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A Note on Global Existence, Blow-Up and Orbital Stability of Standing Waves for the Schrödinger Equation with Mixed Nonlinearities

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Abstract

This study is devoted to the global existence, blow-up and orbital stability of standing waves for the Schrödinger equation with mixed nonlinearities. Firstly, we derive some criteria for global existence and blow-up of the solutions by making use of the ground state and scaling techniques. Secondly, by taking advantage of the refined compactness argument, scaling techniques and the variational characterization of ground state solutions, we explore the L^2-concentration phenomenon and limiting behavior of blow-up solutions in the L^2-critical case with $\mu=\pm 1$ and $1<q<p=1+\frac{4}{N}$. Finally, we research the orbital stability of standing waves in the cases with $\mu=-1$, $1<q<p\le 1+\frac{4}{N}$ or $\mu=1$, $1+\frac{4}{N}=p<q<\frac{N+2}{N-2}$, by combining the variational methods, profile decomposition and the arguments of concentration compactness. Our study complements the conclusions of some known works.

Keywords

Nonlinear Schrödinger Equation, Mixed Nonlinearities, Blow-Up, Global Existence, Orbital Stability of Standing Waves

1. Introduction

In the current paper, we undertake a comprehensive investigation of the Cauchy problem of the following Schrödinger equation with mixed power-type nonlinearities

$$\begin{cases} iu_t + \Delta u = -|u|^{p-1} u + \mu |u|^{q-1} u, \ (t,x) \in [0,T) \times \mathbb{R}^N \\ u(0,x) = u_0 \in H^1(\mathbb{R}^N), \ x \in \mathbb{R}^N \end{cases} \tag{1}$$

where $u(t,x):[0,T)\times\mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < T \le \infty$, $N \ge 3$ is the space dimension, $\mu \in \mathbb{R}$, $1 < p,q < \frac{N+2}{N-2}$.

It is widely acknowledged that the model (1) with double power-type nonlinearities has a wide range of applications in various physical settings. For example, this equation is regarded as a simplified model that extends the nonlinear Schrödinger equation into a saturated nonlinearity, which is related to describing the Bose superfluids at zero temperature, as detailed in [1] [2]. The double power-type nonlinearities can be regarded as a multi-body correction of the mean field interaction in Bose-Einstein condensates (BEC): when $\mu = -1$ and $1 < q < p = 1 + \frac{4}{N}$, $|u|^{q-1} u$ provides a focused three body attraction, while $|u|^{p-1} u$ gives a saturation effect. If $\mu = 1$ and $p = 1 + \frac{4}{N} < q < \frac{N+2}{N-2}$, then the competitive nonlinearities can describe the synergistic competitive mechanism of Kerr effect (p-term) and higher-order nonlinearity (q-term) in nonlinear optics. Furthermore, it also appears as the leading-order model for propagation of intense laser beams in an isotropic bulk medium, see Section 1.2 in [3] for instance. The extensive applicability of model (1) has attracted more and more attention in both practical applications and theoretical research.

Our main goal of this paper is to investigate the criterion of global existence versus blow-up, the dynamical properties of blow-up solutions and the stability of standing waves for Equation (1).

To this end, we first review some remarkable results about the classical Schrödinger equation with single power nonlinearity

$$iu_t + \Delta u + |u|^{p-1} u = 0, \ (t,x) \in [0,T) \times \mathbb{R}^N. \tag{2}$$

It was proved in [4] that the blow-up solutions to Equation (2) exist in the radial case without the restriction $|x| u_0 \in L^2$. Hmidi and Keraani [5] put forward a refined compactness argument by utilizing the profile decomposition, which provides a novel method on the study of the dynamical properties of blow-up solutions in the L^2-critical case. In [6], Berestycki and Cazenave showed the instability of standing waves in the case $p = 1 + \frac{4}{N}$. Moreover, based on the concentration compactness principle, Cazenave and Lions [7] proved the orbital stability of standing waves in the L^2-subcritical case $1 < p < 1 + \frac{4}{N}$. This idea has been extensively exploited and further developed in the research of other kinds of nonlinear Schrödinger equations, offering an alternative perspective on the stability analysis of standing waves for Equation (1). For further details regarding Equation (2), we refer to the works [4] [7]-[10].

Inspired by the aforementioned literatures, numerous scholars have further researched the Schrödinger equation with mixed power-type nonlinearities, particularly yielding a series of significant results in areas such as local and global existence as well as blow-up, see [11]-[16] for example. In the case $\mu=1$, $p=1+\frac{4}{N}$ and $1<q<1+\frac{4}{N}$, Le Coz *et al.* [13] investigated the existence of a new class of minimal blow up solutions of Equation (1) and discovered that the blow-up rate of blow-up solutions is significantly influenced by the double power nonlinearities. Feng [12] verified the sharp threshold mass of global existence versus blow-up and dynamics of blow-up solutions for Equation (1) by making full use of the scaling arguments, the best constant of Gagliardo-Nirenberg inequality, the refined compactness argument and the variational characterization of the ground state solution to Equation (8). However, as far as we know, when $\mu=-1$, $1<q<p\le 1+\frac{4}{N}$ or $\mu=1$, $1+\frac{4}{N}=p<q<\frac{N+2}{N-2}$, the criterion of blow-up and global existence for Equation (1) has not been discussed clearly yet. In what follows, we shall prove the existence of blow-up solutions and then give the sufficient conditions of global existence and blow-up in these cases. Moreover, we will investigate some dynamical properties of blow-up solutions to Equation (1), including L^2-concentration and limiting profile.

To this purpose, for $\mu=-1$ and $1<q<p=1+\frac{4}{N}$, by using the sharp Gagliardo-Nirenberg inequality, Young's inequality together with scaling techniques, we first obtain the condition for global existence to Equation (1). Meanwhile, the key to determining the threshold of global existence versus blow-up for Equation (1) in the mass-critical setting lies in proving the existence of blow-up solutions. Nevertheless, it's not easy to choose $E(u_0)$ to ensure the second-order derivative $J''(t)<-C<0$ (see (6)). To overcome this obstacle, we show the existence of blow-up solutions by contradiction, together with some scaling arguments. It is worth emphasizing that the scaling invariance plays a key role in studying the dynamics of blow-up solutions. However, the presence of combined nonlinearities strongly influences the variational structure of corresponding energy functional, leading to the loss of scaling invariance of Equation (1). Following the clues of [12] [14] [15], we shall apply the ground state solution of Equation (8) with L^2-critical nonlinearity to describe the limiting behaviours of blow-up solutions at blow-up time. This approach can effectively overcome the difficulty of lacking scaling invariance of Equation (1). On the other hand, when $\mu=-1$, $1<q<p<1+\frac{4}{N}$ or $\mu=1$, $1+\frac{4}{N}=p<q<\frac{N+2}{N-2}$, the global existence of solutions to Equation (1) can be easily demonstrated by taking advantage of interpolation inequality and Young's inequality.

We are also deeply interested in the orbital stability of standing waves for Equation (1). As we know, the instability or stability of standing waves for Equation (1) attracts increasing attention. Soave [17] obtained several results concerning the existence or non-existence and stability or instability of ground states with prescribed L^2-norm in the Sobolev critical case by taking advantage of variational ideas. In [18], the author pointed that the concentration-compactness principle established in [19] [20] can be applied to derive the stability of standing waves for Equation (1) with $\mu=-1$ and $1<q<p\le 1+\frac{4}{N}$, but the complete proof were not given. In the case $\mu=-1$ and $1<q<p=1+\frac{4}{N}$, Bai and Zhang [21] established a new compactness principle, based on the technique of Schwartz symmetrization, to show the orbital stability of small solitons to Equation (1). In addition, Fukaya and Hayashi [22] studied the strong instability for all frequencies when $1+\frac{4}{N}\le q<\frac{N+2}{N-2}$ and the instability for small frequencies when $1<q<1+\frac{4}{N}$. In particular, they were the first to give the results on stability properties of algebraic standing waves. Whereafter, Jeanjean and Le [23] proved that there exist standing waves which are located at a mountain-pass level of the energy functional. We refer the readers to [21] [24]-[29] (see also the references therein) for the Schrödinger equation with combined nonlinearities. It is worth to point out that the stability of standing waves for Equation (1) in the aforementioned works are also of significant value in physics. Nevertheless, for $\mu=1$ and $1+\frac{4}{N}=p<q<\frac{N+2}{N-2}$, the orbital stability of standing waves of Equation (1) with competitive nonlinearities is still left open.

It is worth noting that, to prove the stability of standing waves for Equation (1), one may encounter two main challenges. One comes from the combined nonlinear terms, which cause the lack of scaling invariance and change the variational structure of energy functional. The other one is the loss of compactness. To get across the obstacles, we shall take advantage of the profile decomposition and concentration-compactness principle and scaling techniques. More precisely, in the case $\mu=-1$ and $1<q<p\le 1+\frac{4}{N}$, we shall demonstrate that the standing waves of Equation (1) are orbitally stable by using the profile decomposition, which is different from the ideas used in [18] [21]. While for $\mu=1$, $1+\frac{4}{N}=p<q<\frac{N+2}{N-2}$, greatly inspired by Feng and Zhu [30], where the stability issues of standing waves for fractional Schrödinger equation with mixed power-type nonlinearities were studied, we utilize the arguments of concentration-compactness to show the orbital stability of standing waves for Equation (1) with competitive nonlinearities.

This paper is structured as below. In section 2, some preliminaries are given. In section 3, the criterion for global existence versus blow-up is established. In section 4, we focus on the dynamics of blow-up solutions. In section 5, we address the stability of standing waves.

Notations. Throughout this manuscript, to simplify the marks, we abbreviate $\int_{\mathbb{R}^N}\cdot\,\mathrm{d}x$ by $\int\cdot\,\mathrm{d}x$ and use $\|\cdot\|_{H^1}$ to denote $\|\cdot\|_{H^1(\mathbb{R}^N)}$ and replace $\|\cdot\|_{L^2(\mathbb{R}^N)}$ by

$\|\cdot\|_2$. Meanwhile, we utilize C to represent a positive constant that may be different from line to line. $\Sigma := \left\{u \in H^1\left(\mathbb{R}^N\right); xu \in L^2\left(\mathbb{R}^N\right)\right\}$ denotes the energy space equipped with the norm $\|\varphi\|_\Sigma := \left(\|u\|_{H^1}^2 + \|xu\|_2^2\right)^{\frac{1}{2}}$.

2. Preliminaries

In this section, we recall some crucial preliminary results that will be used later. In order to study the global existence versus blow-up as well as the stability of standing waves, we require the well-posedness of solution to Equation (1). Based on [16] [31], we first have the following local well-posedness of Equation (1).

Proposition 1. [16] [31] Let $u_0 \in H^1$, $\mu \in \mathbb{R}$ and $1 < p, q < \frac{N+2}{N-2}$. Then there exists $T = T\left(\|u_0\|_{H^1}\right)$ such that Equation (1) admits a unique solution $u(t,x) \in C\left([0,T), H^1\right)$. Assume that $[0,T)$ is the maximal time interval such that the solution $u(t,x)$ is well-defined. If $T < \infty$, then $\lim_{t\to T}\|u(t,x)\|_{H^1} = \infty$ (blow-up). Moreover, for any $t \in [0,T)$, the following conservation laws of mass and energy hold

$$\|u(t,x)\|_2 = \|u_0\|_2, \tag{3}$$

$$E(u(t,x)) = E(u_0), \tag{4}$$

where the energy functional is defined by

$$E(u(t,x)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{\mu}{q+1}\|u\|_{q+1}^{q+1}.$$

Especially when $p = 1 + \frac{4}{N}$,

$$E(u(t,x)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{\mu}{q+1}\|u\|_{q+1}^{q+1}. \tag{5}$$

Next, by some simple calculations, we are able to derive the second-order derivative of virial quantity for the Cauchy problem (1), which plays a crucial role in the analysis of the existence of blow-up solutions.

Proposition 2. Let $u_0 \in \Sigma$, $\mu \in \mathbb{R}$, $1 < p, q < \frac{N+2}{N-2}$ and $u(t,x)$ be a solution of problem (1) in $C\left([0,T); H^1\right)$. Set $J(t) = \int |x|^2 |u(t,x)|^2 \,\mathrm{d}x$, then we get that

$$J''(t) = 8\int |\nabla u|^2 \,\mathrm{d}x - \frac{4N(p-1)}{p+1}\int |u|^{p+1} \,\mathrm{d}x + \frac{4\mu N(q-1)}{q+1}\int |u|^{q+1} \,\mathrm{d}x.$$

Especially when $p = 1 + \frac{4}{N}$,

$$\begin{aligned} J''(t) &= 8\int |\nabla u|^2 \,\mathrm{d}x - \frac{16}{2+\frac{4}{N}}\int |u|^{2+\frac{4}{N}} \,\mathrm{d}x + \frac{4\mu N(q-1)}{q+1}\int |u|^{q+1} \,\mathrm{d}x. \\ &= 16E(u_0) + \frac{4\mu N(q-1) - 16\mu}{q+1}\|u\|_{q+1}^{q+1}. \end{aligned} \tag{6}$$

Now we recall some useful lemmas.

Lemma 3. [32] [33] Let $u \in \Sigma$, then one has that

$$\int |u|^2 \, \mathrm{d}x \le \frac{2}{N} \left(\int |\nabla u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int |x|^2 |u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{7}$$

Lemma 4. [32] Let $N \ge 2$, $0 < \alpha < \frac{4}{N-2}$. Then for any $u \in H^1(\mathbb{R}^N)$, we have the sharp Gagliardo-Nirenberg inequality

$$\int |u|^{\alpha+2} \, \mathrm{d}x \le C_{\alpha,N} \|u\|_2^{\alpha+2-\frac{N\alpha}{2}} \|\nabla u\|_2^{\frac{N\alpha}{2}},$$

where $C_{\alpha,N} = \frac{\alpha+2}{2\|Q\|_2^{\alpha}}$ and $Q(x)$ is the ground state solution of the elliptic equation

$$-\Delta\varphi + \varphi = |\varphi|^{\alpha} \varphi. \tag{8}$$

In particular, in the L^2-critical case $\alpha = \frac{4}{N}$, then $C_{\alpha,N} = \frac{N+2}{N} \|Q\|_2^{-\frac{4}{N}}$ and the following Pohožaev identity holds

$$\|\nabla Q\|_2^2 = \frac{N}{2+N} \int |Q|^{\frac{4}{N}+2} \, \mathrm{d}x. \tag{9}$$

3. Global Existence and Blow-Up

As we know, it is crucial in physics to determine the conditions under which the condensate becomes unstable and collapses (blow-up) or exists for all time (global existence). Therefore, we are concerned with the criterion of global existence and blow-up for Equation (1). For the L^2-critical Schrödinger equation with defocusing L^2-subcritical perturbation, the sharp threshold mass of blow-up and global existence for Equation (1) has been obtained in [12]. Now, for the focusing L^2-subcritical case, it is of particular interest whether there exists a sharp threshold of blow-up and global existence for Equation (1). To solve this problem, there exists a major difficulty that the second-order derivative of $J(t) = \int |x|^2 |u(t,x)|^2 \, \mathrm{d}x$ is the following form:

$$J''(t) = 16E(u_0) + \frac{16 - 4N(q-1)}{q+1} \|u\|_{q+1}^{q+1}.$$

Since $\frac{16 - 4N(q-1)}{q+1} \|u\|_{q+1}^{q+1} > 0$, it is hard to choose $E(u_0)$ to ensure the existence of blow-up solutions. In the following, we will argue the sharp criterion of global existence and blow-up for Equation (1) by contradiction together with scaling techniques.

It is worth mentioning that for the next Theorem 5, the conclusions on the global existence and blow-up of Equation (1) in the case of $\mu = 1$ have been proven in Feng [12]. However, there are few literatures discussing the case of $\mu = -1$. In Theorem 5, we will consider these two cases and provide a proof for $\mu = -1$.

Theorem 5. Assume that $u_0 \in H^1$, $\mu = \pm 1$, $p = 1 + \frac{4}{N}$ and $1 < q < 1 + \frac{4}{N}$. Then we have the following facts hold:

1) Global existence: If $\|u_0\|_2 < \|Q\|_2$, then the solution $u(t,x)$ of Equation (1) exists globally in $t \in [0,+\infty)$.

2) Blow-up: If the initial data $u_0 = c\rho^{\frac{N}{2}} Q(\rho x)$ satisfies $|x|u_0 \in L^2$, where the constant c satisfying $|c| > 1$, and the real number $\rho > 0$. Then the corresponding solution $u(t,x)$ of the Cauchy problem (1) blows up in finite time.

Proof. 1) Firstly, from (3) - (5) and combining Lemma 4, we have the following estimate

$$
\begin{aligned}
E(u_0) &= E(u(t)) \\
&= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\
&\geq \left(\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}}\right)\|\nabla u\|_2^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\
&\geq \left(\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}}\right)\|\nabla u\|_2^2 - C\|u_0\|_2^{q+1-\frac{N(q-1)}{2}}\|\nabla u\|_2^{\frac{N(q-1)}{2}}.
\end{aligned} \tag{10}
$$

when $\alpha = \frac{4}{N}$, $C_{\alpha,N} = \frac{2+N}{N}\|Q\|_2^{-\frac{4}{N}}$, we have

$$\int |u|^{\alpha+2}\,\mathrm{d}x = \|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \leq \frac{2+N}{N}\frac{\|u\|_2^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2,$$

and

$$\frac{N}{2+N}\cdot\frac{1}{2}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} = \frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \leq \frac{1}{2}\frac{\|u\|_2^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}}\|\nabla u\|_2^2.$$

Since $1 < q < 1 + \frac{4}{N}$, then $\frac{N(q-1)}{2} < 2$. Hence, we deduce Young's inequality that, for any $0 < \varepsilon < \frac{1}{2}$, there exists a constant $C(\varepsilon, M)$ such that

$$C\|u_0\|_2^{q+1-\frac{N(q-1)}{2}}\|\nabla u\|_2^{\frac{N(q-1)}{2}} \leq \varepsilon\|\nabla u\|_2^2 + C\left(\varepsilon, \|u_0\|_2\right).$$

Thus, combining (10), one obtains that

$$E(u_0) = E(u(t)) \geq \left(\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}} - \varepsilon\right)\|\nabla u\|_2^2 - C\left(\varepsilon, \|u_0\|_2\right),$$

which means

$$E(u_0) + C\left(\varepsilon, \|u_0\|_2\right) \geq \left(\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}} - \varepsilon\right)\|\nabla u\|_2^2.$$

Let ε be small enough and by the fact $\|u_0\|_2 < \|Q\|_2$, then we conclude that $\|\nabla u\|_2^2$ is uniformly bounded for all $t \in [0,+\infty)$. Therefore, we have that the solution $u(t,x)$ of Equation (1) exists globally.

2) Assume by contradiction that the corresponding solution $u(t,x)$ exists globally with $T = +\infty$ and there exists $C > 0$ such that

$$\sup_{t\in[0,+\infty)} \|u(t)\|_{H^1} \le C. \tag{11}$$

Since $u_0 = c\rho^{\frac{N}{2}} Q(\rho x)$ and using the Pohožaev identities (9), it follows that

$$\begin{aligned} E(u_0) &= \frac{|c|^2 \rho^2}{2}\|\nabla Q\|_2^2 - \frac{|c|^{2+\frac{4}{N}}\rho^2}{2+\frac{4}{N}}\|Q\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - \frac{|c|^{q+1}\rho^{\frac{N(q-1)}{2}}}{q+1}\|Q\|_{q+1}^{q+1} \\ &= -\frac{|c|^2\rho^2}{2}\left(|c|^{\frac{4}{N}}-1\right)\|\nabla Q\|_2^2 - \frac{|c|^{q+1}\rho^{\frac{N(q-1)}{2}}}{q+1}\|Q\|_{q+1}^{q+1}. \end{aligned} \tag{12}$$

Then by the conservation of mass and interpolating between $L^2(\mathbb{R}^N)$ and $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, together with the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we have

$$\begin{aligned} \|u(t)\|_{q+1} &\le \|u(t)\|_2^{\frac{2N-(N-2)(q+1)}{2(q+1)}} \|u(t)\|_{\frac{2N}{N-2}}^{\frac{N(q-1)}{2(q+1)}} \\ &\le C\|u_0\|_2^{\frac{2N-(N-2)(q+1)}{2(q+1)}} \|u(t)\|_{H^1}^{\frac{N(q-1)}{2(q+1)}}. \end{aligned}$$

From (11) and $\|u_0\|_2 = |c|\|Q\|_2$, we get

$$\|u(t)\|_{q+1}^{q+1} \le C\left(|c|\|Q\|_2\right)^{\frac{2N-(N-2)(q+1)}{2}}.$$

This together with (6) and (12), one has that

$$\begin{aligned} J''(t) &= 16E(u_0) + \frac{16-4N(q-1)}{q+1}\|u\|_{q+1}^{q+1} \\ &= -8|c|^2\rho^2\left(|c|^{\frac{4}{N}}-1\right)\|\nabla Q\|_2^2 - \frac{16|c|^{q+1}\rho^{\frac{N(q-1)}{2}}}{q+1}\|Q\|_{q+1}^{q+1} \\ &\quad + \frac{16-4N(q-1)}{q+1}\|u\|_{q+1}^{q+1} \\ &< -8|c|^2\rho^2\left(|c|^{\frac{4}{N}}-1\right)\|\nabla Q\|_2^2 - \frac{16|c|^{q+1}\rho^{\frac{N(q-1)}{2}}}{q+1}\|Q\|_{q+1}^{q+1} \\ &\quad + \frac{C}{q+1}\left(|c|\|Q\|_2\right)^{\frac{2N-(N-2)(q+1)}{2}} \\ &< -\frac{16|c|^{q+1}\rho^{\frac{N(q-1)}{2}}}{q+1}\|Q\|_{q+1}^{q+1} + \frac{C}{q+1}\left(|c|\|Q\|_2\right)^{\frac{2N-(N-2)(q+1)}{2}}. \end{aligned}$$

Now, taking ρ such that

$$\rho^{\frac{N}{2}(q-1)} > \frac{C\left(c\|Q\|_2\right)^{\frac{2N-(N-2)(q+1)}{2}}}{16|c|^{q+1}\|Q\|_{q+1}^{q+1}},$$

then

$$J''(t) < -\nu < 0,$$

for all $t \in [0,+\infty)$ with some constant $\nu > 0$. Thus there must exist finite time $\tilde{T} < +\infty$ such that

$$\lim_{t\to\tilde{T}} J(t) = 0.$$

Then by Lemma 3, $\lim_{t\to\tilde{T}}\|u(t)\|_{H^1} = +\infty$, which gives a contradiction to (11). Thus, we conclude that the solution $u(t,x)$ of Equation (1) blows up in finite time.

Remark 1. In [12] (see Theorem 5), Feng demonstrated the sharp threshold of global existence and blow-up for Equation (1) in the case $\mu = 1$, $p = 1+\frac{4}{N}$ and $1 < q < 1+\frac{4}{N}$, which infer that the critical mass about the initial data for global existence and blow-up is the same in the two cases.

Theorem 6. Assume that $u_0 \in H^1$, $\mu = 1$, $p = 1+\frac{4}{N}$ and $1+\frac{4}{N} < q < \frac{N+2}{N-2}$, then the solution $u(t,x)$ of Equation (1) exists globally.

Proof. Since $1+\frac{4}{N} < q < \frac{N+2}{N-2}$, using the interpolation inequality, for $u \in H^1$, there exists $0 < \theta = \frac{2(q+1)}{(N+2)(q-1)} < 1$ such that $\frac{1}{2+\frac{4}{N}} = \frac{\theta}{q+1} + \frac{1-\theta}{2}$ and

$$\|u\|_{2+\frac{4}{N}} \le \|u\|_2^{1-\theta}\|u\|_{q+1}^{\theta}. \tag{13}$$

Taking advantage of Young's inequality, mass conservation and (13), for $0 < \varepsilon < \frac{1}{q+1}$, there exists a constant $C(\varepsilon, q, N, \|u_0\|_2) > 0$ such that

$$\frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \le C(\varepsilon, q, N, \|u_0\|_2) + \varepsilon\|u\|_{q+1}^{q+1}.$$

This together with energy conservation, it follows that

$$\begin{aligned} E(u_0) = E(u(t)) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{N}}\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 + \left(\frac{1}{q+1} - \varepsilon\right)\|u\|_{q+1}^{q+1} - C(\varepsilon, q, N, \|u_0\|_2), \end{aligned}$$

which means

$$E(u_0) + C(\varepsilon, q, N, \|u_0\|_2) \ge \frac{1}{2}\|\nabla u\|_2^2.$$

Thus, we obtain the boundedness of $\|\nabla u\|_2^2$ for $t \in [0,+\infty)$, which implies that the solution $u(t,x)$ to Equation (1) is global and bounded. This completes the proof of Theorem 6.

Theorem 7. Assume that $u_0 \in H^1$, $\mu = -1$ and $1 < p,q < 1+\frac{4}{N}$, then the solution $u(t,x)$ of Equation (1) exists globally.

Proof. We deduce by Young's inequality that, for any $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{4}$

$$C\|u_0\|_2^{p+1-\frac{N(p-1)}{2}} \|\nabla u\|_2^{\frac{N(p-1)}{2}} \le \varepsilon_1 \|\nabla u\|_2^2 + C\left(\varepsilon_1, \|u_0\|_2\right),$$

$$C\|u_0\|_2^{q+1-\frac{N(q-1)}{2}} \|\nabla u\|_2^{\frac{N(q-1)}{2}} \le \varepsilon_2 \|\nabla u\|_2^2 + C\left(\varepsilon_2, \|u_0\|_2\right).$$

Then, we have

$$\begin{aligned} E(u_0) &= E(u(t)) \\ &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &\ge \frac{1}{2}\|\nabla u\|_2^2 - C_{p,N}\|u_0\|_2^{p+1-\frac{N(p-1)}{2}}\|\nabla u\|_2^{\frac{N(p-1)}{2}} \\ &\quad - C_{q,N}\|u_0\|_2^{q+1-\frac{N(q-1)}{2}}\|\nabla u\|_2^{\frac{N(q-1)}{2}} \\ &\ge \left(\frac{1}{2} - \varepsilon_1 - \varepsilon_2\right)\|\nabla u\|_2^2 - C\left(p,q,\varepsilon_1,\varepsilon_2,\|u_0\|_2,\|Q\|_2\right), \end{aligned}$$

from which we infer that

$$E(u_0) + C\left(p,q,\varepsilon_1,\varepsilon_2,\|u_0\|_2,\|Q\|_2\right) \ge \left(\frac{1}{2} - \varepsilon_1 - \varepsilon_2\right)\|\nabla u\|_2^2.$$

Since $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{4}$, then $\frac{1}{2} - \varepsilon_1 - \varepsilon_2 > 0$, thus one can conclude that $\|\nabla u\|_2^2$ is uniformly bounded for all $t \in [0, +\infty)$, which yields that the solution $u(t,x)$ of Equation (1) exists globally.

4. Dynamics of Blow-Up Solutions in the L^2-Critical Case

In this section, we investigate the dynamical properties of blow-up solutions of Equation (1) with $\mu = \pm 1$, $p = 1+\frac{4}{N}$ and $1 < q < 1+\frac{4}{N}$. To this aim, we first review a refined compactness conclusion established in Hmidi and Keraani [5].

Lemma 8. Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and satisfy

$$\limsup_{n\to\infty}\|\nabla v_n\|_2 \le M, \quad \limsup_{n\to\infty}\|v_n\|_{2+\frac{4}{N}} \ge m.$$

Then, there exists $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(x+x_n) \rightharpoonup V \text{ weakly in } H^1(\mathbb{R}^N),$$

with

$$\|V\|_2 \ge \left(\frac{N}{N+2}\right)^{\frac{N}{4}} \frac{m^{\frac{N}{2}+1}}{M^{\frac{N}{2}}}\|Q\|_2,$$

where $Q(x)$ is the ground state solution of Equation (8).

Using the refined compactness lemma, we are able to establish the following concentration property of blow-up solutions to Equation (1).

Theorem 9. (L^2 -concentration) Let $u_0 \in H^1$, $\mu = \pm 1$, $p = 1 + \frac{4}{N}$ and $1 < q < 1 + \frac{4}{N}$. Assume that $u(t,x)$ be a corresponding solution of Equation (1) which blows up in finite time T, and $g(t):[0,T) \mapsto \mathbb{R}$ be a real-valued nonnegative function such that $g(t)\|\nabla u(t)\|_2 \to +\infty$ as $t \to T$. Then there exists a function $x(t) \in \mathbb{R}^N$ for $t < T$ such that

$$\liminf_{t\to T} \int_{|x-x(t)|\le g(t)} |u(t,x)|^2 \,\mathrm{d}x \ge \int |Q(x)|^2 \,\mathrm{d}x, \tag{14}$$

where $Q(x)$ is the ground state solution of (8).

Proof. Take

$$\rho_n := \frac{\|\nabla Q\|_2}{\|\nabla u(t_n)\|_2} \text{ and } v_n(x) = \rho_n^{\frac{N}{2}} u(t_n, \rho_n x), \tag{15}$$

where $\{t_n\}_{n=1}^{\infty} \subseteq [0,T)$ is an arbitrary time sequence and $t_n \to T$ as $n \to \infty$. Then the following hold:

$$\begin{aligned} &\|v_n\|_2 = \|u(t_n)\|_2 = \|u_0\|_2, \\ &\|\nabla v_n\|_2 = \rho_n \|\nabla u(t_n)\|_2 = \|\nabla Q\|_2. \end{aligned} \tag{16}$$

Next, we define the functional

$$G(\phi) = \frac{1}{2}\int |\nabla \phi|^2 \,\mathrm{d}x - \frac{1}{2+\frac{4}{N}} \int |\phi|^{2+\frac{4}{N}} \,\mathrm{d}x,$$

then

$$\begin{aligned} G(v_n) &= \frac{1}{2}\int |\nabla v_n(x)|^2 \,\mathrm{d}x - \frac{1}{2+\frac{4}{N}} \int |v_n(x)|^{2+\frac{4}{N}} \,\mathrm{d}x \\ &= \rho_n^2 \left(\frac{1}{2}\int |\nabla u(t_n,x)|^2 \,\mathrm{d}x - \frac{1}{2+\frac{4}{N}} \int |u(t_n,x)|^{2+\frac{4}{N}} \,\mathrm{d}x \right) \\ &= \rho_n^2 \left(E(u_0) + \frac{\mu}{q+1} \int |u(t_n,x)|^{q+1} \,\mathrm{d}x \right). \end{aligned} \tag{17}$$

From Lemma 4, we found

$$\begin{aligned} |G(v_n)| &\le \rho_n^2 \left(|E(u_0)| + \frac{1}{q+1} \int |u(t_n,x)|^{q+1} \,\mathrm{d}x \right) \\ &\le \frac{\|\nabla Q\|_2^2 |E(u_0)|}{\|\nabla u(t_n)\|_2^2} + C \frac{\|\nabla Q\|_2^2 \|\nabla u(t_n)\|_2^{\frac{N(q-1)}{2}}}{\|\nabla u(t_n)\|_2^2}. \end{aligned}$$

Hence, from $\|\nabla u(t_n)\|_2 \to +\infty$ as $n \to +\infty$ and $1 < q < 1 + \frac{4}{N}$, we can infer that $|G(v_n)| \to 0$ as $n \to +\infty$, which yields

$$\int |v_n(x)|^{2+\frac{4}{N}} \mathrm{d}x \to \left(1+\frac{2}{N}\right)\|\nabla Q\|_2^2. \tag{18}$$

Take $M = \|\nabla Q\|_2$ and $m^{2+\frac{4}{N}} = \left(1+\frac{2}{N}\right)\|\nabla Q\|_2^2$, then

$$\limsup_{n\to\infty}\|\nabla v_n\|_2 \le M,\ \limsup_{n\to\infty}\|v_n\|_{2+\frac{4}{N}} \ge m.$$

By Lemma 8, there exist $V \in H^1(\mathbb{R}^N)$ and $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(\cdot + x_n) = \rho_n^{\frac{N}{2}} u(t_n, \rho_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^1(\mathbb{R}^N), \tag{19}$$

and

$$\|V\|_2 \ge \|Q\|_2. \tag{20}$$

Therefore, using the weakly lower semi-continuous of the L^2-norm, one has the following inequality

$$\begin{aligned}\liminf_{n\to\infty}\int_{|x|\le R}|v_n(t_n, x+x_n)|^2\mathrm{d}x &= \liminf_{n\to\infty}\int_{|x|\le R}\left|\rho_n^N u(t_n, \rho_n(x+x_n))\right|^2\mathrm{d}x \\ &\ge \int_{|x|\le R}|V|^2\mathrm{d}x, \text{ for any } R>0.\end{aligned} \tag{21}$$

By the assumption of Theorem 9, we have

$$\lim_{n\to\infty}\frac{g(t_n)}{\rho_n} = \lim_{n\to\infty}\frac{g(t_n)\|\nabla u(t_n)\|_2}{\|Q\|_2} = \infty,$$

thus for sufficiently large n, we get $R\rho_n < g(t_n)$. Combining (19) and (21), it follows that

$$\begin{aligned}&\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le g(t_n)}|u(t_n,x)|^2\mathrm{d}x \\ &\ge \liminf_{n\to\infty}\int_{|x-x_n|\le R\rho_n}|u(t_n,x)|^2\mathrm{d}x \\ &= \liminf_{n\to\infty}\int_{|x|\le R}\rho_n^N|u(t_n,\rho_n(x+x_n))|^2\mathrm{d}x \\ &\ge \int_{|x|\le R}|V|^2\mathrm{d}x, \text{ for every } R>0.\end{aligned}$$

This and (20) infer that

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le g(t_n)}|u(t_n,x)|^2\mathrm{d}x \ge \int|V|^2\mathrm{d}x \ge \|Q\|_2^2.$$

Furthermore, owing to the arbitrariness of the sequence $\{t_n\}_{n=1}^{\infty}$, one has that

$$\liminf_{t\to T}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le g(t)}|u(t,x)|^2\mathrm{d}x \ge \|Q\|_2^2. \tag{22}$$

For every $t\in[0,T)$, it is easy to show that the function $k(y) := \int_{|x-y|\le g(t)}|u(t,x)|^2\mathrm{d}x$ is continuous and $\lim_{|y|\to\infty}k(y)=0$. Thus, there exists a function $x(t)\in\mathbb{R}^N$ such that for any $t\in[0,T)$

$$\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le g(t)}|u(t,x)|^2\,\mathrm{d}x=\int_{|x-x(t)|\le g(t)}|u(t,x)|^2\,\mathrm{d}x.$$

Hence, this together with (22) leads to (14).

Theorem 10. (limiting profile) Let $u_0\in H^1$, $\mu=\pm1$, $p=1+\frac{4}{N}$ and $1<q<1+\frac{4}{N}$. Assume that $\|u_0\|_2=\|Q\|_2$ and $u(t,x)$ be a corresponding solution of Equation (1) which blows up in finite time T. Then there exists a function $x(t)\in\mathbb{R}^N$ and $\theta(t)\in[0,2\pi)$ such that

$$\rho^{\frac{N}{2}}(t)u(t,\rho(t)(\cdot+x(t)))\mathrm{e}^{i\theta(t)}\to Q \text{ strongly in } H^1,\ \text{as } t\to T, \tag{23}$$

where $\rho(t)=\frac{\|\nabla Q\|_2}{\|\nabla u(t)\|_2}$.

Proof. According to Theorem 9, we get $\|V\|_2^2\ge\|Q\|_2^2$ (see (20)), which together with mass conservation (3) implies

$$\|Q\|_2^2\le\|V\|_2^2\le\liminf_{n\to\infty}\|v_n\|_2^2\le\liminf_{n\to\infty}\|u(t_n)\|_2^2=\|u_0\|_2^2=\|Q\|_2^2.$$

Therefore, we have

$$\lim_{n\to\infty}\|v_n\|_2^2=\|V\|_2^2=\|Q\|_2^2. \tag{24}$$

Combining with (19), we claim that $v_n(\cdot+x_n)$ is bounded in H^1 and

$$v_n(\cdot+x_n)\to V \text{ strongly in } L^2 \text{ as } n\to\infty. \tag{25}$$

By Gagliardo-Nirenberg inequality (see Lemma 4), one has

$$\begin{aligned}\|v_n(\cdot+x_n)-V\|_{2+\frac{4}{N}}^{2+\frac{4}{N}}&\le\frac{N+2}{N}\frac{\|v_n(\cdot+x_n)-V\|_2^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}}\|\nabla(v_n(\cdot+x_n))-V\|_2^2\\&\le C\|v_n(\cdot+x_n)-V\|_2^{\frac{4}{N}}.\end{aligned} \tag{26}$$

Then, we deduce from (25) and (26) that

$$v_n(\cdot+x_n)\to V \text{ in } L^{2+\frac{4}{N}} \text{ as } n\to\infty. \tag{27}$$

Now, we will show that

$$v_n(\cdot+x_n)\to V \text{ strongly in } H^1 \text{ as } n\to\infty. \tag{28}$$

On the one hand, from (18), (27) and Lemma 4, we can estimate as below

$$\frac{1}{2}\int|\nabla Q(x)|^2\,\mathrm{d}x=\frac{1}{2+\frac{4}{N}}\int|V(x)|^{2+\frac{4}{N}}\,\mathrm{d}x\le\frac{1}{2}\frac{\|V\|_2^{\frac{4}{N}}}{\|Q\|_2^{\frac{4}{N}}}\|\nabla V\|_2^2=\frac{1}{2}\|\nabla V\|_2^2. \tag{29}$$

On the other hand, this together with (16), we have

$$\|\nabla V\|_2\le\liminf_{n\to\infty}\|\nabla v_n(\cdot+x_n)\|_2=\|\nabla Q\|_2\le\|\nabla V\|_2,$$

which indicates (28) holds and

$$\|V\|_{H^1}=\|Q\|_{H^1}.$$

Hence, we get

$$G(V)=\frac{1}{2}\int\left|\nabla V(x)\right|^{2}\mathrm{d}x-\frac{1}{2+\frac{4}{N}}\int\left|V(x)\right|^{2+\frac{4}{N}}\mathrm{d}x=0.$$

Up to now, the properties of the profile V can be summarized as below,

$$\|V\|_{2}=\|Q\|_{2},\ \|\nabla V\|_{2}=\|\nabla Q\|_{2},\ G(V)=0.$$

Therefore, the variational characterization of the ground state implies that there exists $\theta\in[0,2\pi)$ and $x_{0}\in\mathbb{R}^{N}$ such that

$$V(x)=\mathrm{e}^{i\theta}Q(x+x_{0})$$

and

$$\rho^{\frac{N}{2}}u(t_{n},\rho_{n}(\cdot+x_{0}))\to\mathrm{e}^{i\theta}Q(\cdot+x_{0})\text{ strongly in }H^{1}\text{ as }n\to\infty.$$

Since $\{t_{n}\}_{n=1}^{\infty}$ is an arbitrary sequence, we claim that there exist two functions $\theta(t)\in[0,2\pi)$ and $x(t)\in\mathbb{R}^{N}$ such that

$$\rho^{\frac{N}{2}}\mathrm{e}^{i\theta(t)}u\big(t,\rho(t)(x+x(t))\big)\to Q\text{ strongly in }H^{1}\text{ as }t\to T.$$

Therefore, the conclusion (23) holds.

Remark 2. For $\mu=1$ and $1<q<p=1+\frac{4}{N}$, Feng [12] obtained the dynamics of blow-up solutions, including L^{2}-concentration, location of L^{2}-concentration point, limiting profile and the blow-up rate. Our conclusions in Theorems 9 and 10 can be seen as complements to the corresponding ones in [12].

5. Orbital Stability of Standing Waves

In this part, we consider the orbital stability of standing waves of Equation (1). In particular, the standing wave solutions to Equation (1) are solutions of the form $\mathrm{e}^{iwt}v(x)$, where $w\in\mathbb{R}$ is a frequency and $v\in H^{1}$ is a solution to the stationary equation

$$-\Delta v+wv-|v|^{p-1}v-\mu|v|^{q-1}v=0. \tag{30}$$

In addition, to research the orbital stability of standing waves, we first establish the variational problem as follows:

$$d_{M}:=inf\{E(v);v\in S\}, \tag{31}$$

where

$$S=:\left\{v\in H^{1};\|v\|_{2}^{2}=M\right\},\text{ for }M>0.$$

In what follows, we denote the set of whole minimizers to (31) by

$$S_{M}:=\left\{v\in H^{1};E(v)=d_{M},\|v\|_{2}^{2}=M\right\},$$

which is also called the set of normalized ground states. Then, for any $v(x)\in S_{M}$, by the Euler-Lagrange theorem, we infer that there exists $w>0$ such that $v(x)$ is a solution of (30) and we generally call $\mathrm{e}^{iwt}v(x)$ as the orbit of $v(x)$. On the other hand, if $v\in S_{M}$, that is, u is a minimizer of d_{M}, then $\mathrm{e}^{iwt}v\in S_{M}$, *i.e.*,

$e^{iwt}v$ is also a minimizer of d_M. Now, we give out the definition of orbital stability of the set S_M.

Definition 1. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for any u_0 satisfying

$$\inf_{v \in S_M} \|u_0 - v\|_\Sigma < \delta,$$

the corresponding solution $u(t,x)$ of (1) satisfies

$$\inf_{v \in S_M} \|u(t,x) - v\|_\Sigma < \varepsilon, \ \forall\, t > 0,$$

then the set S_M is called orbitally stable.

In order to investigate the compactness of any minimizing sequence for (31), we introduce the corresponding profile decomposition of bounded sequences in H^1.

Lemma 11. [5] Let $N \geq 2$, $1 < p < 1 + \frac{4}{N-2}$ and $\{v_n\}$ be a bounded sequence in H^1. Then, there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$), a family $\{x_n^j\}_{n=1}^\infty$ of sequences in $\mathbb{R}^N$ and a sequence $\{V^j\}_{j=1}^\infty$ in H^1 such that

1) for every $m \neq j$, $|x_n^m - x_n^j| \to +\infty$, as $n \to \infty$;

2) for every $l \geq 1$ and every $x \in \mathbb{R}^N$, $v_n(x)$ can be decomposed as

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + r_n^l(x), \tag{32}$$

with $\limsup_{n\to\infty} \|r_n^l\|_q \to 0$ as $l \to \infty$ for every $q \in \left[2, \frac{N+2}{N-2}\right)$. Moreover,

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|r_n^l\|_2^2 + o(1), \tag{33}$$

$$\|\nabla v_n\|_2^2 = \sum_{j=1}^{l} \|\nabla V^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1), \tag{34}$$

$$\|v_n\|_{p+1}^{p+1} = \sum_{j=1}^{l} \|V_n^j\|_{p+1}^{p+1} + \|r_n^l\|_{p+1}^{p+1} + o(1), \tag{35}$$

where $o(1) = o_n(1) \to 0$ as $n \to \infty$.

The main result of this section is the following orbital stability of standing waves for Equation (1) with $\mu = -1$, $1 < q < p \leq 1 + \frac{4}{N}$ or $\mu = 1$, $1 + \frac{4}{N} = p < q < \frac{N+2}{N-2}$.

Theorem 12. Let $N \geq 2$ and Q be the ground state solution of the L^2-critical elliptic Equation (8), then the set S_M is not empty, and it is orbitally stable in the following cases:

1) $\mu = -1$, $1 < q < p = 1 + \frac{4}{N}$ and $M \in \left(0, \|Q\|_2^2\right)$;

2) $\mu = -1$, $1 < p, q < 1 + \frac{4}{N}$ and any $M > 0$;

3) $\mu = 1$, $1 + \frac{4}{N} = p < q < \frac{N+2}{N-2}$ and $M \in \left(\|Q\|_2^2, \infty\right)$.

To prove Theorem 12, we first apply the profile decomposition and concentration-compactness principle to derive the key conclusion as below.

Proposition 13. Let $N \ge 2$ and Q be the ground state solution of elliptic equation (8), if one of the following conditions hold:

1) $\mu = -1$, $1 < q < p = 1 + \frac{4}{N}$ and $M \in \left(0, \|Q\|_2^2\right)$;

2) $\mu = -1$, $1 < p, q < 1 + \frac{4}{N}$ and any $M > 0$;

3) $\mu = 1$, $1 + \frac{4}{N} = p < q < \frac{N+2}{N-2}$ and $M \in \left(\|Q\|_2^2, \infty\right)$,

then there exists $v_0 \in H^1$ such that $d_M = E(v_0)$.

Proof. We first show the part 1) and 2) of Proposition 13. The main argument is to prove that the variational problem (31) is well-defined and every minimizing sequence for (31) is bounded in H^1. For case 3), inject the Gagliardo-Nirenberg inequality (see Lemma 4) into the energy functional $E(v)$, one has the following estimate

$$
\begin{aligned}
E(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - \frac{1}{q+1}\|v\|_{q+1}^{q+1} \\
&\ge \left(\frac{1}{2} - \frac{\|v\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}}\right)\|\nabla v\|_2^2 - C_{q,N}\|v\|_2^{q+1-\frac{N(q-1)}{2}}\|\nabla v\|_2^{\frac{N(q-1)}{2}}.
\end{aligned}
\tag{36}
$$

Since $1 < q < 1 + \frac{4}{N}$, we get $\frac{N(q-1)}{2} < 2$. It follows from Young's inequality that for any $0 < \varepsilon < \frac{1}{2}$, there exists a constant $C\left(\varepsilon, \|Q\|_2, M\right)$ such that

$$
C_{q,N}\|v\|_2^{q+1-\frac{N(q-1)}{2}}\|\nabla v\|^{\frac{N(q-1)}{2}} \le \varepsilon\|\nabla v\|_2^2 + C\left(\varepsilon, \|Q\|_2, M\right),
$$

this together with (36), which implies

$$
E(v) \ge \left(\frac{1}{2} - \frac{\|v\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}} - \varepsilon\right)\|\nabla v\|_2^2 - C\left(\varepsilon, \|Q\|_2, M\right).
$$

Therefore, combining the hypothesis $\|v\|_2^2 = M < \|Q\|_2^2$, we infer

$$
E(v) \ge -C\left(\varepsilon, \|Q\|_2, M\right).
$$

Regarding case 2), similarly, one can discover that for any $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{4}$ and $M > 0$

$$
\begin{aligned}
E(v) &\ge \left(\frac{1}{2} - \varepsilon_1 - \varepsilon_2\right)\|\nabla v\|_2^2 - C\left(p, q, \varepsilon_1, \varepsilon_2, M, \|Q\|_2\right) \\
&\ge -C\left(p, q, \varepsilon_1, \varepsilon_2, M, \|Q\|_2\right).
\end{aligned}
$$

which means that $E(v)$ has a finite lower bound and the variational problem (31) is well-defined.

Secondly, we shall show that every minimizing sequence of (31) is bounded in

H^1. Let $\{v_n\}_{n=1}^{+\infty}$ be the minimizing sequence of the variational problem (31) such that

$$E(v_n) \to d_M, \ \|v_n\|_2^2 \to M \ \text{ as } n \to \infty. \tag{37}$$

It follows from (37) that for n large enough, $E(v_n) < d_M + 1$. Thus, in case 1), for all $0 < \varepsilon < \frac{\|v\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}}$, we have

$$\left(\frac{1}{2} - \frac{\|v\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}} - \varepsilon\right)\|\nabla v\|_2^2 \le d_M + 1 + C\left(\varepsilon, \|Q\|_2, M\right).$$

In terms of case 2), for all $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{4}$, we have

$$\left(\frac{1}{2} - \varepsilon_1 - \varepsilon_2\right)\|\nabla v\|_2^2 \le d_M + 1 + C\left(p, q, \varepsilon_1, \varepsilon_2, M, \|Q\|_2\right).$$

This implies that $\{v_n\}_{n=1}^{+\infty}$ is bounded in H^1. Now, let $v \in H^1$ be a fixed function. Set $v_\nu = \nu^{\frac{N}{2}} v(\nu x)$. We get

$$\|v_\nu\|_2^2 = \|v\|_2^2 = M$$

and

$$\begin{aligned} E(v_\nu) &= \frac{1}{2}\|\nabla v_\nu\|_2^2 - \frac{1}{p+1}\|v_\nu\|_{p+1}^{p+1} - \frac{1}{q+1}\|v_\nu\|_{q+1}^{q+1} \\ &= \frac{\nu^2}{2}\|\nabla v\|_2^2 - \frac{\nu^{\frac{N(p-1)}{2}}}{p+1}\|v\|_{p+1}^{p+1} - \frac{\nu^{\frac{N(q-1)}{2}}}{q+1}\|v\|_{q+1}^{q+1}. \end{aligned}$$

For cases 1) and 2), that is $1 < p \le 1 + \frac{4}{N}$ and $1 < q < 1 + \frac{4}{N}$, since $\frac{N(q-1)}{2} < 2$, one can choose a sufficiently small $\nu > 0$ such that

$$\frac{\nu^2}{2}\|\nabla v\|_2^2 - \frac{\nu^{\frac{N(q-1)}{2}}}{q+1}\|v\|_{q+1}^{q+1} < 0,$$

which means $E(v_\nu) < 0$. Therefore, we get $d_M < 0$. For n large enough, there exists a small $\delta > 0$ such that

$$\begin{aligned} \frac{1}{p+1}\|v_n\|_{p+1}^{p+1} + \frac{1}{q+1}\|v_n\|_{q+1}^{q+1} &= \frac{1}{2}\|\nabla v_n\|_2^2 - E(v_n) \\ &\ge \delta - d_M, \end{aligned}$$

which implies for n large enough, there exists a constant $C_0 > 0$ such that

$$\frac{1}{p+1}\|v_n\|_{p+1}^{p+1} + \frac{1}{q+1}\|v_n\|_{q+1}^{q+1} \ge C_0. \tag{38}$$

Thirdly, based on the above conclusions, we apply Lemma 11 to the minimizing sequence $\{v_n\}_{n=1}^{+\infty}$. Up to a subsequence, $v_n(x)$ can be decomposed as

$$v_n(x)=\sum_{j=1}^{l}\tau_{x_n^j}V^j(x)+r_n^l, \tag{39}$$

with $\limsup_{n\to\infty}\left\|r_n^l\right\|_{q+1}\to 0$ as $l\to\infty$ for every $q\in\left[1,\frac{N+2}{N-2}\right)$. It follows from (39) and (33) - (35) that

$$E(v_n)=\sum_{j=1}^{l}E(V^j)+E(r_n^l)+o(1),\ \text{as } n\to\infty \text{ and } l\to\infty. \tag{40}$$

Using the scaling transform $V_{\lambda_j}^j(x)=\lambda_j V^j(x)$ with $\lambda_j=\frac{\sqrt{M}}{\left\|V^j\right\|_2}>1$. For every $V^j(1\le j\le l)$, it's clear that

$$\left\|V_{\lambda_j}^j\right\|_2^2=M \text{ and } \lambda_j^{p-1}-1,\lambda_j^{q-1}-1\ge\lambda_j^{\min\{p,q\}-1}-1. \tag{41}$$

Then, inject $V_{\lambda_j}^j(x)=\lambda_j V^j(x)$ into energy functional $E\left(V_{\lambda_j}^j\right)$.

$$\begin{aligned}E\left(V_{\lambda_j}^j\right)&=\frac{\lambda_j^2}{2}\left\|\nabla V^j\right\|_2^2-\frac{\lambda_j^{p+1}}{p+1}\left\|V^j\right\|_{p+1}^{p+1}-\frac{\lambda_j^{q+1}}{q+1}\left\|V^j\right\|_{q+1}^{q+1}\\&=\lambda_j^2E(V^j)-\frac{\lambda_j^2\left(\lambda_j^{p-1}-1\right)}{p+1}\left\|V^j\right\|_{2+\frac{4}{N}}^{2+\frac{4}{N}}-\frac{\lambda_j^2\left(\lambda_j^{q-1}-1\right)}{q+1}\left\|V^j\right\|_{q+1}^{q+1},\end{aligned}$$

which means that

$$E(V^j)=\frac{E\left(V_{\lambda_j}^j\right)}{\lambda_j^2}+\frac{\lambda_j^{p-1}-1}{p+1}\left\|V^j\right\|_{p+1}^{p+1}+\frac{\lambda_j^{q-1}-1}{q+1}\left\|V^j\right\|_{q+1}^{q+1}. \tag{42}$$

Similarly, for the term $E\left(r_n^l\right)$, we get the estimate as below

$$\begin{aligned}E\left(r_n^l\right)&=\frac{\left\|r_n^l\right\|_2^2}{M}E\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}r_n^l\right)+\frac{\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}\right)^{p-1}-1}{p+1}\left\|r_n^l\right\|_{p+1}^{p+1}\\&\quad+\frac{\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}\right)^{q-1}-1}{q+1}\left\|r_n^l\right\|_{q+1}^{q+1}+o(1)\\&\ge\frac{\left\|r_n^l\right\|_2^2}{M}E\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}r_n^l\right)+o(1).\end{aligned} \tag{43}$$

From (41), we obtain $\left\|V_{\lambda_j}^j\right\|_2^2=M=\left\|\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}r_n^l\right\|_2^2$. By the definition of d_M, one has

$$E\left(V_{\lambda_j}^j\right)\ge d_M \text{ and } E\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}r_n^l\right)\ge d_M. \tag{44}$$

Since $\sum_{j=1}^{l}\left\|V^j\right\|_2^2$ is convergent, there exists $j_0\ge 1$ such that

$$\inf_{j\geq 1}\lambda_j^{\min\{p,q\}-1}=\lambda_{j_0}^{\min\{p,q\}-1}=\left(\frac{\sqrt{M}}{\left\|V^{j_0}\right\|_2}\right)^{\min\{p,q\}-1}. \quad (45)$$

It follows from (40) - (45) that

$$\begin{aligned}E(v_n)&\geq\sum_{j=1}^{l}\left(\frac{E\left(V_{\lambda_j}^j\right)}{\lambda_j^2}+\frac{\lambda_j^{p-1}-1}{p+1}\left\|V^j\right\|_{p+1}^{p+1}+\frac{\lambda_j^{q-1}-1}{q+1}\left\|V^j\right\|_{q+1}^{q+1}\right)\\&\quad+\frac{\left\|r_n^l\right\|_2^2}{M}E\left(\frac{\sqrt{M}}{\left\|r_n^l\right\|_2}r_n^l\right)+o(1)\\&\geq\sum_{j=1}^{l}\frac{d_M}{\lambda_j^2}+\inf_{j\geq 1}\frac{\lambda_j^{p-1}-1}{p+1}\left(\sum_{j=1}^{l}\left\|V^j\right\|_{p+1}^{p+1}\right)\\&\quad+\inf_{j\geq 1}\frac{\lambda_j^{q-1}-1}{q+1}\left(\sum_{j=1}^{l}\left\|V^j\right\|_{q+1}^{q+1}\right)+\frac{\left\|r_n^l\right\|_2^2}{M}d_M+o(1)\\&\geq\sum_{j=1}^{l}\frac{d_M}{\lambda_j^2}+\frac{\left\|r_n^l\right\|_2^2}{M}d_M+o(1)\\&\quad+\inf_{j\geq 1}\left(\lambda_j^{\min\{p,q\}-1}-1\right)\left(\frac{1}{p+1}\left\|v_n\right\|_{p+1}^{p+1}+\frac{1}{q+1}\left\|v_n\right\|_{q+1}^{q+1}\right).\end{aligned} \quad (46)$$

Let $n\to\infty$ and $l\to\infty$ in (46), combining (38) and (45), one has that

$$d_M\geq d_M+C_0\left(\left(\frac{\sqrt{M}}{\left\|V^{j_0}\right\|_2}\right)^{\min\{p,q\}-1}-1\right),$$

which yields

$$\left\|V^{j_0}\right\|_2^2\geq M.$$

But from (33), we have $\left\|V^{j_0}\right\|_2^2\leq M$. Hence, there exists only one term $V^{j_0}\neq 0$ in (39) such that $\left\|V^{j_0}\right\|_2^2=M$. Moreover, we infer from (33) - (35) that $E\left(V^{j_0}\right)=d_M$, which indicates that the infimum of the variational problem (31) is attained at V^{j_0}. Thus the cases 1) and 2) of Proposition 33 are proved.

In what follows, we use the arguments of concentration-compactness to demonstrate the part 3) of Proposition 33. We shall divide the proof into three steps.

Step 1. $-\infty<d_M<0$ for all $M\in\left(\|Q\|_2^2,\infty\right)$. Firstly, since $1+\frac{4}{N}<q<\frac{N+2}{N-2}$, from the interpolation inequality, for $v\in H^1$, there exists $0<\theta=\frac{2(q+1)}{(N+2)(q-1)}<1$ such that $\frac{1}{2+\frac{4}{N}}=\frac{\theta}{q+1}+\frac{1-\theta}{2}$ and

$$\|v\|_{2+\frac{4}{N}}\leq\|v\|_2^{1-\theta}\|v\|_{q+1}^{\theta}.$$

From $\|v\|_2^2=M$ and Young' inequality, for arbitrary ε, there exists a constant $C(\varepsilon,q,N,M)>0$ such that

$$\frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \le C(\varepsilon, p, N, M) + \varepsilon\|v\|_{q+1}^{q+1}.$$

Take $0<\varepsilon<\frac{1}{p+1}$, one has that

$$\begin{aligned} E(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{1}{q+1}\|v\|_{q+1}^{q+1} \\ &= \frac{1}{2}\|\nabla v\|_2^2 + \left(\frac{1}{q+1}-\varepsilon\right)\|v\|_{q+1}^{q+1} - C(\varepsilon, q, N, M), \end{aligned} \tag{47}$$

Thus, for any $M>0$, we have $d_M > -\infty$. Next, we show the following inequalities hold

$$d_M \ge 0, \text{ for all } M \in \left(0, \|Q\|_2^2\right] \tag{48}$$

and

$$d_M < 0, \text{ for all } M \in \left(\|Q\|_2^2, +\infty\right). \tag{49}$$

By the Lemma 4 and $\|v\|_2^2 = M \le \|Q\|_2^2$, we can easily get

$$\begin{aligned} E(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{1}{q+1}\|v\|_{q+1}^{q+1} \\ &\ge \left(\frac{1}{2} - \frac{\|v\|_2^{\frac{4}{N}}}{2\|Q\|_2^{\frac{4}{N}}}\right)\|\nabla v\|_2^2 \\ &\ge 0. \end{aligned} \tag{50}$$

Thus, we infer that (48) holds.

On the other hand, for $M > \|Q\|_2^2$, taking $v^\lambda = \lambda^{\frac{N}{2}} v(\lambda x)$, it follows that

$$\begin{aligned} E(v^\lambda) &= \frac{\lambda^2}{2}\|\nabla v\|_2^2 - \frac{\lambda^2}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{\lambda^{\frac{N(q-1)}{2}}}{q+1}\|v\|_{q+1}^{q+1} \\ &= \lambda^2\left(\frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}}\right) - \frac{\lambda^{\frac{N(q-1)}{2}}}{q+1}\|v\|_{q+1}^{q+1}. \end{aligned} \tag{51}$$

Moreover, we set $v=\rho Q$ and $\rho = \left(\frac{M}{\|Q\|_2^2}\right)^{\frac{1}{2}} > 1$ such that $\|v\|_2^2 = M$. Therefore, we deduce from the Pohožaev identities (9) that

$$\begin{aligned} \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2+\frac{4}{N}}\|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} &= \frac{\rho^2}{2}\|\nabla Q\|_2^2 - \frac{\rho^{2+\frac{4}{N}}}{2+\frac{4}{N}}\|Q\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \\ &= \frac{\rho^2}{2}\|\nabla Q\|_2^2\left(1-\rho^{\frac{4}{N}}\right) \\ &< 0. \end{aligned}$$

This together with (51) implies that $E\left(v^{\lambda}\right)<0$ for sufficiently small $\lambda>0$. Then, we infer from $\left\|v^{\lambda}\right\|_{2}^{2}=\|v\|_{2}^{2}=M$ that (49) holds.

Step 2. In the following, we will show that every minimizing sequence for (31) is bounded in H^{1} and bounded from below in $L^{2+\frac{4}{N}}$. Let $\left\{v_{n}\right\}_{n=1}^{+\infty}$ be the minimizing sequence, then $\left\|v_{n}\right\|_{2}^{2}=M$, combining (47), we have $\left\{v_{n}\right\}$ that is bounded in H^{1}. In addition, since $d_{M}<0$, we have $E\left(v_{n}\right) \leq \frac{1}{2} d_{M}$ for n large enough. Furthermore, together with (47), we obtain

$$-\frac{1}{2} d_{M} \leq-E\left(v_{n}\right) \leq \frac{1}{2+\frac{4}{N}}\left\|v_{n}\right\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} \tag{52}$$

Therefore, $\left\{v_{n}\right\}$ is bounded in H^{1} and bounded from below in $L^{2+\frac{4}{N}}$.

Step 3. The compactness of minimizing sequence $\left\{v_{n}\right\}$ occurs. Let $\left\{v_{n}\right\}_{n=1}^{+\infty}$ be the minimizing sequence for (31). Firstly, we deduce from (49) that there exists $\xi>0$ such that $\left\|\nabla v_{n}\right\|_{2} \geq \xi$. If overwise, then

$$\left\|\nabla v_{n}\right\|_{2} \rightarrow 0 \text { as } n \rightarrow \infty. \tag{53}$$

From the Gagliardo-Nirenberg inequality (see Lemma 4), we deduce from (50) that

$$\begin{aligned}
\left(\frac{1}{2}-\frac{\left\|v_{n}\right\|_{2}^{\frac{4}{N}}}{2\|Q\|_{2}^{\frac{4}{N}}}\right)\left\|\nabla v_{n}\right\|_{2}^{2} &\leq E\left(v_{n}\right) \\
&=\frac{1}{2}\left\|\nabla v_{n}\right\|_{2}^{2}-\frac{1}{2+\frac{4}{N}}\left\|v_{n}\right\|_{2+\frac{4}{N}}^{2+\frac{4}{N}}+\frac{1}{q+1}\left\|v_{n}\right\|_{q+1}^{q+1} \\
&\leq \frac{1}{2}\left\|\nabla v_{n}\right\|_{2}^{2}+\frac{1}{q+1}\left\|v_{n}\right\|_{q+1}^{q+1} \\
&\leq \frac{1}{2}\left\|\nabla v_{n}\right\|_{2}^{2}+C(q, N, M)\left\|\nabla v_{n}\right\|_{2}^{\frac{N(q-1)}{2}} .
\end{aligned}$$

This together with (53), we infer that

$$E\left(v_{n}\right) \rightarrow 0 \text { as } n \rightarrow \infty,$$

which is a contradiction with

$$E\left(v_{n}\right) \rightarrow d_{M}<0 \text { as } n \rightarrow \infty.$$

Therefore, S can be rewritten as

$$S=\left\{v \in H^{1}:\|v\|_{2}^{2}=M,\|\nabla v\|_{2} \geq \xi\right\},$$

then,

$$d_{M}=\inf \{E(v): v \in S\}.$$

In addition, take $\bar{v}=v\left(\mu^{-\frac{1}{N}} x\right)$ and $\mu>1$, we have $\|\bar{v}\|_{2}^{2}=\mu\|v\|_{2}^{2}=\mu M$.

Thus, we note

$$S_0 = \left\{ v \in H^1 : \|v\|_2^2 = \mu M, \|\nabla v\|_2 \geq \xi \right\}$$

and

$$\overline{S} = \left\{ \overline{v} \in H^1 : \|\overline{v}\|_2^2 = \mu M, \|\nabla \overline{v}\|_2 \geq \mu \xi \right\}.$$

It's clear that

$$\forall v \in S \Leftrightarrow \overline{v} \in \overline{S} \subset S_0.$$

Therefore,

$$\begin{aligned} d_{\mu M} &= \inf \left\{ E(v) : v \in S_0 \right\} \\ &\leq \inf \left\{ E(\overline{v}) : \overline{v} \in \overline{S} \right\} \\ &= \inf \left\{ E(\overline{v}) : v \in S \right\}, \end{aligned} \tag{54}$$

combining $\|\nabla v_n\|_2 \geq \xi$, one has that

$$\begin{aligned} E(\overline{v}) &= \frac{\mu^{1-\frac{N}{2}}}{2} \|\nabla v\|_2^2 - \frac{\mu}{2+\frac{4}{N}} \|v\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + \frac{\mu}{q+1} \|v\|_{q+1}^{q+1} \\ &= \mu E(v) - \frac{1}{2}\left(\mu - \mu^{1-\frac{N}{2}} \right) \|\nabla v\|_2^2 \\ &\leq \mu E(v) - \frac{\xi^2}{2}\left(\mu - \mu^{1-\frac{N}{2}} \right) < \mu E(v). \end{aligned}$$

Taking the infimum, for all $M > \|Q\|_2^2$, $\mu > 1$, we deduce from (54) that

$$d_{\mu M} \leq \inf \left\{ E(\overline{v}) : v \in S \right\} < \mu \inf \left\{ E(v) : v \in S \right\} = \mu d_M. \tag{55}$$

In fact, if $\beta > \|Q\|_2^2$ and $0 < M - \beta \leq \|Q\|_2^2$, then $\frac{M}{\beta} > 1$ and $\frac{\beta}{M-\beta} > 1$. Moreover, together with (55), we have

$$\begin{aligned} d_M &= d_{\frac{M}{\beta}\cdot\beta} < \frac{M}{\beta} d_\beta = d_\beta + \frac{M-\beta}{\beta} d_\beta \\ &= d_\beta + \frac{M-\beta}{\beta} d_{\frac{\beta}{M-\beta}\cdot M-\beta} \\ &< d_\beta + d_{M-\beta}. \end{aligned} \tag{56}$$

Now, let us apply the concentration compactness principle in H^1 (see Lemma III.1 in [19]) to the minimizing sequence $\{v_n\}$. Firstly, we infer that vanishing cannot occur. If not, $v_n \to 0$ strongly in L^r, $r \in (2, 2^*)$ (see [20], Lemma I.1), it follows that $E(v_n) > 0$ which contradicts to (52).

Next, we show dichotomy cannot occur. Assume by contradiction that there exists a constant $\eta \in (0, M)$, a subsequence, still denoted by $\{v_n\}$ and two bounded sequences $\{v_n^1\}$, $\{v_n^2\} \subset H^1$ such that $n \to \infty$

$$\|v_n - v_n^1 - v_n^2\|_r \to 0, \text{ for } 2 \leq r < 2^*,$$

$$\left\|v_n^1\right\|_2^2 \to \eta,\ \left\|v_n^2\right\|_2^2 \to M-\eta,\ \operatorname{dist}\left(\operatorname{supp} v_n^1, \operatorname{supp} v_n^2\right) \to \infty, \tag{57}$$

$$\liminf_{n\to\infty}\left(\left\|\nabla v_n\right\|_2^2 - \left\|\nabla v_n^1\right\|_2^2 - \left\|\nabla v_n^2\right\|_2^2\right) \ge 0.$$

These imply that

$$\liminf_{n\to\infty}\left(E\left(v_n\right) - E\left(v_n^1\right) - E\left(v_n^2\right)\right) \ge 0,$$

which means that

$$E\left(v_n\right) \ge E\left(v_n^1\right) + E\left(v_n^2\right).$$

Therefore, we obtain

$$\limsup_{n\to\infty}\left(E\left(v_n^1\right) + E\left(v_n^2\right)\right) \le \lim_{n\to\infty} E\left(v_n\right) = d_M.$$

On the other hand, combining (57), it is easy to get

$$\liminf_{n\to\infty} E\left(v_n^1\right) \ge d_\eta \ \text{ and } \ \liminf_{n\to\infty} E\left(v_n^2\right) \ge d_{M-\eta}.$$

Thus, we derive

$$\begin{aligned} d_\eta + d_{M-\eta} &\le \liminf_{n\to\infty} E\left(v_n^1\right) + \liminf_{n\to\infty} E\left(v_n^2\right) \\ &\le \liminf_{n\to\infty}\left(E\left(v_n^1\right) + E\left(v_n^2\right)\right) \\ &\le \limsup_{n\to\infty}\left(E\left(v_n^1\right) + E\left(v_n^2\right)\right) \le d_M, \end{aligned}$$

which is a contradiction with (56). Therefore, using the facts that both vanishing and dichotomy cannot occur, then we conclude that compactness holds. Thus, we deduce that there exist $\{y_n\} \subseteq \mathbb{R}^N$, $\tilde{v}_n = v_n(\cdot + y_n)$ and some $v \in H^1$ such that

$$\tilde{v}_n \to \tilde{v} \ \text{ strongly in } L^2.$$

For $2 \le r < 2^*$, combining with the Hölder and Sobolev inequality, we have

$$\begin{aligned} \left\|\tilde{v}_n - \tilde{v}\right\|_r^r &\le \left\|\tilde{v}_n - \tilde{v}\right\|_2^{2(1-\alpha)} \left\|\tilde{v}_n - \tilde{v}\right\|_{2^*}^{2^*\alpha} \\ &\le C\left\|\tilde{v}_n - \tilde{v}\right\|_2^{2\alpha} \to 0 \ \text{ for some } \alpha \in (0,1). \end{aligned}$$

Thus, together with the weak lower semicontinuity of the H^1 norm and definition of d_M, we get

$$d_M \le E\left(\tilde{v}\right) \le \lim_{n\to\infty} E\left(\tilde{v}_n\right) = \lim_{n\to\infty} E\left(v_n\right) = d_M,$$

which means $E(\tilde{v}) = d_M$. In particular, $E(\tilde{v}_n) \to E(\tilde{v})$, which implies $\left\|\nabla\tilde{v}_n\right\|_2 \to \left\|\nabla\tilde{v}\right\|_2$. Hence, we conclude that $\tilde{v}_n \to \tilde{v}$ strong in H^1. This shows that any minimizing sequence for d_M have the relative compactness.

In the final, we show the orbital stability of standing waves to Equation (1) in terms of Proposition 13.

Proof of Theorem 12. According to Theorem 6, we know that the corresponding solution $u(t,x)$ of Equation (1) exists globally under the assumptions. We argue by contradiction. Assume that there exist ε_0 and a sequence $\{u_{0,n}\}_{n=1}^{\infty}$ such that

$$\inf_{\varphi\in S_M}\left\|u_{0,n}-v\right\|_{H^1}<\frac{1}{n}, \tag{58}$$

and there also exists a time sequence $\{t_n\}_{n=1}^{\infty}$ such that the solution sequence $\{u_n(t_n)\}_{n=1}^{\infty}$ of Equation (1) satisfies

$$\inf_{v\in S_M}\left\|u_n(t_n)-v\right\|_{H^1}\geq\varepsilon_0. \tag{59}$$

Owing to (58) and Lemma 13, we thereby discover

$$\int\left|u_n(t_n)\right|^2\mathrm{d}x=\int\left|u_{0,n}\right|^2\mathrm{d}x\to\int|v|^2\mathrm{d}x=M \tag{60}$$

and

$$E\left(u_n(t_n)\right)\to E(v)=d_M. \tag{61}$$

It follows from (60), (61) and conservation laws that $\{u_n(t_n)\}_{n=1}^{\infty}$ is a minimizing sequence of the variational problem (31). Therefore, combining the argument of Proposition 13, there exists $v_0\in S_M$ such that

$$\left\|u_n(t_n)-v_0\right\|_{H^1}\to 0,\ \text{as } n\to\infty.$$

which contradicts with (59). Then we derive the desired result.

Remark 3. In [18], Soave not only proved the existence of the normalized ground state for Equation (1), but also demonstrated the relative compactness of all the minimizing sequences for d_M by using Lions' concentration-compactness principle [19] [20], as well as the validity of the strict sub-additivity for $M\mapsto d_M$. As a result, when $\mu=-1$ and $1<q<p=1+\frac{4}{N}$, they obtained the same conclusion of part 1) of Theorem 12 by taking advantage of the relative compactness of minimizing sequences in H^1 up to translations, according to the classical Cazenave-Lions' argument [7]. Our approach differs from theirs.

Remark 4. For $\mu=-1$ and $1<q\leq p=1+\frac{4}{N}$, Soave [18] mentioned that the orbital stability of standing waves to Equation (1) can be proved by using concentration-compactness principle. In [21], Bai and Zhang established a new framework and constructed a novel compactness lemma to prove the stability of small solitons of Equation (1). While in this study, we show that the standing waves are orbitally stable by making use of profile decomposition technique.

Statement

The authors declare that no conflict of competing interests exists.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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