2\right) \end{aligned}}$$

$$= \frac{1}{5!}abcde\frac{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}{\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\left(a^2+b^2+c^2+d^2\right)}$$

$$= \frac{1}{5!}abcde = V\left(A_2A_3A_4A_5A_6A_1\right)$$

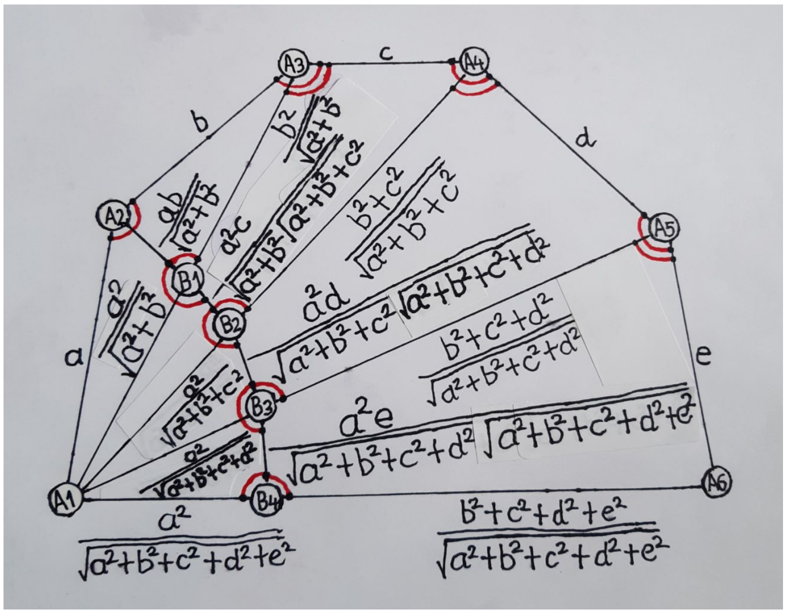

Figure 8. Coxeter partition in 5D.

This result completes the proof of the Coxeter partition of a Pyth simplex in 5D space.

To construct a 6D Pyth simplex, add the triangle $(A_1A_6A_7)$ to side (A_1A_6) of the 5D Pyth simplex in **Figure 8**, and mark a new partition point B_5 on side (A_1A_6).

The respective measures of segments:

$(A_6A_7) = f$;

$$\left(A_1B_5\right) = \frac{a^2}{\sqrt{a^2+b^2+c^2+d^2+e^2+f^2}};$$

$$\left(B_5A_7\right) = \frac{b^2+c^2+d^2+e^2+f^2}{\sqrt{a^2+b^2+c^2+d^2+e^2+f^2}};$$

$$\left(B_5B_6\right) = \frac{a^2 f}{\sqrt{a^2+b^2+c^2+d^2+e^2}\sqrt{a^2+b^2+c^2+d^2+e^2+f^2}}.$$

The volume calculations can be made using the pattern of the previous cases (see also Chapter 3 below).

3. Generalization to *n*-Dimensional Space: An *n*-Dimensional Pyth Simplex Partitioned in *n* Pyth Simplexes

The Principle of Recursion for the *n*-Dimensional Pyth Simplex

Suppose that the Coxeter partition is settled for the (n-1)-dimensional simplex $A_1A_2A_3\cdots A_{n-1}A_n$ .with partition points $B_1, B_2, B_3, \cdots B_{n-3}B_{n-2}$.

In the n-dimensional space, triangle $(A_1A_nA_{n+1})$ is added to side (A_1A_n) of the (n – 1)D Pyth simplex (**Figure 9**). Mark a new partition point B_{n-1} as the foot of the perpendicular dropped from point B_{n-2} onto side (A_1A_{n+1}).

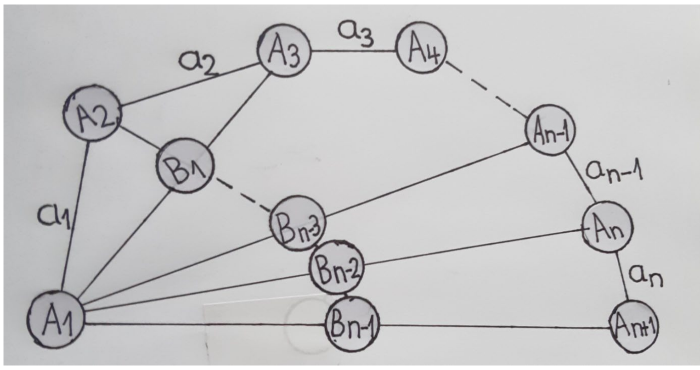

Figure 9. Coxeter partition in (n)D.

Consider the Pyth simplex with vertices $A_1A_2A_3\cdots A_{n-1}A_nA_{n+1}$, and sides $a_1a_2a_3\cdots a_{n-1}a_n$. The partitioning of this Pyth simplex with partition points $B_1, B_2, B_3, \cdots B_{n-2}B_{n-1}$ can be carried out by applying the algorithm in the previous cases (**Table 1**).

Table 1. Vertices of the n Pyth simplexes of the Coxeter division of the original (n)D Pyth simplex.

Simplex 1	A_2	B_1	B_2	B_3	B_4	$\cdots$	B_{n-3}	B_{n-2}	B_{n-1}	A_1
Simplex 2	A_2	B_1	B_2	B_3	B_4	$\cdots$	B_{n-3}	B_{n-2}	B_{n-1}	A_{n+1}
Simplex 3	A_2	B_1	B_2	B_3	B_4	$\cdots$	B_{n-3}	B_{n-2}	A_n	A_{n+1}
Simplex 4	A_2	B_1	B_2	B_3	B_4	$\cdots$	B_{n-3}	A_{n-1}	A_n	A_{n+1}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Simplex (n – 2)	A_2	B_1	B_2	B_3	A_5	$\cdots$	A_{n-2}	A_{n-1}	A_n	A_{n+1}
Simplex (n – 1)	A_2	B_1	B_2	A_4	A_5	$\cdots$	A_{n-2}	A_{n-1}	A_n	A_{n+1}
Simplex (n)	A_2	B_1	A_3	A_4	A_5	$\cdots$	A_{n-2}	A_{n-1}	A_n	A_{n+1}

The respective measures of segments in triangle $(A_1A_nA_{n+1})$:

$(A_nA_{n+1}) = a_n$;

$$(A_1B_{n-2}) = \frac{a_1^2}{\sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2}};$$

$$(B_{n-2}A_n) = \frac{a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2}{\sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2}};$$

$$(B_{n-2}B_{n-1}) = \frac{a_1^2 a_n}{\sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2}\sqrt{a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2}}.$$

The calculation of the volumes of n pieces of (n)-dimensional Pyth simplexes can be done by using the divisions in Table 1.

4. Conclusions and Background

The starting idea of the article comes from spherical geometry. If point P is on line l, then the polar p of point P is perpendicular to line e. This means that the incidence of a point and a straight line is equivalent to the perpendicularity of two straight lines. In this interpretation, a general triangle is a 6-cycle of incidence of vertices and sides $V_1s_1V_2s_2V_3s_3$, but a right triangle can be interpreted as a 5-cycle of vertices and sides $V_1s_1V_2V_3s_3$, where sides V_2, V_3 are perpendicular to each other. In this sense, a right triangle is not a special case of a general triangle, but conversely, the general triangle is a composite shape formed from right triangles.

I tried to apply this approach in 2D spherical, hyperbolic and plane geometry [7], and in higher-dimensional Euclidean geometry [8]. I proposed the right triangle, the Lambert quadrilateral, the spherical Napier pentagram, and the hyperbolic Napier pentagon as simplexes in 2D geometries, and the Pythagorean solids as simplexes in multidimensional Euclidean geometry. I used the Coxeter trisection to show that moving from general simplexes to Pythagorean simplexes as basic building blocks of multidimensional geometry leads to thought-provoking questions and reasonable answers. Only later did I become acquainted with Hadwiger's conjecture and Tschirpke's results in this field, which were based on graph theory and classification of dihedral angles, quite different from my own reasoning.

Clearly, the proofs in the present article apply only to the Pythagorean subset of general simplexes. However, I hope that Pythagorean simplexes will prove to be a useful tool for asking pertinent questions and solving certain problems in multidimensional geometry.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Lénárt, I. (2024) The Coxeter Trisection and the Hadwiger Conjecture in Multidimensional Spaces. *Journal of Applied Mathematics and Physics*, **12**, 4301-4321. https://doi.org/10.4236/jamp.2024.1212264

[2] Schläfli, L. (1901) Theorie der vielfachen Kontinuität. Cornell University Library Historical Math Monographs 2010 (in German), 10-22.

[3] Tschirpke, K. (1994) The Dissection of Five-Dimensional Simplexes into Orthoschemes. *Beiträge zur Algebra und Geometrie*, **35**, 1-11.

[4] Petrie, J.F. (1938) Petrie Polygons. In: Coxeter, H.S.M., du Val, P., Flather, H.T., Petrie, J.F., Eds., *The Fifty-Nine Icosahedra*, University of Toronto Studies, Mathematical Series, 1-26.

[5] Coxeter, H.S.M. (1989) Trisecting an Orthoscheme. *Computers & Mathematics with Applications*, **17**, 59-71. https://doi.org/10.1016/0898-1221(89)90148-x

[6] Hadwiger, H. (1956) Ungelöste Probleme. *Elemente der Mathematik*, **11**, 109-110.

[7] Lénárt, I. (2020) The Algebra of Projective Spheres on Plane, Sphere and Hemisphere. *Journal of Applied Mathematics and Physics*, **8**, 2286-2333. https://doi.org/10.4236/jamp.2020.810171

[8] Lénárt, I. (2022) The Right Triangle as the Simplex in 2D Euclidean Space, Generalized to *n* Dimensions. *Journal of Applied Mathematics and Physics*, **10**, 2837-2850. https://doi.org/10.4236/jamp.2022.109189