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Review of the Dynamics of Falling, Accreting, Spherical Drops Acted Upon by Air Resistance

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Abstract

In this review a detailed treatment of the dynamics of an accreting raindrop falling through stationary mist under the action of gravity and air resistance, is presented. There are two contributions to the air resistance: one term proportional to the velocity and one proportional to the square of the velocity. It is discussed when it is a good approximation to neglect one of these terms. Also, a new form of the solution of the equation of motion is presented when accretion is neglected, but both the linear and quadratic contributions to the air resistance are present. The paper is written in a style making it useful in the teaching of classical dynamics, including in between calculations and providing several examples where the equation of motion of the drop, and the equation for the change of the droplet's mass due to accretion, can be solved analytically in terms of elementary functions.

Keywords

Accreting Raindrop, Gravity, Air Resistance, Dynamics

1. Introduction

There have been many experimental investigations of falling droplets [1]-[3] and further references are found in these articles. Analytical solutions to the motion of a falling droplet acted upon by the usual friction terms in the equation of motion, that are respectively linear and quadratic in the velocity, have been published [4]-[10], but a review article on this topic does not seem to exist. Also, the treatment of several proportionality constants and the published form of the solutions of the equation of motion for different cases can be improved. A presentation of theoretical aspects of this topic, where different cases are treated with a unified notation, is the main object of the present review.

Some of the articles [4]-[10] are concerned with a mass-collecting raindrop falling through a mist, neglecting air resistance. In order to give a useful update for students I will here give a review where the main results in these articles are deduced in a rather detailed way.

Then a new form of the equation of motion of a non-accreting raindrop falling under the action of gravity and air resistance will be deduced. Earlier results of different cases are reviewed in a unified way, and some new forms of the solution of the equation of motion of falling droplets are presented. The formulae are illustrated graphically and compared to observed data.

In the present article it is assumed that the droplets are spherical during the motion.

The paper is organised with the following main topics.

- *Accretion without resistance—raindrops falling through a mist.* This section contains a detailed discussion of how the velocity and acceleration of the drop depends upon time under different assumptions for the rate of accretion. Some new expressions for the velocity are found. The section contains several examples suitable for teaching of classical dynamics.
- *How the terminal velocity of a falling droplet depends upon its radius.* Here the main focus is upon how the mass of the drop depends upon time during the motion. Also the terminal velocity of the drop is discussed, and different models are compared with earlier published results of measurements.
- *Resistance without accretion—slowly falling droplet with linear air resistance*. The velocity as a function of the height the droplet has fallen, is calculated by integrating Newton's 2. Law. Then the velocity, as found in this way, of a water droplet with radius 0.35 mm, is compared with the results of measurements performed by W. Ji *et al.* [2]. There is good agreement for the droplet with radius 0.35 mm.
- *Rapidly falling droplet—quadratic air resistance*. Again the velocity is calculated as a function of the height the droplet has fallen, by integrating Newton's 2. Law. This time the calculated velocity is larger than the corresponding velocity measured by W. Ji *et al.* [2].
- *Droplet falling with both linear and quadratic air resistance.* Here a new form of the solution of Newton's 2. Law is deduced, and a height-velocity relationship is deduced. There is reasonably good agreement with the results obtained in the measurements performed by W. Ji *et al.* [2], although not as good as with only the contribution to the air resistance which is linear in velocity.
- *Effect of variation of air density with height upon the motion of a droplet.* Her two cases are considered: *A droplet falling in an isothermal atmosphere*, and *a droplet falling in an adiabatic atmosphere*.
- *Effects of vertical motion of the air upon the motion of a droplet.*
- *Accretion and resistance.* This section gives the most general treatment including both accretion and air resistance. A general formula for the acceleration is deduced, and the acceleration is constant for a drop which falls from an initial

position where the velocity and mass of the droplet vanish.

- *Energy considerations.* A general formula for the rate of change of the mechanical energy of an accreting drop falling under the action of air resistance, is deduced.

Index-notation:

v_{FF} : Free fall motion.

$v_{(1+\beta)\mathrm{A}}$: Drop falling with accretion. The index β means that the rate of mass accretion, $\dot{m}$ is proportional to the β -power of the velocity, $\dot{m} \propto v_{(1+\beta)\mathrm{A}}^{\beta}$. Then the accretion causes a friction-like term in the equation of motion which is proportional to $\dot{m}v$, and hence to $v_{(1+\beta)\mathrm{A}}^{1+\beta}$.

The index A means that accretion of mass from mist is taken into account.

$v_{\alpha(1+\beta)\mathrm{A}}$: Drop falling with accretion. Here $\dot{m} \propto m^{\alpha} v_{\alpha(1+\beta)\mathrm{A}}^{\beta}$, and hence the “accretion term” in the equation of motion of the droplet is proportional to $m^{\alpha} v_{\alpha(1+\beta)\mathrm{A}}^{1+\beta}$.

$v_{1\mathrm{R}}$: Drop falling without accretion with air resistance proportional to the velocity. The index R means that air resistance is taken into account.

$v_{2\mathrm{R}}$: Drop falling without accretion with resistance proportional to the square of the velocity.

$v_{12\mathrm{R}}$: Drop falling without accretion. Resistance depending linearly on velocity and velocity squared.

$v_{..\mathrm{AR}}$: Drop falling with both accretion and air resistance.

$v_{..\mathrm{T}}$: Terminal velocity.

2. Accretion without Resistance—Raindrops Falling Through Mist

We are going to compare several cases of falling droplets with accretion of matter from the mist they fall through and air resistance. In some cases, it turns out that the motion of the droplets are close to the motion of a freely falling object, with a velocity falling distance-relationship

$$v_{\mathrm{FF}} = \sqrt{2gh} \tag{1}$$

The different cases will be compared by calculating the velocities when the droplets have fallen 1 m and 10 m. Inserting $g = 9.81\ \mathrm{m/s^2}$ we have $v_{\mathrm{FF}}(1) = 4.43\ \mathrm{m/s^2}$ and $v_{\mathrm{FF}}(10) = 14.0\ \mathrm{m/s^2}$.

As noted by K. K. Krane [4] the point of departure for analysing the motion of a raindrop collecting matter while falling through a mist, is the general version of Newton’s 2. Law valid for a body with a mass which varies during the motion,

$$F_{ext} = (mv_{\mathrm{A}})^{\cdot} = m\dot{v}_{\mathrm{A}} + v_{\mathrm{A}}\dot{m} . \tag{2}$$

Here we have used the notation that a dot represents differentiation with respect to time. The mechanism of accretion is inelastic collisions of very small mist-particles with the drop.

Neglecting air resistance and the buoyancy due to the upward force on the droplet by the surrounding air, Newton’s 2. Law applied to the droplet takes the form

$$m\dot{v}_{\mathrm{A}}+v_{\mathrm{A}}\dot{m}=mg\ , \tag{3}$$

where the acceleration of gravity is assumed to be constant during the considered motion.

The acceleration, $a_{\mathrm{A}}=\dot{v}_{\mathrm{A}}$, of the droplet is

$$a_{\mathrm{A}}=g-\frac{\dot{m}}{m}v_{\mathrm{A}}\, , \tag{4}$$

We see that the mass accretion acts against gravity as a resistance to the motion. Hence in some cases the acceleration will decrease with increasing velocity, and the droplet will reach a terminal velocity $v_{\beta mt}$. In the cases with air resistance and vanishing accretion one defines the terminal velocity by the condition that the acceleration of the droplet vanishes, $a_{\beta m}=0$. With vanishing air resistance and non-vanishing accretion this gives a terminal velocity

$$v_{\mathrm{AT}}=\frac{mg}{\dot{m}}\, . \tag{5}$$

Here a problem appears. In the general case with accretion, $m/\dot{m}$ is not constant. For example with a constant rate of mass increase, $\dot{m}=\text{constant}$, m increases with time, and hence v_{0mt} increases with time. This means that with accretion, putting $a_{\beta m}=0$ in the equation of motion is not a valid procedure for finding the terminal velocity. The equation of motion has to be solved, and if the velocity approaches a constant velocity, this is the terminal velocity. This will be illustrated below.

Krane [4] considered several cases. One was that the accreted mass is proportional to the height, *h*, which the droplet has fallen. A droplet falling from an initial state with vanishing velocity, and with $m_0=0$ (*i.e.* with an initial mass which is so small that it can be neglected in the equation of motion of the droplet) then has the mass

$$m=\lambda_{02}h\ , \tag{6}$$

where λ_{02} is a positive constant measured in kg/m. The index 0 means that the rate of accretion, $\dot{m}$, is independent of the mass of the droplet, and the index 2 means that the rate of accretion is proportional to the velocity of the droplet, so that it gives rise to a term in the equation of motion of the droplet, which is proportional to the square of the velocity. Since the droplet's mass comes from accretion, λ_{02} is the accreted mass per unit distance which the droplet falls. Hence

$$\dot{m}=\lambda_{02}\dot{h}=\lambda_{02}v_{2\mathrm{A}}\, . \tag{7}$$

In this case Equation (3) takes the form

$$h\ddot{h}+\dot{h}^2=gh\, ,\quad i.e.\quad \left(h\dot{h}\right)^{\bullet}=gh\quad \text{or}\quad h\dot{h}\left(h\dot{h}\right)^{\bullet}=gh^2\dot{h}\, . \tag{8}$$

Note that the motion of the drop does not depend upon the constant λ_{02}. Integration with the initial condition $\dot{h}(0)=0$ gives

$$\frac{1}{2}\left(h\dot{h}\right)^2 = \frac{g}{3}h^3 \quad \text{or} \quad h^{-1/2}\dot{h} = \sqrt{\frac{2g}{3}} . \tag{9}$$

New integration with the initial condition $h(0)=0$ leads to

$$h = \frac{g}{6}t^2 , \quad v_{2\mathrm{A}} = \dot{h} = \frac{g}{3}t , \quad a_{2\mathrm{A}} = \dot{v}_{2\mathrm{A}} = \frac{g}{3}, \quad v_{2\mathrm{A}} = \sqrt{\frac{2}{3}gh} . \tag{10}$$

In this case, with constant acceleration, there is no terminal velocity. Also, it follows from Equations (7) and (10) that in this case the rate of accretion of mass upon the droplet is a linear function of time, which gives a constant acceleration of the droplet. In may be noted that in this case the velocity at a given height is $v_{2\mathrm{A}} = 0.58\, v_{\mathrm{FF}}$.

Adawi [5] followed up Krane's article and tried to integrate the equation of motion for a drop where the rate of increase of mass due to accretion was given by

$$\dot{m} = \lambda_{\alpha 2} m^{\alpha} v_{\alpha 2\mathrm{A}} , \tag{11}$$

where $\lambda_{\alpha 2}$ is measured in $\mathrm{kg}^{1-\alpha}/\mathrm{m}$, and α is a dimensionless constant with value $\alpha \le 1$. The case considered by Krane [4] has $\alpha = 0$. The general solution of Equation (2) with the rate of mass accretion given in Equation (10) required numerical integration.

Inserting $v_{\alpha 2m} = \dot{h}$ into Equation (10) and integrating with the initial condition $m(0) = m_0$ gives

$$m = \left[m_0^{1-\alpha} + \lambda_{\alpha 2}\left(1-\alpha\right)h \right]^{\frac{1}{1-\alpha}} , \quad \alpha \ne 1 . \tag{12}$$

In the case $\alpha = 1$ Equation (11) takes the form

$$\dot{m} = \lambda_{12} m v_{12\mathrm{A}} = \lambda_{12} m \dot{h} . \tag{13}$$

Integration of Equation (13) with the initial condition $m(0) = m_0$ gives

$$m = m_0 \mathrm{e}^{\lambda_{12} h} , \tag{14}$$

where λ_{12} has dimension m^{-1}. The physical meaning of λ_{12} is that $\lambda_{12} = \ln 2/h_2$, where h_2 is the height that the droplet must fall to double its mass, $m(h_2) = 2m_0$. Hence Equation (14) can be written

$$m = 2^{h/h_2} m_0 . \tag{15}$$

In this case Equation (4) takes the form

$$\dot{v}_{12\mathrm{A}} + \lambda_{12} v_{12\mathrm{A}}^2 = g . \tag{16}$$

The solution of this equation with the initial condition $v_{2m}(0) = 0$ is

$$v_{12\mathrm{A}} = v_{12\mathrm{AT}} \tanh\left(\sqrt{\lambda_{12} g}\, t\right), \quad v_{12\mathrm{AT}} = \sqrt{\frac{g}{\lambda_{12}}} . \tag{17}$$

The value of λ_{12} is calculated in Equation (28) with the result $\lambda_{12} = 2.1 \times 10^{-3}\ \mathrm{m}^{-1}$, giving $v_{12\mathrm{AT}} = 69\ \mathrm{m/s}$. Equation (16) shows that the velocity of the droplet approaches the value $v_{12\mathrm{AT}}$, but does not reach $v_{12\mathrm{AT}}$ in a finite time. In this case $v_{12\mathrm{AT}}$ as given in Equation (17) is an asymptotic terminal velocity of the droplet.

With $h(0)=h_0$ the position of the droplet as a function of time is

$$h=\frac{1}{\lambda_{12}}\ln\cosh\left(\sqrt{\lambda_{12}g}\,t\right). \tag{18}$$

It follows from Equations (16) and (17) that the velocity of the droplet as a function of the height it has fallen is

$$v_{12\text{A}}=v_{12\text{AT}}\sqrt{1-\mathrm{e}^{-2\lambda_{12}h}}\,. \tag{19}$$

For $h<10\ \text{m}$ the exponent in Equation (19) obeys $2\lambda_{12}h<4.2\times10^{-2}$. Hence in this region the expression (19) for the velocity can with good approximation be represented by a series expansion to 2. order in $\lambda_{12}h$. Using Equation (1) this gives

$$v_{12\text{A}}\approx v_{\text{FF}}\left(1-\frac{1}{2}\lambda_{12}h\right). \tag{20}$$

Equation (20) shows that in this region the motion of the droplet is very close to free fall motion. Here the velocities used in our comparison are
$v_{12\text{A}}(1)=4.425\ \text{m/s}$ and $v_{12\text{A}}=13.86\ \text{m/s}$, very close to the free fall velocities $v_{\text{FF}}(1)=4.43\ \text{m/s}^2$ and $v_{\text{FF}}(10)=14.0\ \text{m/s}^2$.

Inserting the expression (17) for the velocity of the droplet and performing the integration with the initial condition $m(0)=m_0$ gives the mass as a function of time and falling height,

$$m=m_0\cosh\left(\sqrt{\lambda_{12}g}\,t\right). \tag{21}$$

B. G. Dick [6] gave a further development of the same problem. The mass of the droplet and its rate of change is

$$m=\frac{4\pi}{3}\rho_{\text{w}}r^3,\quad \dot{m}=4\pi\rho_{\text{w}}r^2\dot{r}, \tag{22}$$

where ρ_{w} is the density of water. Assuming that the rate of mass accretion is proportional to the mass of mist that the droplet passes through per second, *i.e.* the product of the cross section area, velocity and density of the mist, ρ_{mist}, we have

$$\dot{m}=\pi\rho_{\text{mist}}r^2\dot{h}\,. \tag{23}$$

It follows from Equations (22) and (23) that

$$\dot{r}=\gamma_0\dot{h},\quad \gamma_0=\frac{\rho_{\text{mist}}}{4\rho_{\text{w}}}. \tag{24}$$

Integration of this equation with the initial condition $r(0)=r_0$ gives

$$r=r_0+\gamma_0 h\,. \tag{25}$$

Note that Equation (22) corresponds to the case $\alpha=2/3$ in Equation (11). Then Equation (12) takes the form

$$m=\left[m_0^{1/3}+\left(\lambda_{(2/3)2}/3\right)h\right]^3. \tag{26}$$

Putting $\dot{m}$ in Equations (11) and (23) equal to each other and inserting γ_0

from Equation (24), lead to

$$\lambda_{(2/3)2} = \gamma_0 \left(36\pi\rho_{\mathrm{w}}\right)^{1/3}, \tag{27}$$

For a typical value of the mist in a cloud $\gamma_0 = 2.5\times10^{-7}$ and $\lambda_{(2/3)2} = 6\times10^{-5}\ \mathrm{kg}^{1/3}/\mathrm{m}$.

We shall need the value of λ_{12} in Equation (20). It is related to γ_0 by means of Equations (11) with $\alpha = 1$, together with Equations (22) and (24) with $\dot{h} = v$,

$$\dot{m} = \lambda_{12} m v = \lambda_{12}\frac{4\pi}{3}\rho_{\mathrm{w}} r^3 \dot{h} = 4\pi\rho_{\mathrm{w}} r^2 \dot{r} = 4\pi\rho_{\mathrm{w}} r^2 \gamma_0 \dot{h}. \tag{28}$$

It follows that

$$\lambda_{12} = \frac{3\gamma_0}{r}. \tag{29}$$

For a droplet with radius $r = 0.35\ \mathrm{mm}$ this gives $\lambda_{12} = 2.1\times10^{-3}\ \mathrm{m}^{-1}$.

Inserting Equations (22)-(24) into Equation (3) gives the equation of motion of the radius of the drop

$$r\ddot{r} + 3\dot{r}^2 = \gamma_0 g r. \tag{30}$$

Using that

$$r\ddot{r} + 3\dot{r}^2 = \frac{1}{r_2}\left(r^3\dot{r}\right)^{\bullet}, \tag{31}$$

Equation (30) can be written as

$$\left(r^3\dot{r}\right)^{\bullet} = \gamma_0 g r^3. \tag{32}$$

Multiplying by $r^3\dot{r}$ this equation takes the form

$$r^3\dot{r}\left(r^3\dot{r}\right)^{\bullet} = \gamma_0 g r^6 \dot{r}. \tag{33}$$

It follows from Equation (11) that $\dot{r} = 0$ for $r = 0$. Hence we integrate Equation (33) with the initial condition $\dot{r}(0) = 0$ and get

$$\dot{r} = \sqrt{\frac{2}{7}\gamma_0 g}\, r^{\frac{1}{2}}. \tag{34}$$

Integrating this equation with $r(0) = 0$ gives

$$r = \frac{1}{2}at^2, \quad \dot{r} = at, \quad a = \frac{\gamma_0}{7}g. \tag{35}$$

in agreement with a result noted by Dick [6]. Hence

$$m = \frac{\pi}{6}\rho_{\mathrm{w}} a^3 t^6, \quad \dot{m} = \pi\rho_{\mathrm{w}} a^3 t^5. \tag{36}$$

Inserting this into Equation (2) gives the equation of motion of the droplet

$$\ddot{h} + \frac{6}{t}\dot{h} = g. \tag{37}$$

The solution of this equation with the initial condition $\dot{h}(0) = 0$ is that the velocity and acceleration of the droplet are

$$\dot{h} = \frac{g}{7}t, \quad \ddot{h} = \frac{g}{7}. \tag{38}$$

The motion here is similar to that considered in Equation (10), with constant acceleration in both cases, although the rate of accretion is different. In the case considered in Equation (10) the mass is a linear function of the height which the droplet has fallen, and the acceleration of the droplet is $g/3$, while in the present case the radius of the droplet is a linear function of the height, and the acceleration is $g/7$.

The study of falling, accreting droplets was followed up by B. F. Edwards, J. W. Wilder and E. E. Scime in 2001 in an article [8] with title "Dynamics of falling raindrops", where they considered a raindrop which grows in size as it falls through a mist of suspended water droplets. Assuming that the rate of increase of the radius is proportional to the velocity of the droplet as in Equation (20) they wrote the equation of motion of the droplet in the form

$$r\frac{\mathrm{d}}{\mathrm{d}r}\frac{\dot{h}^2}{r} = \frac{2g}{\gamma_0} - 7\frac{\dot{h}^2}{r}, \tag{39}$$

They discussed the velocity radius-relationship of the droplets, but did not solve Equation (39). Introducing $y = \dot{h}^2/r$ and the boundary condition $\dot{h}(r_1) = v_1$, one finds the solution

$$\dot{h} = \sqrt{v_1^2\left(\frac{r_1}{r}\right)^6 + \frac{g}{7\gamma_0}(r - r_1)}. \tag{40}$$

Differentiation with respect to time and use of Equation (23) gives the acceleration as a function of the radius

$$\ddot{h} = \frac{g}{7} + \frac{6\gamma_0 v_1^2 r_1^6}{r^7}. \tag{41}$$

The droplet approaches a motion with a constant terminal acceleration $\ddot{h}_t = g/7$.

The authors confined their attention to spherical drops with radii 0.4 mm < r < 1.0 mm. Then, unless r is extremely close to r_1, the velocity of the drop is of the order $\dot{h} \sim \sqrt{g/7\gamma_0} \approx 56\ \mathrm{m/s}$ which is an unreasonably large velocity. Hence the approximation of neglecting air resistance is not realistic.

In 2019 A. D. Sokal [9] revisited the problem of giving a description of falling raindrops accreting mass during the motion. He assumed that the rate of accretion is given by the general relationship

$$\dot{m} = \lambda_{\alpha\beta} m^{\alpha} v^{\beta}_{\alpha(1+\beta)\mathrm{A}}, \text{ no summation,} \tag{42}$$

where α, β are dimensionless constants, and $\lambda_{\alpha\beta}$ is measured in $\mathrm{kg}^{1-\alpha} \cdot \mathrm{m}^{1-\beta} \cdot \mathrm{s}^{\beta-2}$. For $\beta = 2$ we have $\lambda_{\alpha\beta} = \lambda_{\alpha 2}$ where $\lambda_{\alpha 2}$ is given in Equation (11).

It may be noted [9] that $\alpha = 2/3, \beta = 1$ corresponds to growth of the raindrop proportional to the surface area, and that $\alpha = 2/3, \beta = 2$ corresponds to growth of the raindrop proportional to the volume swept out by the drop along the path, which was considered above in Equations (25)-(33), leading to a constant acceleration $g/7$ for the drop.

It may be noted from Equation (5) that the requirement that a vanishing acceleration shall lead to a constant terminal velocity is that $m/\dot{m} = \text{constant}$, which corresponds to the case $(\alpha, \beta) = (1, 0)$. Putting

$$\dot{m} = \lambda_{11} m \tag{43}$$

Into Equation (5) gives the terminal velocity

$$v_{11\text{AT}} = \frac{g}{\lambda_{11}}. \tag{44}$$

The value of λ_{11} can be estimated by assuming that the rate of change of the droplet's mass due to accretion with Equation (43) is equal to the rate with Equation (13) for a certain velocity v_{12m1}. Then Equation (11) gives

$$\lambda_{11} = \lambda_{12} v_{12\text{A}}. \tag{45}$$

With the value $\lambda_{12} = 2.1 \times 10^{-3}\ \text{m}^{-1}$, and for a typical droplet velocity a hundred meters below its starting point, $v_{12\text{A}} = 30\ \text{m/s}$, this gives $\lambda_{11} = 6.3 \times 10^{-2}\ \text{s}^{-1}$ and $v_{11\text{AT}} = 156\ \text{m/s}$.

Solving Equation (42) gives an exponential increase of the mass of the drop,

$$m = 2^{\frac{t}{T_2}} m_0, \quad T_2 = \frac{\ln 2}{\lambda_{11}} = \frac{v_{1\text{AT}}}{g} \ln 2, \tag{46}$$

where $m_0 = m(0)$, and T_2 is the time taken to double the mass of the drop. This rate of accretion can only be realized under very special circumstances for a brief time $t \ll T_2$. Putting Equation (43) into the equation of motion (4) of the droplet gives

$$\dot{v}_{11\text{A}} = g - \lambda_{11} v_{11\text{A}}. \tag{47}$$

Solving this with the initial condition $v_{11\text{A}}(0) = 0$ gives

$$v_{11\text{A}} = v_{11\text{AT}} \left(1 - \mathrm{e}^{-\lambda_{11} t}\right). \tag{48}$$

where $v_{11\text{AT}}$ is given in Equation (44). This shows that the velocity v_{11mt} is only obtained as a limit in the far future. Using Equation (44) and that $v_{11m} = \dot{h}_{11m}$, and integrating Equation (48) with the initial condition $h_{11m}(0) = 0$, we get

$$h_{11\text{A}} = \frac{v_{11\text{AT}}^2}{g} \left(t + \mathrm{e}^{-\lambda_{11} t} - 1\right). \tag{49}$$

Equations (48) and (49) gives the height-velocity relation

$$h_{11\text{A}} = -\frac{v_{11\text{AT}}^2}{g} \left[\frac{v_{11\text{A}}}{v_{11\text{AT}}} + \ln\left(1 - \frac{v_{11\text{A}}}{v_{11\text{AT}}}\right)\right]. \tag{50}$$

This is plotted with $v_{11\text{A}}$ as a function of $h_{11\text{A}}$ in **Figure 1**.

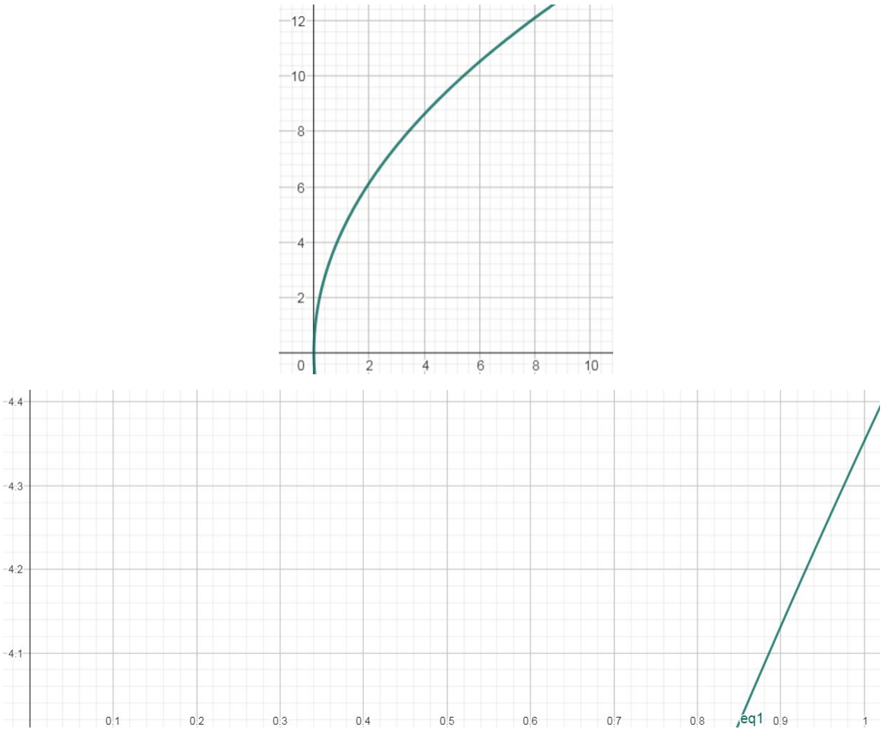

Figure 1. In the upper part of the figure the velocity $v_{11\mathrm{A}}$ given in Equation (48) plotted as a function of $h_{11\mathrm{A}}$. The lower figure shows a magnified part of the curve in order to read off $v_{11\mathrm{A}}(1) = 4.36\ \mathrm{m/s}$ with sufficient accuracy to make a comparison with the free fall velocity $v(1) = 4.43\ \mathrm{m/s}$.

It may be noted that in this case the motion is close to free fall, where $v_{\mathrm{FF}} = \sqrt{2gh}$, which gives $v(1) = 4.43\ \mathrm{m/s}$ and $v(10) = 14.0$, while it is seen from the graph in **Figure 1** that $v_{11\mathrm{A}}(1) = 4.36\ \mathrm{m/s}$ and $v_{11\mathrm{A}}(10) = 12.1\ \mathrm{m/s}$. The reason for this is that in this region $\lambda_{11} v_{11\mathrm{A}} \ll g$ in Equation (45) because the velocity is much smaller than the terminal velocity, 156 m/s.

Let us look at the motion during the first moments when $v_{11\mathrm{A}} \ll v_{11\mathrm{AT}}$. Making a series expansion of the expression (50) to 2. order in $v_{11\mathrm{A}}/v_{11\mathrm{AT}}$ and using Equation (42) gives

$$v_{11\mathrm{A}} \approx \sqrt{2\lambda_{11} v_{11\mathrm{AT}} h} = \sqrt{2gh} = v_{\mathrm{FF}}, \tag{51}$$

corresponding to free fall motion with constant acceleration *g*, as is also seen from Equation (47) in this approximation.

We now go back to the general case with unspecified values of α and β in Equation (42). Using the chain rule for differentiation,

$$\dot{v}_{\alpha\beta\mathrm{A}} = \frac{\mathrm{d}v_{\alpha\beta\mathrm{A}}}{\mathrm{d}m}\dot{m}, \tag{52}$$

Sokal [8] noted that Equation (2) with the assumption (42) takes the form

$$v_{\alpha\beta m}^{\beta} \frac{\mathrm{d}v_{\alpha\beta m}}{\mathrm{d}m} + \frac{v_{\alpha\beta m}^{\beta}}{m} = \frac{g}{\lambda_{\alpha\beta}} m^{-\alpha} \tag{53}$$

Making the substitution $w = v_{\alpha\beta\mathrm{A}}^{\beta}$ the equation takes the form

$$\frac{\mathrm{d}w}{\mathrm{d}m} + \frac{\beta}{m}w = \frac{\beta g}{\lambda_{\alpha\beta}}m^{-\alpha}. \tag{54}$$

He solved this first order linear differential equation for $v_{\alpha\beta m}(m)$ with the initial condition that $v_{\alpha\beta\mathrm{A}}(m_0) = 0$ and found

$$v_{\alpha\beta\mathrm{A}} = Km^{\frac{1-\alpha}{\beta}}\left[1-\left(m_0/m\right)^{1+\beta-\alpha}\right]^{\frac{1}{\beta}}, \quad K = \left(\frac{\beta}{1+\beta-\alpha}\frac{g}{\lambda_{\alpha\beta}}\right)^{\frac{1}{\beta}}. \tag{55}$$

Differentiating this expression for the velocity with respect to time and using Equation (40) gives the acceleration

$$a_{\alpha\beta m} = \frac{1}{1+\beta-\alpha}\left[1-\alpha+\beta\left(\frac{m_0}{m}\right)^{1+\beta-\alpha}\right]g. \tag{56}$$

If the initial mass of the drop is so small that it can be neglected, $m_0 = 0$, the droplet has a constant acceleration

$$a_{\alpha\beta\mathrm{A}} = \frac{1-\alpha}{1+\beta-\alpha}g. \tag{57}$$

It may be noted that in this case there will be a terminal velocity with vanishing acceleration only if $\alpha = 1$ which was described in Equations (12)-(20) for the case $\beta = 2$ and Equations (43)-(51) for $\beta = 1$.

Inserting $\alpha = 2/3, \beta = 1$ into Equations (55) and (56) give respectively

$$v_{(2/3)1\mathrm{A}} = \frac{3}{4}\left(m^{\frac{1}{3}} - \frac{m_0^{4/3}}{m}\right)\frac{g}{\lambda_{(2/3)1}}, \quad a_{1\mathrm{A}} = \left[1+3\left(\frac{m_0}{m}\right)^{4/3}\right]\frac{g}{4}. \tag{58}$$

In this case Equation (42) gives

$$m = \left(m_0^{1/3} + \frac{\lambda_{(2/3)1}}{3}t\right)^3, \tag{59}$$

and hence,

$$\begin{aligned} v_{(2/3)1\mathrm{A}} &= \frac{g}{4}t + \frac{3gm_0^{1/3}}{4\lambda_{(2/3)1}}\left[1-\left(1+\frac{\lambda_{(2/3)1}}{3m_0^{1/3}}t\right)^{-4}\right], \\ a_{(2/3)1\mathrm{A}} &= \frac{g}{4} + g\left(1+\frac{\lambda_{(2/3)1}}{3m_0^{1/3}}t\right)^{-5}, \quad m_0 \neq 0. \end{aligned} \tag{60}$$

With $m_0 = 0$ Equations (59) and (60) give

$$m = \left(\frac{\lambda_{(2/3)1}}{3}t\right)^3, \quad v_{(2/3)1\mathrm{A}} = \frac{g}{4}t, \quad a_{(2/3)1m} = \frac{g}{4}. \tag{61}$$

Inserting $\alpha = 2/3, \beta = 2$ into Equations (55) and (56) gives

$$v_{(2/3)2\mathrm{A}} = \left[\frac{6}{7}\frac{g}{\lambda_{(2/3)2}}\left(m^{\frac{1}{3}} - \frac{m_0^{7/3}}{m^2}\right)\right]^{1/2}, \quad a_{(2/3)2\mathrm{A}} = \left[1+6\left(\frac{m_0}{m}\right)^{7/3}\right]\frac{g}{7}. \tag{62}$$

In this case Equation (42) takes the form

$$\dot{m} = \lambda_{(2/3)2} m^{2/3} v_{(2/3)2\text{A}} . \tag{63}$$

Inserting the expression (62) for the velocity gives a differential equation which cannot be solved analytically in terms of elementary functions. However, with $m_0 = 0$ in Equation (62), integration of Equation (63) gives

$$m = \left(\frac{\lambda_{(2/3)2} g}{42} \right)^3 t^6 , \tag{64}$$

and then the expressions (60) for the velocity and acceleration of the droplet reduce to

$$v_{(2/3)2\text{A}} = \frac{g}{7} t, \quad a_{(2/3)2\text{A}} = \frac{g}{7} . \tag{65}$$

Similar results were obtained by C. E. Mungan [10] by using the momentum of the droplet as a dependent variable.

Finally in the case $\alpha = 1, \beta = 1$ Equations (55) and (56) take the form

$$v_{11m} = \frac{g}{\lambda_{11}} \left(1 - \frac{m_0}{m} \right), \quad a_{11m} = \frac{m_0}{m} g . \tag{66}$$

Russell Herman has written a very nice chapter on the fall of raindrops [11], developing further the results in [8] and [9], taking air drag into consideration.

Herman suggested that it is more natural to make the radius the dynamic variable than the mass, and assumed the accretion rate to take the form

$$\dot{r} = \gamma_{\mu\nu} r^\mu v^\nu_{\alpha(1+\nu)\text{A}} , \text{ no summation} \tag{67}$$

It follows from Equations (40) and (65) that

$$\dot{m} = 4\pi \rho_\text{w} r^2 \gamma_{\mu\beta} r^\mu v^\beta_{(1+\beta)m} = \lambda_{\alpha\beta} m^\alpha v^\beta_{(1+\beta)m} = \lambda_{\alpha\beta} \left(\frac{4\pi}{3} \rho_\text{w} \right)^\alpha r^{3\alpha} v^\beta_{(1+\beta)m} , \tag{68}$$

showing that $\mu = 3\alpha - 2$, or

$$\alpha = \frac{1}{3} (\mu + 2) . \tag{69}$$

This means that if the first lower index of the velocity is taken from Equation (67), one must use the prescript

$$\dot{r} = \gamma_{\mu\nu} r^\mu v^\nu_{[(1/3)(\mu+2)](1+\nu)\text{A}} , \tag{70}$$

in order to have a notation consistent with that used in connection with Equation (42), and meaning that $\dot{r} \propto r^\mu$ corresponds to $\dot{m} \propto m^{(1/3)(\mu+2)}$. Hence, for example $\alpha = 2/3$ and $\mu = 0$ represent the same physical situation, which is also the case for $\alpha = 1$ and $\mu = 1$. Furthermore Equation (68) gives

$$\lambda_{\left[\frac{1}{3}(2+\mu)\right]\beta} = 3 \left(\frac{4\pi}{3} \rho_\text{w} \right)^{\frac{1}{3}(1-\mu)} \gamma_{\mu\beta} . \tag{71}$$

For $\mu = 1$ this relationship takes the form

$$\lambda_{1\beta} = 3\gamma_{1\beta} . \tag{72}$$

Note also the similarity of the relationship (71) for $\mu = 0$ and that in Equation (27).

This shows that with the assumption (65) the two cases considered above take the following forms:

- Rate of increase of the radius of the raindrop proportional to the surface area: $\mu = 0, \nu = 0$.
- Growth of the raindrop is proportional to the volume swept out along the path: $\mu = 0, \nu = 1$.

Herman further noted that with the accretion formula (67), the equation of motion (3) of a spherical accreting droplet falling without air resistance, takes the form

$$\dot{v}_{\alpha(1+\nu)\mathrm{A}} = g - 3\gamma_{\mu\nu} r^{\mu} v_{\alpha(1+\nu)\mathrm{A}}^{1+\nu} , \tag{73}$$

where α is given in Equation (69). He then considered the case $\mu = \nu = 0$. Then $\dot{r} = \gamma_{00}$, which gives

$$r = r_0 + \gamma_{00} t , \tag{74}$$

where r_0 is the initial radius of the droplet. In this case Equation (73) reduces to

$$\dot{v}_{(2/3)1\mathrm{A}} + \frac{3\gamma_{00}}{r} v_{(2/3)1\mathrm{A}} = g . \tag{75}$$

Using that

$$\dot{v}_{(2/3)1\mathrm{A}} = \gamma_{00} \left(\mathrm{d}v_{(2/3)1\mathrm{A}} / \mathrm{d}r \right) , \tag{76}$$

this equation takes the form

$$\frac{\mathrm{d}v_{(2/3)1\mathrm{A}}}{\mathrm{d}r} + \frac{3}{r} v_{(2/3)1\mathrm{A}} = \frac{g}{\gamma_{00}} . \tag{77}$$

Since

$$\frac{\mathrm{d}v_{(2/3)1\mathrm{A}}}{\mathrm{d}r} + \frac{3}{r} v_{(2/3)1\mathrm{A}} = \frac{1}{r^3} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^3 v_{(2/3)1\mathrm{A}} \right) , \tag{78}$$

Equation (77) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^3 v_{(2/3)1m} \right) = \frac{g}{\gamma_{00}} r^3 . \tag{79}$$

Integration with $v_{(2/3)1m}(r_0) = 0$ gives

$$v_{(2/3)1\mathrm{A}} = \frac{g}{4\gamma_{00}} r \left[1 - \left(\frac{r_0}{r} \right)^4 \right] . \tag{80}$$

Inserting this into Equation (75) gives the acceleration

$$\dot{v}_{(2/3)1\mathrm{A}} = \frac{g}{4} \left[1 + 3 \left(\frac{r_0}{r} \right)^4 \right] . \tag{81}$$

Hence the acceleration approaches a constant "terminal acceleration" $a_T = g/4$.

Inserting Equation (74) into Equation (80) gives the velocity as a function of time

$$v_{(2/3)\mathrm{1A}}=\frac{g}{4\gamma_{00}}\left(r_0+\gamma_{00}t\right)\left[1-\left(\frac{r_0}{r_0+\gamma_{00}t}\right)^4\right]. \tag{82}$$

Due to the relationship (72) with $\beta=0$ this expression is identical to Equation (60).

We shall now consider the case that the accretion is proportional to the volume of mist swept out by the raindrop during the motion, $\alpha=2/3,\beta=1$ or $\mu=0,\nu=1$. In this case the equation of motion, Equation (73), of the droplet takes the form

$$\dot{v}_{(2/3)\mathrm{1A}}+\frac{3\gamma_{01}}{r}v_{(2/3)\mathrm{1A}}^2=g\,. \tag{83}$$

Since

$$\dot{v}_{(2/3)\mathrm{1A}}=\dot{r}\frac{\mathrm{d}v_{(2/3)\mathrm{1A}}}{\mathrm{d}r}=\gamma_{01}v_{(2/3)\mathrm{1A}}\frac{\mathrm{d}v_{(2/3)\mathrm{1A}}}{\mathrm{d}r}, \tag{84}$$

Equation (83) takes the form

$$v_{\{2/3\}\mathrm{1A}}\frac{\mathrm{d}v_{(2/3)\mathrm{1A}}}{\mathrm{d}r}+\frac{3}{r}v_{(2/3)\mathrm{1A}}^2=\frac{g}{\gamma_{01}}\,. \tag{85}$$

Using that

$$v_{(2/3)\mathrm{1A}}\frac{\mathrm{d}v_{(2/3)\mathrm{1A}}}{\mathrm{d}r}+\frac{3}{r}v_{(2/3)\mathrm{1A}}^2=\frac{1}{2r^6}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^6v_{(2/3)\mathrm{1A}}^2\right), \tag{86}$$

Equation (83) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^6v_{(2/3)\mathrm{1A}}^2\right)=\frac{2g}{\gamma_{01}}r^6\,. \tag{87}$$

Integration with $v_{(2/3)\mathrm{1}m}\left(r_0\right)=0$ gives

$$v_{(2/3)\mathrm{1A}}=\sqrt{\frac{2g}{7\gamma_{01}}r\left[1-\left(\frac{r_0}{r}\right)^7\right]}\,. \tag{88}$$

Inserting this into Equation (73) gives the acceleration

$$\dot{v}_{(2/3)\mathrm{1A}}=\frac{g}{7}\left[1+6\left(\frac{r_0}{r}\right)^7\right]. \tag{89}$$

In this case there is no terminal velocity since the motion approaches a state with constant "terminal acceleration" $a_T=g/7$.

Inserting Equation (25) into Equation (87) leads to the velocity-height relation

$$v_{(2/3)\mathrm{1A}}=\sqrt{\frac{2gr_0}{7\gamma_0}\left(1+x\right)\left[1-\left(1+x\right)^{-7}\right]},\quad x=\frac{\gamma_0h}{r_0}, \tag{90}$$

where $\gamma_0=2.5\times10^{-7}$. A typical initial mass of a raindrop is $r_0=2\times10^{-5}\,\mathrm{m}$. Hence for $h\ll r_0/\gamma_0=80\,\mathrm{m}$ we can approximate the expression by a series expansion to 2. order in x. This gives

$$v_{(2/3)1A} \approx v_{FF}\left(1-\frac{3}{2}\frac{\gamma_0 h}{r_0}\right), \tag{91}$$

which leads to $v_{(2/3)1A}(1)=4.37\ \text{m/s}$ and $v_{(2/3)1A}(10)=11.4\ \text{m/s}$, showing again that accretion causes only a very small deviation from free fall motion during the first ten meters that the droplet falls, but with increasing deviation from free fall, the larger the falling distance, and hence the velocity, is.

Since $v_{0A}=\dot{h}$ Equation (80) is a differential equation whose solution gives the position of the droplet as a function of time. The solution involves an integral of the form $\int\frac{\mathrm{d}x}{\sqrt{1-x^7}}$ which cannot be expressed in terms of elementary functions.

For later comparison we also consider the two cases $(\mu,\nu)=(1,0)$ and $(\mu,\nu)=(1,1)$. In the first case Equation (70) shows then the rate of change of the radius of the droplet is proportional to its radius, but independent of its velocity, $\dot{r}=\gamma_{10}r$. Now Equation (72) gives $\lambda_{10}=3\gamma_{10}$, and the equation of motion Equation (73) takes the form

$$\dot{v}_{11A}=g-3\gamma_{10}v_{11A}\,. \tag{92}$$

This has the same form as the equation of motion Equation (47) with solution given in Equation (48). This equation of motion will be further discussed in section 4 in connection with air resistance proportional to the velocity.

In the second case with $(\mu,\nu)=(1,1)$ the rate of increase of the radius is $\dot{r}=\gamma_{11}r\,v_{2A}$, Equation (72) gives $\lambda_{11}=3\gamma_{11}$, and the equation of motion (73) takes the form

$$\dot{v}_{12A}=g-3\gamma_{11}v_{12A}^2\,. \tag{93}$$

In this case there is a terminal velocity

$$v_{12AT}=\sqrt{\frac{g}{3\gamma_{11}}}\,. \tag{94}$$

The solution of Equation (84) will be given in section 5 where air resistance proportional to the square of the velocity will be discussed.

3. How the Terminal Velocity of a Falling Droplet Depends Upon Its Radius

In Equations (92)-(96) we shall not use indices on the velocity because the discussion here is not related to a certain situation, but is valid generally. We now consider a droplet with mass m falling vertically with a velocity v in air where the vertical velocity of the air is so small that it can be neglected. The effects of the vertical motion of the air is considered in section 7. The droplet is acted upon by gravity, mg, and air resistance, often called drag. The air resistance is usually given by a sum of two contributions, f_{r1} and f_{r2}. The term f_{r1} is linear in the velocity of the droplet. It is given by Stokes' law and is dominating at small velocities, so small that the shape of the drop can be assumed to be spherical.

$$f_{r1} = -6\pi \eta_{\text{air}} r v \, , \tag{95}$$

where η_{air} is the viscosity of the air. Under usual atmospheric conditions $\eta_{\text{air}} = 1.8 \times 10^{-5} \text{ kg/s} \cdot \text{m}$.

In this section we shall consider a droplet with radius $r = 0.1$ mm = 10^{-4} m. This gives $6\pi r \eta_{\text{air}} = 3.4 \times 10^{-8} \text{ kg/s}$.

The mass of the droplet with radius $r = 0.1$ mm is $m = (4/3)\pi \rho_{\text{w}} r^3 = 4.2 \times 10^{-9} \text{ kg}$ or 4.2 micrograms. In order to perform a comparison with published measurements [2]. I shall also consider a droplet with radius 0.35 mm, which has the mass 180 microgram.

The other contribution to the air resistance is dominating at larger velocities and is quadratic in the velocity, $f_{r2} \propto v_{12}^2$. In The Physics Hypertextbook [12] Glenn Elert has argued for this in the following way. The point of departure is Bernoullis principle which says that for an element of a fluid in a stationary flow

$$\frac{1}{2}\rho v^2 + \rho g h + p = \text{constant} \, . \tag{96}$$

This is usually taken as an expression of energy conservation: The sum kinetic-, potential- and pressure energy per volume is constant along a stream line of the fluid. It is also formulated in terms of pressure: The sum of dynamic-, gravity- and static pressure is constant. The dynamic pressure is the pressure due to the velocity of the fluid,

$$p_{\text{d}} = \frac{1}{2}\rho v^2 \, . \tag{97}$$

For a droplet moving fairly rapidly through air the main contribution to the air resistance comes from the dynamic pressure. Since pressure is force per area this gives

$$f_{2\text{R}} = -p_{\text{d}} A = -\frac{1}{2}\rho_{\text{air}} A v^2 \, , \tag{98}$$

where ρ_{air} is the density of the air, and A is the effective cross section area of the droplet normal to the direction of motion of the droplet. If one assumes that this is equal to the geometric area, $A = \pi r^2$, where r is the radius of the droplet. Then

$$f_{2\text{R}} = -\frac{\pi}{2}\rho_{\text{air}} r^2 v^2 \, . \tag{99}$$

We now go back to specific cases of falling droplet. In the next paragraphs we shall neglect accretion, and hence A is not included among the indices of the velocities. Let us consider a droplet with a velocity such that both $f_{1\text{R}}$ and $f_{2\text{R}}$ should be taken into account. Hence the air resistance is

$$f_{\text{R}} = -6\pi \eta_{\text{air}} r v_{12\text{R}} - \frac{\pi}{4}\rho_{\text{air}} r^2 v_{12\text{R}}^2 \, . \tag{100}$$

We are considering a droplet falling a short distance so it is a good approximation to consider g as a constant during the motion. With the air resistance given by Equation (97) Newton's 2. Law applied to the droplet then takes the form

$$mg-6\pi\eta_{\mathrm{air}}r\,v_{12\mathrm{R}}-\frac{\pi}{4}\rho_{\mathrm{air}}r^{2}v_{12\mathrm{R}}^{2}=m\dot{v}_{12\mathrm{R}}\,. \tag{101}$$

Consider a droplet falling from rest. Initially the air resistance is very small, and the droplet accelerates. When the velocity increases the air resistance becomes larger and eventually becomes equal to the weight of the droplet. Then the sum of the forces upon the droplet is zero, and its velocity becomes constant. This is the terminal velocity of the droplet.

The terminal velocity with only air resistance which is linear in the velocity, is

$$v_{1\mathrm{RT}}=\frac{mg}{6\pi\eta_{\mathrm{air}}r}\,. \tag{102}$$

This is the maximal velocity of the droplet with only linear air resistance. It represents the strength of the linear resistance in an inverse way: The larger $v_{1\mathrm{RT}}$ is, the smaller is the linear air resistance. Inserting $m=(4/3)\pi\rho_{\mathrm{w}}r^{3}$ gives

$$v_{1\mathrm{RT}}=\frac{2}{9}\frac{\rho_{\mathrm{w}}}{\eta_{\mathrm{air}}}gr^{2}\,. \tag{103}$$

For droplets with radius $r=0.1\,\mathrm{mm}$ and 0.35 mm this gives respectively $v_{1\mathrm{RT}}=1.21\,\mathrm{m/s}$ and 14.8 m/s. Including the buoyancy due to the air, this expression is generalized to

$$v_{1\mathrm{RT}}=\frac{2}{9}\frac{\rho_{\mathrm{w}}-\rho_{\mathrm{air}}}{\eta_{\mathrm{air}}}gr^{2} \tag{104}$$

A typical density of air in a cloud is $\rho_{\mathrm{air}}=1.2\,\mathrm{kg/m^{3}}$ while the density of water is $\rho_{\mathrm{w}}=1.0\times10^{3}\,\mathrm{kg/m^{3}}$. Hence, $\rho_{\mathrm{air}}\ll\rho_{\mathrm{w}}$, and therefore the buoyancy will be neglected further on.

With only the quadratic term for the air resistance the terminal velocity is

$$v_{2\mathrm{RT}}=\frac{2}{r}\sqrt{\frac{mg}{\pi\rho_{\mathrm{air}}}}=4\sqrt{\frac{1}{3}\frac{\rho_{\mathrm{w}}}{\rho_{\mathrm{air}}}gr}\,. \tag{105}$$

For droplets with radius $r=0.1\,\mathrm{mm}$ and 0.35 mm this gives respectively $v_{2\mathrm{RT}}=2.7\,\mathrm{m/s}$ and 3.9 m/s. This velocity represents the strength of the quadratic air resistance: the larger $v_{2\mathrm{RT}}$ is, the smaller is the quadratic air resistance.

Expressed in terms of the terminal velocities (102) and (105) the equation of motion, (101), of the droplet can be written

$$\dot{v}_{12\mathrm{R}}=-g\left(\frac{v_{12\mathrm{R}}^{2}}{v_{2\mathrm{RT}}^{2}}+\frac{v_{12\mathrm{R}}}{v_{1\mathrm{RT}}}-1\right). \tag{106}$$

The two friction terms are equal for the velocity

$$v_{\mathrm{eqR}}=\frac{24\eta_{\mathrm{air}}}{\rho_{\mathrm{air}}r}=\frac{v_{2\mathrm{RT}}^{2}}{v_{1\mathrm{RT}}}\,. \tag{107}$$

The equation of motion of the droplet can now be expressed in terms of $v_{2\mathrm{RT}}$ and v_{eqR} as follows,

$$\dot{v}_{12\mathrm{R}}=-\frac{g}{v_{2\mathrm{RT}}^{2}}\left(v_{12\mathrm{R}}^{2}+v_{\mathrm{eqR}}v_{12\mathrm{R}}-v_{2\mathrm{RT}}^{2}\right). \tag{108}$$

Focusing upon the radial dependence we may write

$$v_{\mathrm{eqR}} = \frac{c_{\mathrm{eqR}}}{r}, \qquad c_{\mathrm{eqR}} = 24\frac{\eta_{\mathrm{air}}}{\rho_{\mathrm{air}}} = 3.2\times10^{-4}\,\frac{\mathrm{m}^2}{\mathrm{s}}. \tag{109}$$

For droplets with radius $r = 0.1\,\mathrm{mm}$ and 0.35 mm this gives respectively $v_{\mathrm{eqR}} = 3.2\,\mathrm{m/s}$ and $v_{\mathrm{eqR}} = 1.0\,\mathrm{m/s}$.

With both friction terms present in the equation of motion, (105), and using Equation (108), Newton's 1. Law gives for the terminal velocity,

$$v_{12\mathrm{RT}}^2 + v_{\mathrm{eqR}} v_{12\mathrm{RT}} - v_{2\mathrm{RT}}^2 = 0. \tag{110}$$

The positive solution, meaning downwards velocity of the droplet, of this equation is

$$v_{12\mathrm{RT}} = \frac{1}{2}\left(\sqrt{v_{\mathrm{eqR}}^2 + 4v_{2\mathrm{RT}}^2} - v_{\mathrm{eqR}}\right), \tag{111}$$

or, by using Equation (104),

$$v_{12\mathrm{RT}} = \frac{v_{\mathrm{eqR}}}{2}\left(\sqrt{1+\left(\frac{2v_{1\mathrm{RT}}}{v_{2\mathrm{RT}}}\right)^2} - 1\right). \tag{112}$$

For the droplets with radius $r = 0.1\,\mathrm{mm}$ and 0.35 mm Equation (112) gives the terminal velocities $v_{12\mathrm{RT}} = 0.93\,\mathrm{m/s}$ and $v_{12\mathrm{RT}} = 3.35\,\mathrm{m/s}$.

Stokes' law is usually said to be valid for droplets with radii less than 0.1 mm and small velocities when there is laminar flow of air at the surface of the droplet. Then the drag force is dominated by friction. For greater velocities the flow of air is non-laminar behind the droplet, and then the pressure behind the droplet becomes lower than at the front side of the droplet. This causes a pressure dominated contribution to the air resistance which is quadratic in the velocity, often called pressure drag.

In this connection J. van Boxel [13] writes that small droplets with diameter < 0.05 mm are spherical, and the flow around the drops can be considered to be laminar. For these droplets the terminal velocity can be found from Stokes' law. Larger drops fall faster, and the flow around the drops becomes turbulent. Therefore Stokes' law will fail for drops with diameter > 0.1 mm. The transition to a turbulent flow regime is characterized by the dimensionless Reynolds number, Re, which expresses (it is not equal to) the ratio between the air resistance with turbulent and laminar air motion around the droplet,

$$\mathrm{Re} = \frac{2\rho_{\mathrm{air}} r v}{\eta_{\mathrm{air}}}, \tag{113}$$

where $\eta_{\mathrm{air}} = 1.8\times10^{-5}\,\mathrm{Pa\cdot s}$ is the dynamical viscosity of air at 20^0C. It may be noted that the Reynolds number can be expressed as

$$\mathrm{Re} = 24\frac{v}{v_{\mathrm{eqR}}}. \tag{114}$$

The ratio of the quadratic and linear air resistance terms in Equation (98) are $\mathrm{Re}/24$ in agreement with the formulae in ref. [14], where the quadratic term of the air resistance is written as (neglecting the vertical velocity of the air),

$$f_{2\mathrm{R}} = -\frac{1}{2}\lambda_{\mathrm{d}} A_{\mathrm{wd}} \rho_{\mathrm{air}} v_{12\mathrm{R}}^2 . \tag{115}$$

Here λ_{d} is the drag coefficient, and A_{wd} is the surface area of a water droplet in the direction of motion, $A_{\mathrm{wd}} = 2\pi r^2$. The authors further write that

$$\lambda_{\mathrm{d}} = \frac{24}{\mathrm{Re}} \text{ for } \mathrm{Re} \le 1, \quad \text{and} \quad \lambda_{\mathrm{d}} = \frac{24}{\mathrm{Re}}\left(1 + 0.14\,\mathrm{Re}^{0.7}\right) \text{ for } \mathrm{Re} > 1. \tag{116}$$

Let us consider how the terminal velocity of a droplet depends upon its radius. Equations (103) and (105) can be written as

$$v_{1\mathrm{RT}} = c_{1\mathrm{RT}} r^2 , \quad c_{1\mathrm{RT}} = \frac{2}{9}\frac{\rho_{\mathrm{w}} g}{\eta_{\mathrm{air}}} = 1.2 \times 10^8 \frac{1}{\mathrm{s} \cdot \mathrm{m}} , \tag{117}$$

and

$$v_{2\mathrm{RT}} = c_{2\mathrm{RT}} \sqrt{r} , \quad c_{2\mathrm{RT}} = 4\sqrt{\frac{\rho_{\mathrm{w}}}{\rho_{\mathrm{air}}}\frac{g}{3}} = 2.1 \times 10^2 \frac{\sqrt{\mathrm{m}}}{\mathrm{s}} . \tag{118}$$

where $c_{2\mathrm{RT}}^2 / c_{1\mathrm{RT}} = c_{\mathrm{eqR}}$.

The expression (112) for the terminal velocity with both the linear and quadratic term for the air resistance may now be written as

$$v_{12\mathrm{RT}} = \frac{c_{\mathrm{eqR}}}{2r}\left(\sqrt{1 + 4\left(\frac{c_{1\mathrm{RT}}}{c_{2\mathrm{RT}}}\right)^2 r^3} - 1\right) . \tag{119}$$

where c_{eqR} is given in Equation (109). For a droplet with radius 0.1 mm $v_{1\mathrm{RT}} = 1.21\ \mathrm{m/s}$, $v_{2\mathrm{RT}} = 1.93\ \mathrm{m/s}$ and $v_{12\mathrm{RT}} = 0.93\ \mathrm{m/s}$. For a droplet with radius 0.35 mm these velocities are $v_{1\mathrm{RT}} = 14.8\ \mathrm{m/s}$, $v_{2\mathrm{RT}} = 3.9\ \mathrm{m/s}$ and $v_{12\mathrm{RT}} = 3.0\ \mathrm{m/s}$. Naturally the terminal velocity is smallest when both the friction terms are present.

The terminal velocity, $v_{12\mathrm{RT}}$, as a function of the radius is plotted in **Figure 2**.

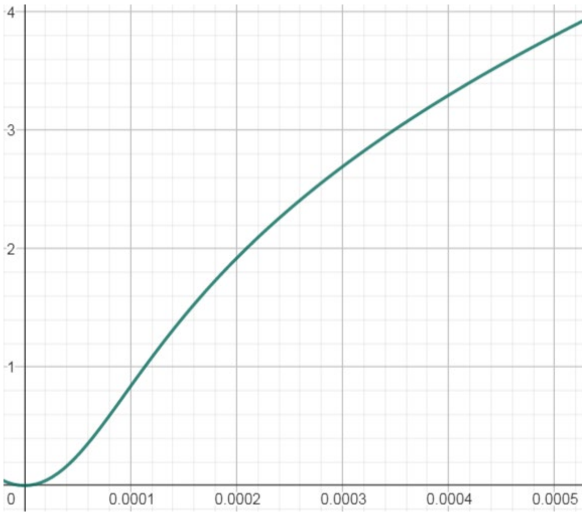

Figure 2. The terminal velocity, $v_{12\mathrm{RT}}$, in m/s as given in Equation (119) as a function of the radius in meter of the droplet for radii $r < 0.5\ \mathrm{mm}$.

In [1] the terminal velocity of small droplet falling in air was measured and plotted as a function of the masses of the droplets. In order to compare with their result we therefore express the terminal velocity in Equation (119) in terms of the mass of the droplets. Using that

$$r^3 = \frac{r_1^3}{m_1} m\,, \tag{120}$$

where $r_1 = 10^{-4}\ \text{m}$, and $m_1 = 4.2 \times 10^{-9}\ \text{kg} = 4.2\ \text{microgram}$, we get

$$v_{12\text{RT}} = a m^{-1/3} \left(\sqrt{1 + bm} - 1\right), \tag{121}$$

where

$$a = \frac{c_{\text{eqR}} m_1^{1/3}}{2 r_1} = 2.6 \frac{\text{m}}{\text{s}} \cdot \text{microgram}^{1/3}, \quad b = 4\left(\frac{c_{1\text{RT}}}{c_{2\text{RT}}}\right)^2 \frac{r_1^3}{m_1} = 1.25\ \text{microgram}^{-1}. \tag{122}$$

The terminal velocity (121) is plotted as a function of the mass of the droplet, as measured in microgram, in **Figure 3**, while the curve found by measurements by Ross Gunn and Gilbert D. Kinzer [1] is shown in **Figure 4**.

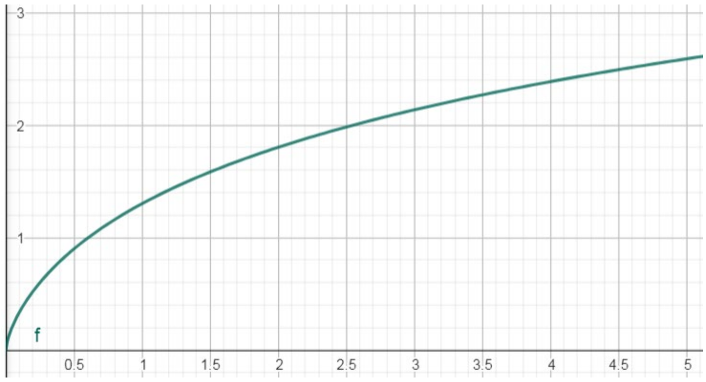

Figure 3. The terminal velocity in m/s as given in Equation (121) as a function of the mass in micrograms of the droplet for masses $m < 4.5$ micrograms.

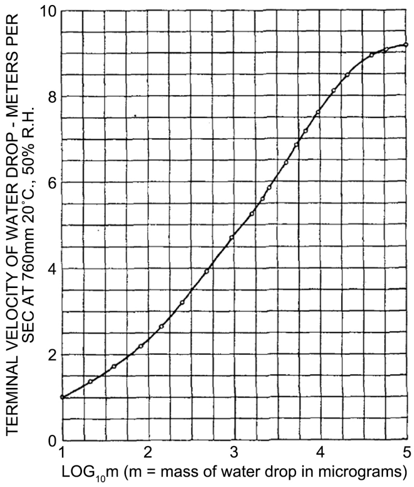

Figure 4. Measured terminal velocity of droplets falling in air as function of the mass of the droplets as measured in micrograms [1].

The theoretical formula (121) gives a smaller velocity than the measured values [1] of the terminal velocity. Adjusting the values of *a* and *b* by requiring the curve in **Figure 2** to coincide with that in **Figure 3** at two points, $v_{12\mathrm{RT}}(m_1)=v_{\mathrm{RT}1}$, $v_{12\mathrm{RT}}(m_2)=v_{\mathrm{RT}2}$, leads to

$$a=\frac{v_{1\mathrm{RT}}m_1^{1/3}}{\sqrt{1+bm_1}-1},\quad b=\frac{A(2-A)m_2-(1-2A)m_1}{\left(m_1-A^2m_2\right)^2},\quad A=\left(\frac{m_1}{m_2}\right)^{1/3}\frac{v_{1\mathrm{RT}}}{v_{2\mathrm{RT}}}. \tag{123}$$

Inserting for illustration $m_1=1\,\mu\mathrm{g}$, $m_2=2\,\mu\mathrm{g}$, $v_{1\mathrm{RT}}=1\,\mathrm{m/s}$, $v_{2\mathrm{RT}}=2.35\,\mathrm{m/s}$ from **Figure 4** gives $a=1.88\frac{\mathrm{m}}{\mathrm{s}}\cdot\mu\mathrm{g}^{1/3}$, $b=1.35\,\mu\mathrm{g}^{-1}$. Since the curves have different shapes other coincidence points will give other values of *a* and *b*.

Accretion increases the mass of the droplet and should therefor make the terminal velocity greater. We shall now investigate the effect of accretion upon the terminal velocity, or whether there will be any terminal velocity at all. Generalizing Equation (110) Newton's 1. Law then takes the form

$$\left(1+\frac{v_{2\mathrm{RT}}^2}{v_{2\mathrm{ART}}^2}\right)v_{12\mathrm{ART}}^2+v_{\mathrm{eqR}}v_{12\mathrm{ART}}-v_{2\mathrm{RT}}^2=0\,. \tag{124}$$

The positive solution of this equation is

$$v_{12\mathrm{ART}}=\frac{v_{\mathrm{eqR}}}{2}\frac{\sqrt{1+4\frac{v_{1\mathrm{RT}}}{v_{\mathrm{eqR}}}\left(1+\frac{v_{2\mathrm{RT}}^2}{v_{2\mathrm{ART}}^2}\right)}-1}{1+\frac{v_{2\mathrm{RT}}^2}{v_{2\mathrm{ART}}^2}}\,. \tag{125}$$

It should be noted from Equation (16) that the limit with vanishing accretion corresponds to an infinitely large value of $v_{2\mathrm{ART}}$. Using Equations (120)-(122) to express $v_{12\mathrm{ART}}$ as a function of the mass of the droplet Equation (125) takes the form

$$v_{12\mathrm{ART}}=a\frac{\sqrt{1+bm\left(1+c_1m^{1/3}\right)}-1}{\left(1+c_1m^{1/3}\right)m^{1/3}}, \tag{126}$$

Using Equations (16), (118) and (120) we obtain

$$c_1=\lambda_{12}\frac{16}{3}\frac{\rho_{\mathrm{w}}}{\rho_{\mathrm{air}}}\frac{r_1}{m_1^{1/3}}\,. \tag{127}$$

$\lambda_{12}=2.1\times10^{-3}\,\mathrm{m}^{-1}$, $\rho_{\mathrm{w}}/\rho_{\mathrm{air}}\approx2.5\times10^3$ ten kilometres above the surface of the Earth, $r_1=10^{-4}\,\mathrm{m}$ and $m_1=4.2\times10^{-9}\,\mathrm{kg}$ give $c_1=1.7\times10^4\,\mathrm{kg}^{-1/3}$.

In the physically most interesting case, where the accretion is proportional to the volume swept out by the falling droplet, $\alpha=2/3$, giving

$$v_{12\mathrm{RT}}=\frac{a}{1+c_{2/3}}\frac{\sqrt{1+b\left(1+c_{2/3}\right)m}-1}{m^{1/3}}, \tag{128}$$

with

$$c_{2/3}=4\frac{\rho_{\mathrm{mist}}}{\rho_{\mathrm{air}}}\,. \tag{129}$$

The density of air ten kilometres above the surface of the Earth is typically $\rho_{\text{air}} = 0.40\ \text{kg}/\text{m}^3$. As noted in Equation (23) $\rho_{\text{mist}} = 4\times 2.5\times 10^{-7}\rho_{\text{w}} = 10^{-3}\ \text{kg}/\text{m}^3$. Hence, an order of magnitude value of $c_{2/3}$ is $c_{2/3} \approx 4\times 10^{-3}$. The small value of $c_{2/3}$ means that the effect of accretion upon the terminal velocity is negligibly small. The physical reason for this is that the density of the mist is around 400 times smaller than the density of the air in a cloud.

4. Resistance without Accretion—Slowly Falling Droplet with Linear Air Resistance

We shall now calculate the falling velocity, $v_{1\text{R}}(t)$, as a function of time of a droplet falling so slowly, $v_{1\text{R}} \ll v_{\text{eqR}}$, that it is sufficient to include the linear term in the expression for the air resistance. Then Newton's 2. Law reduces to

$$\dot{v}_{1\text{R}} = g\left(1 - \frac{v_{1\text{R}}}{v_{1\text{RT}}}\right), \tag{130}$$

where $v_{1\text{RT}}$ is given in Equation (102). The velocity and distance moved by the droplet as a function of time and the velocity-height relationship for this case are given in Equations (48)-(50) with $v_{11\text{AT}}$ replaced by $v_{1\text{RT}}$. The present case has also been studied by J. van Boxel [13]. The velocity-height relationship is

$$h_{1\text{R}} = -\frac{v_{1\text{RT}}^2}{g}\left[\frac{v_{1\text{R}}}{v_{1\text{RT}}} + \ln\left(1 - \frac{v_{1\text{R}}}{v_{1\text{RT}}}\right)\right]. \tag{131}$$

The velocity is plotted from the relationship (131) as a function of the height in Figure 5, which is similar to the initial part of Figure 1. Equation (131) gives $v_{1\text{R}}(1) = 3.85\ \text{m/s}$ and $v_{1\text{R}}(10) = 10.0\ \text{m/s}$.

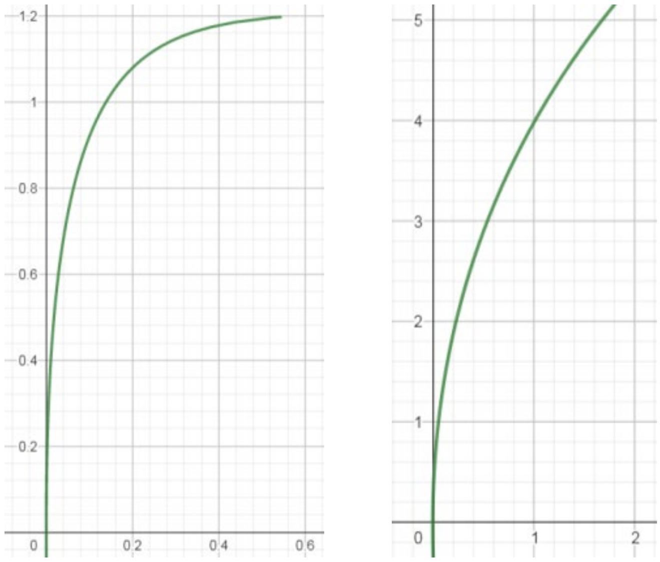

Figure 5. The velocity measured in m/s as a function of height measured in m as given in Equation (101) for a droplet with radius 0.10 mm to the left and radius 0.35 mm to the right.

Note that the equation of motion (130) has the same form as that in Equation (47) with accretion given by Equation (43). Hence with both accretion proportional to the mass and independent of the velocity, and resistance which is linear in the velocity, the equation of motion takes the form

$$\dot{v}_{1\mathrm{AR}} = g\left[1-\left(\frac{1}{v_{11\mathrm{AT}}}+\frac{1}{v_{1\mathrm{RT}}}\right)v_{1\mathrm{AR}}\right]. \tag{132}$$

Equation (132) shows that in this case accretion acts like air resistance. For a droplet with radius $r = 0.35\ \mathrm{mm}$ the ratio accretion/resistance in the equation of motion is

$$\frac{\text{accretion}}{\text{resistance}} = \frac{v_{1\mathrm{RT}}}{v_{11\mathrm{AT}}} = \frac{14.8\ \mathrm{m/s}}{156\ \mathrm{m/s}} = 9.5\times10^{-2}. \tag{133}$$

Hence the effect of accretion upon the motion of the droplet is much less than the effect of resistance. The main physical reason for this is that the density of the mist in a typical cloud is much smaller than the density of air.

5. Rapidly Falling Droplet—Quadratic Air Resistance

This case was considered by G. Feinberg [15], J. Lindemuth [16], G. W. Parker [17], P. Timmermann and J. P. van der Weele [18] and by S. Dey and A. Gorai [19]. We now assume that the velocity, $v_{2\mathrm{R}}(t)$, is so large, $v_{2\mathrm{R}} \gg v_{eq\mathrm{R}}$, that the resistance term which is quadratic in the velocity, dominates. This is not a good approximation for the droplet with radius 0.1 mm, but for the sake of comparison we shall nevertheless consider such a small droplet. Then the equation of motion of the droplet reduces to

$$\dot{v}_{2\mathrm{R}} + \frac{g}{v_{2\mathrm{RT}}^2}v_{2\mathrm{R}}^2 = g. \tag{134}$$

where $v_{2\mathrm{RT}}$ is given in Equation (105). This has the same form as Equation (16) for an accreting droplet where air resistance is neglected. The only difference is that $v_{12\mathrm{AT}} = 69\ \mathrm{m/s}$ in Equation (16) has been replaced by $v_{2\mathrm{RT}} = 3.9\ \mathrm{m/s}$ for a droplet with radius 0.35 mm in Equation (134). Hence, the friction term is $\left(v_{12\mathrm{AT}}/v_{2\mathrm{RT}}\right)^2 = 300$ times larger in the case of air resistance than with only accretion. So, as we have seen earlier, air resistance acts with a much stronger force upon the droplet than the force-like action of accretion.

The solution of Equation (134) with the initial condition $v_{2\mathrm{R}}(0) = 0$ is

$$v_{2\mathrm{R}} = v_{2\mathrm{RT}}\tanh\left(\frac{g}{v_{2\mathrm{RT}}}t\right). \tag{135}$$

Integrating the equation $v_{2\mathrm{R}} = \mathrm{d}h_{2\mathrm{R}}/\mathrm{d}t$ gives the height as a function of time

$$h_{2\mathrm{R}} = \frac{v_{2\mathrm{RT}}^2}{g}\ln\cosh\left(\frac{g}{v_{2\mathrm{RT}}}t\right). \tag{136}$$

It follows from Equations (135) and (136) that in this case the velocity of the droplet as a function of its height is

$$v_{2\mathrm{R}} = v_{2\mathrm{RT}}\sqrt{1-\mathrm{e}^{-(2g/v_{2\mathrm{RT}})h_{2\mathrm{R}}}}\,. \tag{137}$$

This is plotted in **Figure 6**. Equation (137) gives $v_{2\mathrm{R}}(1) \approx v_{2\mathrm{R}}(10) \approx v_{2\mathrm{RT}} = 3.9\ \mathrm{m/s}$.

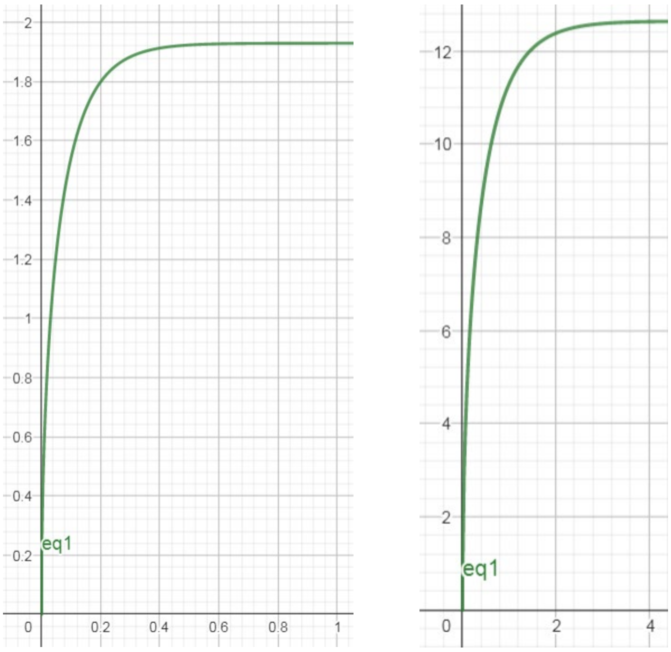

Figure 6. The velocity in m/s as a function of height in m as given in Equation (108) for a droplet with radius 0.10 mm to the left and radius 0.35 mm to the right.

The velocity as function of height for a water droplet with radius 0.35 mm as measured by W. Ji *et al.* [2] is shown in **Figure 7**.

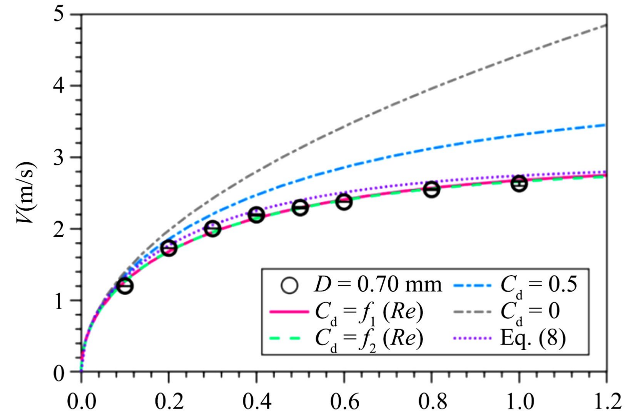

Figure 7. The velocity as function of height for a water droplet with radius 0.35 mm as measured and calculated by W. Ji *et al.* [2] with quadratic air resistance. The dragging coefficient used by Ji is $C_d = 2\lambda_d$, where λ_d is given in Equation (115). $C_d = 0$, the grey curve, corresponds to free fall with $v = \sqrt{2gh}$.

The right hand graph in **Figure 6** shows a larger velocity than by that of W. Ji *et al.* [2], in **Figure 7** for a droplet of the same size.

Differentiating the expression (135) we find that the acceleration of the droplet is

$$a_{2R} = \frac{g}{\cosh^2\left(\frac{g}{v_{2RT}}t\right)}. \tag{138}$$

Comparison with Equation (135) shows that this may be written as

$$a_{2R} = g\left[1-\left(\frac{v_{2R}}{v_{2RT}}\right)^2\right]. \tag{139}$$

This shows how the acceleration decreases rapidly as the velocity approaches the terminal velocity. As seen from Equation (137) and the graphs in **Figure 7**, the velocity approaches very rapidly the terminal velocity and has then a very small acceleration.

6. Droplet Falling with Both Linear and Quadratic Air Resistance

This case has been considered by Jorge Andrade [20], although the expressions he obtained for the velocity, acceleration and position of the droplet are much more complicated than those obtained here. The equation of motion can now be written in the form

$$\frac{\mathrm{d}v_{12R}}{\left(\frac{v_{12R}}{v_{2RT}}\right)^2 + \frac{v_{12R}}{v_{1RT}} - 1} = -g\mathrm{d}t\,. \tag{140}$$

Integrating this with the initial condition $v_{12R}(0) = 0$ gives. (Two different methods of solving this equation are shown in detail in the Appendices A and B.)

$$v_{12R} = \frac{2v_{1RT}}{1+\frac{2\overline{v}}{v_{eqR}}\coth\left(\frac{\overline{v}g}{v_{2RT}^2}t\right)}, \quad \overline{v} = \sqrt{\left(\frac{v_{eqR}}{2}\right)^2 + v_{2RT}^2}\,. \tag{141}$$

Introducing

$$\hat{t} = \frac{t}{t_1}, \quad t_1 = \frac{2v_{1RT}}{k_R g} \tag{142}$$

and the constant

$$k_R = \frac{2\overline{v}}{v_{eqR}} = \sqrt{1+\left(\frac{2v_{2RT}}{v_{eqR}}\right)^2} = \sqrt{1+\left(\frac{2v_{1RT}}{v_{2RT}}\right)^2} \tag{143}$$

the expression (141) for the velocity may be written as

$$v_{12R} = \frac{2v_{1RT}}{1+k_R\coth\hat{t}}. \tag{144}$$

Inserting $v_{1RT} = 14.8\ \mathrm{m/s}$ and $v_{2RT} = 3.9\ \mathrm{m/s}$ gives $k_R = 7.7$ and $t_1 = 0.39\ \mathrm{s}$. This is plotted in **Figure 8**.

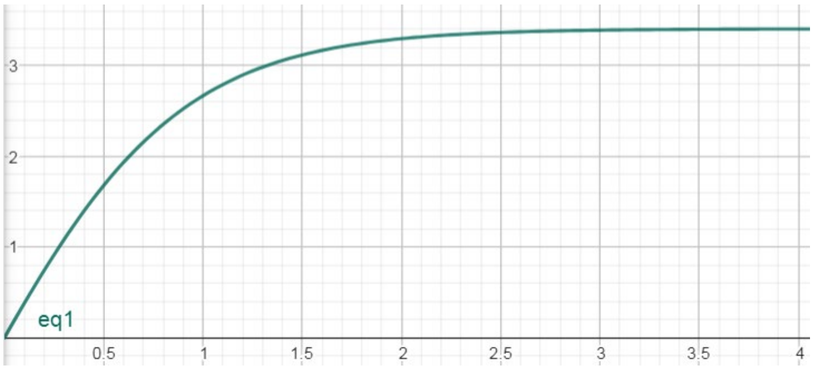

Figure 8. The velocity of a droplet with radius 0.35 mm as function of $\hat{t}$ with both linear and quadratic air resistance.

In the limit of large t we have an asymptotic terminal velocity

$$v_{12\mathrm{RT}}=\lim_{t\to\infty}v_{12\mathrm{R}}=\frac{2v_{1\mathrm{RT}}}{1+k_{\mathrm{R}}}, \tag{145}$$

where the last expression is obtained by using Equation (143) in combination with $v_{\mathrm{eqR}}=v_{2\mathrm{RT}}^2/v_{1\mathrm{RT}}$. However, as seen from the graph in Figure 8, the droplet arrives at a practically constant velocity close to $v_{12\mathrm{RT}}$ already after $\hat{t}=2.5$, *i.e.* already after about one second. Using Equation (107) and Equation (143) for k_v the expression (145) for $v_{12\mathrm{RT}}$ may be written as

$$v_{12\mathrm{RT}}=\frac{v_{\mathrm{eqR}}}{2}\left(\sqrt{1+\left(\frac{2v_{1\mathrm{RT}}}{v_{2\mathrm{RT}}}\right)^2}-1\right)=\left(k_{\mathrm{R}}-1\right)\frac{v_{\mathrm{eqR}}}{2} \tag{146}$$

in agreement with Equation (112) for the terminal velocity. It may also be noted that in the case with only linear resistance $v_{2\mathrm{RT}}\to\infty$, and $\overline{v}\to v_{2\mathrm{RT}}$, so that the expression (146) reduces to (48), and in the case with only quadratic air resistance $v_{1t}\to\infty$, the expression (146) reduces to (131), as shown in appendix A. For droplets with radii 0.1mm and 0.35 mm the constant k_{R} has the values $k_{\mathrm{R}}=1.60$ and 7.7. For a droplet with radius 0.35 mm this gives $v_{12\mathrm{RT}}=3.4\ \mathrm{m/s}$.

Differentiation of the velocity (144) gives the acceleration

$$a_{12\mathrm{R}}=\left(\frac{k_{\mathrm{R}}}{\sinh\hat{t}+k_{\mathrm{R}}\cosh\hat{t}}\right)^2 g. \tag{147}$$

Note that the acceleration decreases steadily from an initial value $a_{12\mathrm{R}}(0)=g$ to an asymptotically vanishing value. With an initial value $h_{12\mathrm{R}}(0)=0$ the height $h_{12\mathrm{R}}$ as a function of time is

$$h_{12\mathrm{R}}=2v_{1\mathrm{RT}}\int_0^t\frac{\mathrm{d}t}{1+k_{\mathrm{R}}\coth\hat{t}}, \tag{148}$$

giving

$$h_{12\mathrm{R}}=h_{\mathrm{RT}}\left[\ln\frac{1+k_{\mathrm{R}}\coth\hat{t}}{\left(\coth\hat{t}+1\right)^{\frac{k_{\mathrm{R}}+1}{2k_{\mathrm{R}}}}\left(\coth\hat{t}-1\right)^{\frac{k_{\mathrm{R}}-1}{2k_{\mathrm{R}}}}}\right],\quad h_{\mathrm{RT}}=\frac{v_{2\mathrm{RT}}^2}{g}. \tag{149}$$

It is shown in Appendix D that this expression can be given the simpler form

$$h_{12\mathrm{RTvs}} = h_{\mathrm{RT}}\left[\ln\left(\cosh\hat{t} + \frac{1}{k_{\mathrm{R}}}\sinh\hat{t}\right) - \frac{1}{k_{\mathrm{R}}}\hat{t}\right]. \tag{150}$$

This is plotted in **Figure 9**.

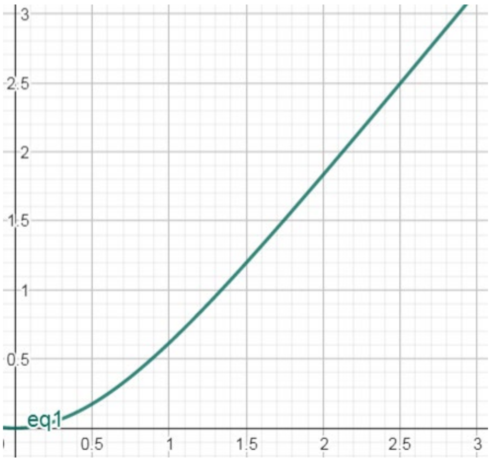

Figure 9. The falling distance of a droplet with radius 0.35 mm in m as a function of $\hat{t}$ with both linear and quadratic air resistance. The straight part of the curve indicates a constant velocity.

Andrade [20] also found expressions for the velocity, acceleration and position of a body falling under both linear and quadratic resistance, but his expressions are much more complicated than those deduced here. From Equation (144) we get

$$1 + k_{\mathrm{R}}\coth\hat{t} = \frac{2v_{1\mathrm{RT}}}{v_{12\mathrm{R}}}. \tag{151}$$

Inserting this into Equation (149) gives the velocity-height relationship

$$h_{12\mathrm{R}} = h_{\mathrm{RT}}\ln\left\{\left[1-\left(k_{\mathrm{R}}+1\right)\frac{v_{12\mathrm{R}}}{2v_{1\mathrm{RT}}}\right]^{\frac{1-k_{\mathrm{R}}}{2k_{\mathrm{R}}}}\left[1+\left(k_{\mathrm{R}}-1\right)\frac{v_{12\mathrm{R}}}{2v_{1\mathrm{RT}}}\right]^{-\frac{1+k_{\mathrm{R}}}{2k_{\mathrm{R}}}}\right\}. \tag{152}$$

This expression contains two singularities, at the velocities

$$v_{12\mathrm{R}1} = \frac{2V_{1\mathrm{RT}}}{1+k_{\mathrm{R}}} \quad \text{and} \quad v_{12\mathrm{R}2} = \frac{2V_{1\mathrm{RT}}}{1-k_{\mathrm{R}}}. \tag{153}$$

For a droplet with radius 0.35 mm $v_{1RT} = 14.8\ \mathrm{m/s}$ and $k_{\mathrm{R}} = 7.7$. Hence $v_{12\mathrm{RT}2} < 0$ which is not physically relevant without vertical air motion. Comparing Equations (153) and (145) it is seen that $v_{12\mathrm{R}1} = v_{12\mathrm{RT}}$. This provides an explanation for the singularity in the expression (152). The droplet uses an infinitely long time to reach the velocity $v_{12\mathrm{R}1}$. Therefore, it must fall an infinitely long distance in order to obtain this velocity, which is of course not physically possible.

The velocity $v_{12\mathrm{R}}$ as given in Equation (152) is plotted as function of $h_{12\mathrm{R}}$ in **Figure 10** for a droplet with radius 0.35 mm.

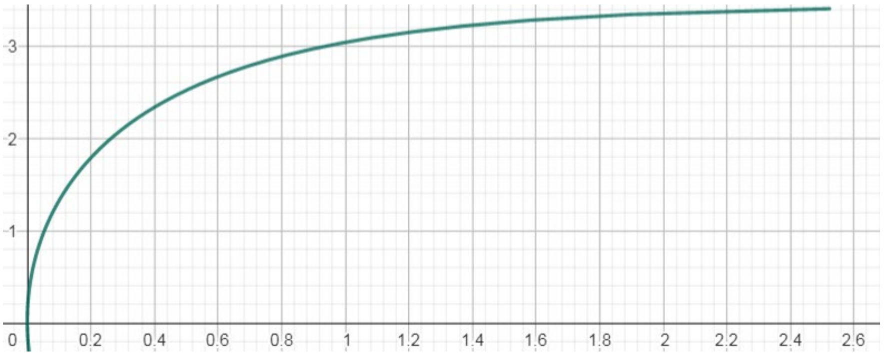

Figure 10. The velocity $v_{12\mathrm{R}}$ in m/s as a function of height (positive in the downwards direction) in m as given in Equation (151) for a droplet with radius 0.35 mm. It is seen that the velocity approaches the asymptotic velocity $v_{12\mathrm{RT}} = 3.4\ \mathrm{m/s}$ already after having fallen about 2.5 m. The following values have been used: $v_{1\mathrm{RT}} = 14.8\ \mathrm{m/s}$, $v_{2\mathrm{RT}} = 3.9\ \mathrm{m/s}$, and $k_{\mathrm{R}} = 7.7$. This velocity is in good agreement with those measured by W. Ji and co-workers [2] and shown in **Figure 6**.

We see from the graph that $v_{12\mathrm{R}}(1) = 2.6\ \mathrm{m/s}$. The graph shows that there is reasonably good agreement with the results obtained in the measurements performed by W. Ji *et al.* [2], although not as good as with only the contribution to the air resistance which is linear in velocity.

7. Effect of Variation of Air Density with Height Upon the Motion of a Droplet

The density of the atmosphere usually decreases with height. Hence we shall now focus upon the effect of increasing air density at the position of the droplet as it falls downwards, upon the motion of the droplet. We shall consider two idealizations, an isothermal atmosphere and an atmosphere with an adiabatic temperature gradient.

7.1. Isothermal Atmosphere

This case has earlier been considered by P. Mohazzabi and J. H. Shea [21], and I shall here review their analysis. In the case of an isothermal atmosphere the density decreases with height in an exponential way,

$$\rho = \rho_0 \mathrm{e}^{-\frac{z}{z_{\mathrm{T}}}}, \tag{154}$$

where ρ_0 is the density at the sea level, and z_{T} is the characteristic height of the isothermal atmosphere,

$$z_{\mathrm{T}} = \frac{RT}{Mg}. \tag{155}$$

Here R is the molar gas constant, T the absolute temperature and M the average molar mass of the air molecules. Inserting the values M = 0.0288 kg/mol and T = 254 K gives $z_T = 7.5\ \mathrm{km}$.

The velocity of a non-accreting droplet moving in this atmosphere under quadratic air resistance will be called $v_{2\text{iso}}$. We here introduce the height z with positive z-direction pointing downwards, as a variable instead of t, utilizing that $\mathrm{d}z = v_{2\text{iso}}\,\mathrm{d}t$. Then using Equations (105) and (154), the equation of motion, Equation (134), of the droplet may be written as

$$\frac{\mathrm{d}v_{2\text{iso}}}{\mathrm{d}t} = v_{2\text{iso}}\frac{\mathrm{d}v_{2\text{iso}}}{\mathrm{d}z} = g\left(1 - \frac{v_{2\text{iso}}^2}{v_{2\text{RT}}^2}\mathrm{e}^{-\frac{z}{z_\text{T}}}\right). \tag{156}$$

This equation is solved in Appendix C. The solution with the initial condition $v_{2\text{iso}}(0) = 0$ is

$$v_{2\text{iso}} = v_{2\text{RT}}\sqrt{p\mathrm{e}^{p\mathrm{e}^{z/z_\text{T}}}\left[Ei(-p) - Ei\left(-p\mathrm{e}^{-z/z_\text{T}}\right)\right]}, \quad p = \frac{2gz_\text{T}}{v_{2\text{RT}}^2}, \tag{157}$$

where Ei is a function called the exponential integral. Inserting $g \approx 10\ \mathrm{m/s^2}$, $z_\text{T} = 7.5\times10^3\ \mathrm{m}$ and $v_{2\text{RT}} = 3.9\ \mathrm{m/s}$ for a droplet with radius 0.35 mm gives $p = 10000$.

7.2. Adiabatic Atmosphere

It is unusual that the atmosphere has the same temperature at different heights. An adiabatic atmosphere is a better approximation to the real atmosphere. The change of density with height of an adiabatic atmosphere is [22]

$$\rho(z) = \rho_0\left(1 - \frac{\gamma-1}{\gamma}\frac{z}{z_\text{T}}\right)^{\frac{1}{\gamma-1}}, \tag{158}$$

where $\gamma = 1.4$ for the standard atmosphere. The upper bound, z_A, of the adiabatic atmosphere is defined by $\rho(z_\text{A}) = 0$, giving

$$z_\text{A} = \frac{\gamma}{\gamma-1}z_\text{T}. \tag{159}$$

With $\gamma = 1.4$ and $z_\text{T} = 7.5\ \text{km}$ this gives $z_\text{T} \approx 26\ \text{km}$.

In this case the equation of motion (134) is replaced by

$$v_{2\text{ad}}\frac{\mathrm{d}v_{2\text{ad}}}{\mathrm{d}z} = \frac{1}{2}\frac{\mathrm{d}v_{2\text{ad}}^2}{\mathrm{d}z} = -g\left[1 - \frac{v_{2\text{ad}}^2}{v_{2\text{RT}}^2}\left(1 - \frac{\gamma-1}{\gamma}\frac{z}{z_\text{T}}\right)^{\frac{1}{\gamma-1}}\right]. \tag{160}$$

The numerical solution of this is shown graphically in **Figure 10**. Comparing **Figure 9** and **Figure 10** it is seen that the velocities of the droplet in isothermal- and adiabatic atmospheres are nearly equal to each other.

Introducing

$$u = v_{2\text{ad}}^2 / v_{2\text{RT}}^2, \quad x = \left(1 - \frac{\gamma-1}{\gamma}\frac{z}{z_T}\right)^{\frac{\gamma}{\gamma-1}}, \tag{161}$$

where $0 \le x \le 1$, Equation (159) takes the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} + pu = \frac{p}{x^{1/\gamma}}. \tag{162}$$

where p is given in Equation (157).

With the initial condition $v_6(0)=0$, *i.e.* $u(1)=0$, the solution of Equation (161) takes the form

$$u=\frac{p}{\mathrm{e}^{px}}\int_1^x\frac{\mathrm{e}^{px}}{x^{1/\gamma}}\mathrm{d}x\,. \tag{163}$$

This can be expressed in terms of the Γ-function,

$$u=p^{1/\gamma}\mathrm{e}^{-px}\left[\Gamma\left(\frac{\gamma-1}{\gamma},-px\right)-\Gamma\left(\frac{\gamma-1}{\gamma},-p\right)\right]. \tag{164}$$

Hence

$$v_{2\mathrm{ad}}=v_{2\mathrm{RT}}\,p^{\frac{1}{2\gamma}}\mathrm{e}^{-\frac{p}{2}x}\left[\Gamma\left(\frac{\gamma-1}{\gamma},-px\right)-\Gamma\left(\frac{\gamma-1}{\gamma},-p\right)\right]^{1/2}. \tag{165}$$

One may wonder why the droplet falling in an adiabatic atmosphere from a given height moves faster than a droplet falling from the same height in an isothermal atmosphere. Introducing $x=z/z_\mathrm{T}$ in Equations (154) and (158), and inserting the value $\gamma=1.4$, the ratios of the density of the adiabatic and isothermal atmospheres at a height z is

$$\frac{\rho_\mathrm{A}}{\rho_\mathrm{T}}=\left(1-\frac{2}{7}x\right)^{2.5}\mathrm{e}^x\,. \tag{166}$$

This is plotted in **Figure 11**. We see from the graph that at a height above 13.5 km the density of the adiabatic atmosphere is smaller than the density of the isothermal atmosphere. Hence, starting above this height the droplets will fall with a greater velocity in an adiabatic atmosphere than those in an isothermal atmosphere.

Figure 11. The ratio $\rho_\mathrm{A}/\rho_\mathrm{T}$ as a function of the height measured in units of $z_\mathrm{T}=7.5\ \mathrm{km}$.

8. Effects of Vertical Motion of the Air Upon the Motion of a Droplet

We now consider a droplet with mass m falling vertically with a velocity $v_{12\mathrm{Rvs}}$ in air with vertical speed (vs) in the upwards direction, $v_\mathrm{air}<0$, as is usual at several places in a cumulus cloud, and acted upon by linear and quadratic air resistance. Typical upwards velocities in a cumulus cloud are from $v_\mathrm{air}=-3\ \mathrm{m/s}$ to $v_\mathrm{air}=-10\ \mathrm{m/s}$. The velocity of the droplet relative to the air is $v_{12\mathrm{Rvs}}-v_\mathrm{air}$. Hence, the frictional force is

$$f_r = -6\pi\eta_{\text{air}} r\left(v_{12\text{Rvs}} - v_{\text{air}}\right) - \frac{\pi}{4}\rho_{\text{air}} r^2\left(v_{12\text{Rvs}} - v_{\text{air}}\right)^2. \tag{167}$$

Then Newton's 2.law applied to a falling droplet takes the form

$$\begin{aligned} m\dot{v}_{12\text{Rvs}} &= mg + 6\pi\eta_{\text{air}} r\, v_{\text{air}} - \frac{\pi}{4}\rho_{\text{air}} r^2 v_{\text{air}}^2 - \left(6\pi\eta_{\text{air}} r - \frac{\pi}{2}\rho_{\text{air}} r^2 v_{\text{air}}\right) v_{12\text{Rvs}} \\ &\quad - \frac{\pi}{4}\rho_{\text{air}} r^2 v_{12\text{Rvs}}^2. \end{aligned} \tag{168}$$

Using Equations (105), (107) and (146) this equation may be written as

$$\dot{v}_{12\text{Rvs}} = \frac{g}{v_{2\text{RT}}^2}\left[v_{12\text{vs}}^2 - \left(2v_{\text{air}} - v_{\text{eqR}}\right)v_{12\text{vs}} - v_{\text{air}}^2 + v_{\text{eqR}} v_{\text{air}} + v_{2\text{RT}}^2\right]. \tag{169}$$

Following the same procedure as in Appendix B and using Equation (107), we find that the solution of this equation with the initial condition $v_{12\text{vs}}(0) = 0$ is

$$v_{12\text{Rvs}} = \frac{2\left(v_{2\text{RT}}^2 + v_{\text{eqR}} v_{\text{air}} - v_{\text{air}}^2\right)}{v_{\text{eqR}} - 2v_{\text{air}} + 2\overline{v}\coth\left(\dfrac{g\overline{v}}{v_{2\text{RT}}^2} t\right)}, \tag{170}$$

where $\overline{v}$ is given in Equation (141). Introducing

$$v_{1\text{RTvs}} = v_{1\text{RT}} - v_{\text{air}} - v_{\text{air}}^2 / v_{\text{eqR}} \tag{171}$$

and

$$k_{\text{air}} = 1 - \frac{2v_{\text{air}}}{v_{\text{eqR}}}, \tag{172}$$

Equation (170) may be written

$$v_{12\text{Rvs}} = \frac{2v_{1\text{RTvs}}}{k_{\text{air}} + k_{\text{R}}\coth\hat{t}}, \tag{173}$$

where k_{R} is given in Equation (143). Equation (171) has the same form as the expression (144) for $v_{12\text{R}}$. **Figure 12** shows $v_{12\text{Rvs}}$ as a function of time for a droplet with radius 0.35 mm and with vertical wind speeds $v_{\text{air}} = 0$, $v_{\text{air}} = -3\ \text{m/s}$ and $v_{\text{air}} = -6\ \text{m/s}$.

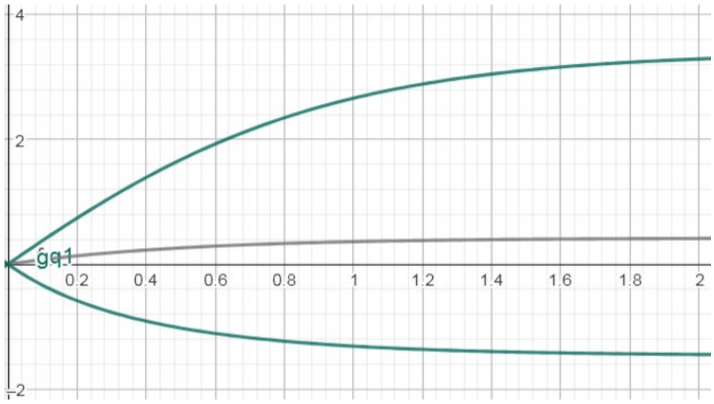

Figure 12. $v_{12\text{Rvs}}$ as a function of $\hat{t} = 0.29t$ for a droplet with radius 0.35 mm for vertical wind speeds $v_{\text{air}} = 0$ (upper curve), $v_{\text{air}} = -3\ \text{m/s}$ (grey curve) and $v_{\text{air}} = -6\ \text{m/s}$ (curve below the x-axis). In the last case the droplet moves upwards due to the strong vertical

wind.

Figure 13 shows $v_{12\mathrm{Rvs}}$ as a function of v_{air} for a droplet with radius 0.35 mm for $\hat{t}=3$ and $\hat{t}=6$, *i.e.* for $t \approx 1\,\mathrm{s}$ and $t \approx 2\,\mathrm{s}$.

The velocity of the droplets vanishes, $v_{12\mathrm{Rvs}}=0$, when the vertical velocity of the air obeys

$$v_{\mathrm{air}}^2 + v_{\mathrm{eqR}} v_{\mathrm{air}} - v_{2\mathrm{RT}}^2 = 0\,, \tag{174}$$

with solutions

$$v_{\mathrm{air}1,2} = \frac{v_{\mathrm{eqR}}}{2}\left(\pm\sqrt{1+\left(\frac{2v_{2\mathrm{RT}}}{v_{\mathrm{eqR}}}\right)}-1\right). \tag{175}$$

Using that $v_{\mathrm{eqR}} = v_{2\mathrm{RT}}^2 / v_{1\mathrm{RT}}$ and comparing with Equation (146) the positive solution, meaning downwards air velocity, is $v_{\mathrm{air}1} = v_{12\mathrm{RT}}$, and the negative, meaning upwards motion of the air, is

$$v_{\mathrm{air}2} = -\left(v_{12\mathrm{RT}} + v_{\mathrm{eqR}}\right). \tag{176}$$

In **Figure 13** we see that the graphs pass the x-axis ($v_{12\mathrm{Rvs}}=0$) at these two air-velocities.

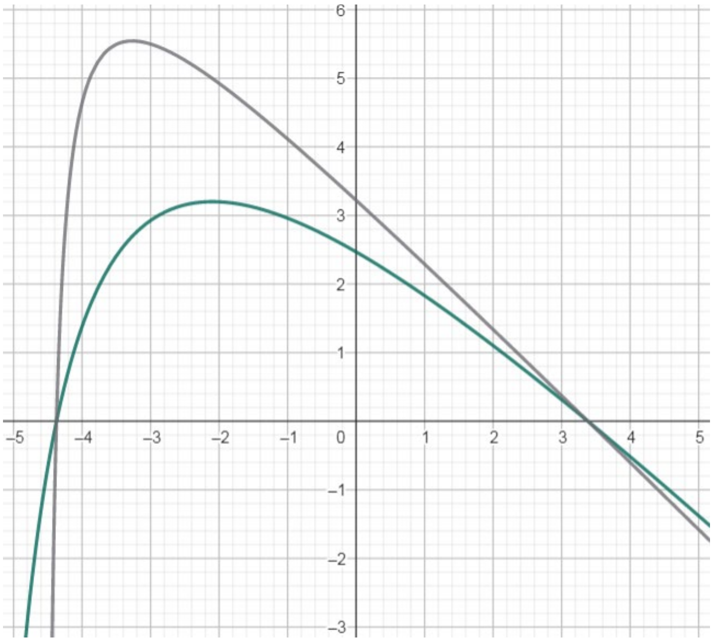

Figure 13. $v_{12\mathrm{Rvs}}$ as a function of v_{air} for $t \approx 1\,\mathrm{s}$ (green) and $t \approx 2\,\mathrm{s}$ (grey). Positive velocity is directed downwards and negative upwards. The physically relevant part of the figure has upwards directed wind speeds. *i.e.* $v_{\mathrm{air}} < 0$, and is the part of the graph in the region $-4.4\,\mathrm{m/s} < v_{\mathrm{air}} < 0$.

Only the latter one is physically relevant, since the droplets cannot stay at rest with downwards air velocity. Inserting $v_{12\mathrm{RT}} = 3.4\,\mathrm{m/s}$ and $v_{\mathrm{eqR}} = 1\,\mathrm{m/s}$ we get $v_{\mathrm{air}2} = -4.4\,\mathrm{m/s}$. If the upwards wind is stronger, the droplets will move upwards, and if it is weaker, they will move downwards.

Equation (173) shows how the expression for $v_{12\mathrm{R}}$ must be generalized to include the effect of vertical motion of the air upon the motion of the droplets. Inserting the values $v_{1\mathrm{RT}} = 14.8\ \mathrm{m/s}$, $v_{\mathrm{eqR}} = 1.0\ \mathrm{m/s}$, $v_{\mathrm{air}} = 4\ \mathrm{m/s}$ for a droplet with radius 0.35 mm gives $v_{1\mathrm{RTvs}} = -5.2\ \mathrm{m/s}$.

The droplet with radius 0.35 mm moving in air with an upward velocity approaches an asymptotic terminal velocity

$$v_{12\mathrm{RTvs}} = \lim_{t\to\infty} v_{12\mathrm{Rvs}} = \frac{2v_{1\mathrm{RTvs}}}{k_{\mathrm{air}} + k_{\mathrm{R}}}. \tag{177}$$

Inserting the values $v_{\mathrm{air}} = -6\ \mathrm{m/s}$, $v_{1\mathrm{RTvs}} = -30.3\ \mathrm{m/s}$, $k_{\mathrm{air}} = 13$ and $k_{\mathrm{R}} = 7.7$ gives $v_{12\mathrm{vsT}} = -1.5\ \mathrm{m/s}$, which is the velocity approached by the droplet, as we see in the curve below the x-axis in Figure 11.

Integrating Equation (173) with the initial condition $h_{12\mathrm{RTvs}}(0) = 0$, the velocity-height relationship for a droplet in a cumulus cloud with vertical motion of the air, is found to have the form

$$h_{12\mathrm{Rvs}} = h_{\mathrm{RT}} \ln\left\{ \frac{\left[1-\left(k_{\mathrm{R}}+k_{\mathrm{air}}\right)\dfrac{v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}\right]^{\frac{k_{\mathrm{air}}-k_{\mathrm{R}}}{2k_{\mathrm{R}}}}}{\left[1+\left(k_{\mathrm{R}}-k_{\mathrm{air}}\right)\dfrac{v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}\right]^{\frac{k_{\mathrm{air}}+k_{\mathrm{R}}}{2k_{\mathrm{R}}}}} \right\}, \tag{178}$$

where h_{RT} is given in Equation (149), $v_{1\mathrm{RTvs}}$ in (170), k_{R} in Equation (143) and k_{air} in Equation (173). Note that $v_{\mathrm{air}} = 0$ gives $k_{\mathrm{air}} = 1$. The details of the deduction are found in Appendix D. There are two singularities, at the velocities,

$$v_{12\mathrm{Rvs1}} = \frac{2v_{1\mathrm{RTvs}}}{k_{\mathrm{R}} + k_{\mathrm{air}}} \quad \text{and} \quad v_{12\mathrm{Rvs2}} = \frac{2v_{1\mathrm{RTvs}}}{k_{\mathrm{air}} - k_{\mathrm{R}}}. \tag{179}$$

The explanation is the same as for Equation (152); that $v_{12\mathrm{Rvs2}}$ is unphysical (since a vertical air velocity with $k_{\mathrm{air}} < k_{\mathrm{R}}$ means that $v_{12\mathrm{Rvs2}}$ is oppositely directed to $v_{1\mathrm{RTvs}}$), and the droplet must fall an infinitely large distance to obtain the asymptotic terminal velocity $v_{12\mathrm{Rvs1}} = v_{12\mathrm{RTvs}}$.

The terminal velocity of the droplets with vertical wind is found by putting $\dot{v}_{12\mathrm{vs}} = 0$ in Equation (169). This gives the positive solution

$$v_{12\mathrm{vsT}} = \frac{1}{2}\left(\sqrt{v_{\mathrm{eqR}}^2 + 4v_{2\mathrm{RT}}^2} - v_{\mathrm{eqR}} - 2v_{\mathrm{air}}\right). \tag{180}$$

Using Equation (146) this equation may be written

$$v_{12\mathrm{vsT}} = v_{12\mathrm{RT}} - v_{\mathrm{air}}, \tag{181}$$

The decrease of the terminal velocity due to the vertical motion of the air, is equal to the velocity of the air.

9. Accretion and Resistance

B. F. Edwards, J. W. Wilder and E. E. Scime [8] have given a fine analysis of accreting droplets falling through mist and acted upon by gravity and air resistance proportional to the square of the velocity. In this case the equation of motion takes the form

$$m\dot{v}_2 = mg - \lambda_d \pi \rho_{air} r^2 v_2^2 - v_2 \dot{m}\,, \tag{182}$$

where λ_d is a drag introduced in Equation (115), and the rate of mass accretion is

$$\dot{m} = \pi \rho_{\text{mist}} r^2 v_2\,. \tag{183}$$

This has the same form as Equation (42) with $\alpha = 2/3$ and $\beta = 1$, which is also the same as the assumption (23) *i.e.* that the rate of mass accretion is equal to the mass of mist that the droplet passes through per second. Hence, the equation of motion takes the form

$$m\dot{v}_2 = mg - \pi\left(\lambda_d \rho_{\text{air}} + \rho_{\text{mist}}\right) r^2 v_2^2\,. \tag{184}$$

The authors wrote that as raindrops grow in radius from $r = 0.1$ mm to $r = 1$ mm within a cloud, their drag coefficients decrease from about $\lambda_d = 5$ to about $\lambda_d = 0.5$. They further wrote that air densities $\rho_{\text{air}} \approx 10^{-3}\ \text{g}\cdot\text{cm}^{-3}$ greatly exceed the mist densities $\rho_{\text{mist}} \approx 10^{-6}\ \text{g}\cdot\text{cm}^{-3}$ typical of terrestrial rain clouds. Hence the effect of accretion upon the motion of the droplet is much smaller than the effect of air resistance.

M.H. Partovi and D.R. Aston [7] developed the study of the dynamics of falling raindrops further in a very nice and comprehensive article. They took into account both accretion, change of shape of the droplet during the fall, and air resistance.

The mass m of the raindrop is increasing with a rate given by Equation (42), written in the form

$$\dot{m} = \rho_{\text{mist}} A v\,, \tag{185}$$

where A is the cross section area of the drop normal to the velocity.

The deviation from a spherical form increases with increasing velocity. Since the mass increases with a rate proportional to the velocity Partovi and Aston chose to represent the deviation in terms of the mass of the drop. For a spherical drop the cross section area of the drop normal to the velocity is proportional to $m^{2/3}$. Hence the deviation from spherical form may be represented by

$$A = \gamma m^{\frac{2}{3}+\varepsilon}\,, \tag{186}$$

where the parameter $\varepsilon = 0$ for a spherical droplet.

Equations (185) and (186) give a simple relationship between the height a drop has fallen and its mass. Using that $v = \dot{h}$ they lead to

$$m^{-\left(\frac{2}{3}+\varepsilon\right)} \dot{m} = \gamma \rho_{\text{mist}} \dot{h}\,. \tag{187}$$

Integration with $m(h_0) = m_0$ gives

$$m = \left[m_0^{\frac{1-3\varepsilon}{3}} + \frac{1-3\varepsilon}{3} \gamma \rho_{\text{mist}} \left(h - h_0\right) \right]^{\frac{3}{1-3\varepsilon}}. \tag{188}$$

If the droplets is created on a condensation nucleus with vanishing small mass, $m_0 = 0$, and this creation position is chosen as origin on the vertical axis, the expression reduces to

$$m=\left(\frac{1-3\varepsilon}{3}\gamma\rho_{\text{mist}}h\right)^{\frac{3}{1-3\varepsilon}}. \tag{189}$$

Partovi and Aston [23] considered a droplet falling, with a velocity $v \gg v_{eq}$, where v_{eq} is given in Equation (107), so that the quadratic term in the expression for the air resistance is dominating. Hence Newton's 2. Law takes the form

$$(mv)^{\cdot}=mg-\frac{1}{4}\rho_{\text{air}}Av^{2}. \tag{190}$$

Inserting Equations (185) and (189) into Equation (190) gives the acceleration

$$\dot{v}=g-\gamma\left(\frac{1}{2}\rho_{\text{air}}+\rho_{\text{mist}}\right)m^{\varepsilon-\frac{1}{3}}v^{2}. \tag{191}$$

Partovi and Aston [23] then defined the constant $\beta=\rho_{\text{air}}/4\rho_{\text{mist}}$, and showed that the equation of motion of the droplet can be written in the form

$$\left(m^{\frac{4}{3}-\varepsilon+\beta}\right)^{\cdot\cdot}=q\left(\frac{4}{3}-\varepsilon+\beta\right)m^{1+\beta}. \tag{192}$$

Taking this as a point of departure they then made a thorough and fascinating analysis of the motion of the droplet.

Mungan [10] has given a further development of the dynamics of a raindrop when both accretion and air resistance are taken account of. He assumed that the air resistance is given by a term $\eta_{\alpha\delta}m^{\alpha}v^{\delta}$ (no summation) so that the equation of motion of the droplet takes the form

$$\dot{p}=mg-\eta_{\alpha\delta}m^{\alpha}v^{\delta}. \tag{193}$$

The constant $\eta_{\alpha\delta}$ has dimension $\text{kg}^{1-\alpha}\cdot\text{s}^{\delta-2}\cdot\text{m}^{1-\delta}$. The evolution of the mass of the droplet and its motion is now found by solving Equations (42) and (193) simultaneously. Note that it is assumed that the rate of accretion and the air resistance are assumed to depend upon the same power of the mass, m^{α}. Mungan gave the following argument for this: "The same exponent α for m is used for the drag and mass accretion terms because both effects are expected to scale similarly with the size of the drop."

We now follow Mungan's way of solving the equations. Assuming that $\lambda_{\alpha\beta}\neq 0$ we can divide Equation (193) by Equation (42), which gives

$$\frac{\mathrm{d}p}{\mathrm{d}m}+\frac{\eta_{\alpha\delta}}{\lambda_{\alpha\beta}}\frac{p}{m}=\frac{g}{\lambda_{\alpha\beta}}m^{\beta-\alpha}p^{1-\beta}. \tag{194}$$

This shows that $\eta_{\alpha\delta}/\lambda_{\alpha\beta}$ must be dimensionless. Inserting the dimensions of $\eta_{\alpha\delta}$ and $\lambda_{\alpha\beta}$ then leads to the requirement $\delta=\beta$. Equation (186) includes the important special cases $\delta=1,\beta=1$ (linear drag with speed-independent mass accretion) and $\delta=2,\beta=2$ (quadratic drag with linear speed accretion).

Noting that

$$\frac{\mathrm{d}p}{\mathrm{d}m}+k_1\frac{p}{m}=m^{-k_1}\frac{\mathrm{d}}{\mathrm{d}m}\left(m^{k_1}p\right),\quad k_1=\frac{\eta_{\alpha\beta}}{\lambda_{\alpha\beta}}, \tag{195}$$

and putting the right hand sides of Equations (194) and (195) equal to each other gives

$$p^{\beta}\mathrm{d}\left(m^{k_1}p\right)=\frac{g}{\lambda_{\alpha\beta}}m^{\beta-\alpha+k_1}\mathrm{d}m\,. \tag{196}$$

In order to have an integrable left hand side of this equation, the factor multiplying the differential of $m^{k_1}p$ have to be $\left(m^{k_1}p\right)^{\beta-1}$. Hence we must multiply each side with $m^{k_1(\beta-1)}$, which gives

$$\left(m^{k_1}p\right)^{\beta-1}\mathrm{d}\left(m^{k_1}p\right)=\frac{g}{\lambda_{\alpha\beta}}m^{\beta(1+k_1)-\alpha}\,. \tag{197}$$

Integration of this equation with the initial condition $p\left(m_0\right)=0$ and inserting $v=p/m$ gives

$$v^{\beta}=\frac{g}{1+\dfrac{1-\alpha}{\beta}+k_1}\left(m^{1-\alpha}-m_0^{1-\alpha}\right). \tag{198}$$

With $m_0=0$ we have

$$v=Km^n,\quad K=\left[g/\left(1+n+k_1\right)\right]^{1/\beta},\quad n=\frac{1-\alpha}{\beta}. \tag{199}$$

Differentiation of Equation (199) gives

$$\dot{v}=nKm^{n-1}\dot{m}\,. \tag{200}$$

Inserting Equation (42) for $\dot{m}$,

$$\dot{v}=nK\lambda_{\alpha\beta}m^{-n\beta}v^{\beta}\,. \tag{201}$$

Finally, inserting Equation (199) leads to

$$\dot{v}=\frac{\lambda_{\alpha\beta}\left(1-\alpha\right)}{\lambda_{\alpha\beta}\left(1+\beta-\alpha\right)+\eta_{\alpha(1+\beta)}\beta^2}g\,. \tag{202}$$

Hence the acceleration with $m_0=0$ is constant. In the case of vanishing air resistance, $\eta_{\alpha(1+\beta)}=0$, this expression reduces to that in Equation (49). Equation (202) combines kinematical and dynamical factors. Those associated with $\lambda_{\alpha\beta}$ are kinematical and due to the assumption of how the rate of mass accretion depend upon the mass and velocity of the droplet, while that associated with $\eta_{\alpha(1+\beta)}$ is dynamical and relates to the air resistance. Greater value of $\eta_{\alpha(1+\beta)}$ means greater air resistance and thus smaller acceleration. However greater value of $\lambda_{\alpha\beta}$, which means greater mass accretion, gives greater acceleration. This is due to the fact that greater acceleration gives greater velocity and thus faster mass accretion. This problem has recently been considered by S. Day and A. Gorai [24] who solved the equation of motion numerically and found the terminal velocity for several cases.

If there is no mass accretion, $\lambda_{\alpha\beta}=0$, and m is constant. Then the equation of motion reduces to

$$\dot{v} = g - \eta_{\alpha\delta} m^{\alpha-1} v^{\delta}. \tag{203}$$

If we do not summarize over δ this equation can be integrated analytically in terms of elementary functions for many integer values of δ, including the physically most important ones, $\delta = 1$ and $\delta = 2$, which were considered above in sections 3 and 4. Summarising over delta up to $\delta = 2$ leads to the result discussed in section 5 with contributions to the air resistance both linear and quadratic in the velocity.

10. Change of Mechanical Energy of Falling Droplets

We shall briefly consider the change of mechanical energy of the droplets during their motion. As is well known the mechanical energy of a freely falling body is conserved. But both accretion and air resistance cause a loss of mechanical energy of the droplets.

In general the rate of loss of mechanical energy of a body of mass m falling with velocity $\dot{h} = -v$ at a height h above the level of zero potential energy in a gravitational field with acceleration of gravity g is

$$\dot{E}_{mech} = \dot{E}_{pot} + \dot{E}_{kin} = (mgh)^{\cdot} + \left(\frac{1}{2}mv^2\right)^{\cdot} = gh\dot{m} - m(g - \dot{v})v. \tag{204}$$

Newton's 2. Law applied to an accreting, falling drop acted upon by air resistance both linear and quadratic in the velocity takes the form

$$m\dot{v} = mg\left(1 - \frac{v}{v_{1\mathrm{RT}}} - \frac{v^2}{v_{2\mathrm{RT}}^2}\right) - v\dot{m}. \tag{205}$$

Inserting this into Equation (204) gives the rate of loss of mechanical energy

$$\dot{E}_{mech} = mg\left(\frac{v}{v_{1\mathrm{RT}}} + \frac{v^2}{v_{2\mathrm{RT}}^2}\right) + (gh - v)\dot{m}. \tag{206}$$

One of the cases considered by Krane [4] was that the accreted mass is proportional to the height, h, which the droplet has fallen. Then the rate of mass accretion is given by Equation (7), which gives

$$\dot{E}_{mech} = \left(\frac{mg}{v_{1\mathrm{RT}}} + \lambda_{02} gh\right)v + \left(\frac{mg}{v_{2\mathrm{RT}}^2} - \lambda_{02}\right)v^2. \tag{207}$$

This shows that if the rate of mass accretion is proportional to the velocity of the droplet , accretion contributes to the loss of mechanical energy of the droplet by two terms; a positive term proportional to the product of the distance the droplet has fallen and its velocity, and a negative term proportional to the square of the velocity. If the mass of the droplet is sufficiently small,

$$m < \lambda_{02} v_{2\mathrm{RT}}^2 / g, \tag{208}$$

The factor multiplying v^2 is negative. Using Equation (6) this condition takes the form

$$gh < v_{2\mathrm{RT}}^2 \,, \tag{209}$$

Furthermore using Equation (105) for $v_{2\mathrm{RT}}$ this condition takes the form

$$h < \frac{16}{3}\frac{\rho_{\mathrm{water}}}{\rho_{\mathrm{air}}} r \,. \tag{210}$$

Using Equation (24) this may be written

$$\gamma_0 > \frac{3}{16}\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{water}}} \,. \tag{211}$$

Since $\gamma_0 = 2.5 \times 10^{-7}$ and typically $\rho_{\mathrm{air}} = 4 \times 10^{-4} \rho_{\mathrm{water}}$, the condition (211) is not fulfilled. This means that both the factors of v and v^2 in Equation (202) are positive during all of the motion even if $m(0) = 0$. The reason is that λ_0 is proportional to the mass of the droplet.

11. Conclusions

In this article I have given a detailed review of the dynamics of accreting droplets falling under the action of gravity and air resistance, focusing on analytical calculations. In between calculations have been included to a degree which makes the article readable by students, not only by experienced teachers and researchers.

In particular I have considered both very small and slowly falling droplets where the contribution to the air resistance coming from Stokes' law, linear in the velocity, is dominating, and faster moving droplets where the contribution which is quadratic in the velocity becomes important.

A new and nice form of the solution of the equation of motion for a non-accreting, spherical droplet acted upon by the full expression for air resistance containing both the linear and quadratic term in the velocity is deduced in section 5. For pedagogical reasons the equation of motion of the droplet for this case has been deduced in two different ways in the appendices A and B.

The treatment in section 6.2 of a droplet falling in an adiabatic atmosphere has not earlier been published. Furthermore the content of sections 7 - 10 is new. Here the effect of vertical motion of the air upon the motion of a droplet is considered. Furthermore, there are detailed deductions of eqs.

Also, in the treatment of several proportionality constants, where earlier treatments did not care about the dimensions of the constants, I have presented a systematic approach where all the constants have correct dimensions. When teaching physics this is an important point, both due to consistency, and the possibility for the students to check the correctness of a calculation by considering the units of their expressions.

Furthermore, the effect upon the motion of different physical assumptions concerning the rate of accretion and how air resistance depends upon the velocity, has been compared. It is found that air resistance is of greater significance for the motion than accretion. In this connection new velocity—falling distance relationships have been deduced including the effect of vertical motion of the air upon the motion of the droplet.

The physics of the topic can be extended, by including effects of for example change of form of the droplets during the motion, turbulence and temperature gradients, and different types of inhomogeneities in the cloud where the raindrops fall. Such extensions will generally require numerical calculations, and are not the topic of the present article.

The focus has been on analytical calculations with good physical motivations. The article provides a systematic exposition of an interesting and easily understandable topic, which gives the students an opportunity of good physical and mathematical training – and the pleasure of seeing how a calculation may increase their physical understanding of a topic.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Deduction of Equation (141)

Equation (140) may be written

$$\frac{\mathrm{d}v_{12\mathrm{R}}}{v_{12\mathrm{R}}^2 + v_{\mathrm{eqR}} v_{12\mathrm{R}} - v_{2\mathrm{RT}}^2} = -\frac{g}{v_{2\mathrm{RT}}^2}\mathrm{d}t \tag{A1}$$

Using either an integrator on internet, or an integral table, or performing the integral by means of the method with partial fractions, one ends up with an answer of the form

$$\frac{1}{2\bar{v}}\ln\frac{v_{12\mathrm{R}} + \frac{v_{\mathrm{eqR}}}{2} - \bar{v}}{v_{12\mathrm{R}} + \frac{v_{\mathrm{eqR}}}{2} + \bar{v}} = -\frac{g}{v_{2\mathrm{RT}}^2}t - C. \tag{A2}$$

where $\bar{v}$ is given in Equation (141). Using the formula

$$\operatorname{arcoth} x = -\frac{1}{2}\ln\frac{x-1}{x+1}, \tag{A3}$$

Equation (A2) can be written as

$$\operatorname{arcoth}\left[\frac{1}{\bar{v}}\left(v_{12\mathrm{R}} + \frac{v_{\mathrm{eqR}}}{2}\right)\right] = \bar{v}\left(\frac{g}{v_{2\mathrm{RT}}^2}t + C\right), \tag{A4}$$

or

$$v_{12\mathrm{R}} = \bar{v}\coth\left(\frac{\bar{v}g}{v_{2\mathrm{RT}}^2}t + \bar{v}C\right) - \frac{v_{\mathrm{eqR}}}{2}. \tag{A5}$$

We consider a droplet falling from rest so that the initial condition is $v_{12\mathrm{R}}(0) = 0$, giving

$$\frac{2\bar{v}}{v_{\mathrm{eqR}}}\coth\left(\bar{v}C\right) = 1. \tag{A6}$$

Hence

$$\bar{v}C = \operatorname{arcoth}\frac{v_{\mathrm{eqR}}}{2\bar{v}}. \tag{A7}$$

Inserting this into Equation (A5) gives

$$v_{12\mathrm{R}} = \bar{v}\coth\left(\frac{\bar{v}g}{v_{2\mathrm{RT}}^2}t + \operatorname{arcoth}\frac{v_{\mathrm{eqR}}}{2\bar{v}}\right) - \frac{v_{\mathrm{eqR}}}{2}. \tag{A8}$$

Using that

$$\coth\left(x+y\right) = \frac{\coth x \coth y + 1}{\coth x + \coth y}, \tag{A9}$$

we finally obtain

$$v_{12\mathrm{R}} = \frac{2v_{1\mathrm{RT}}}{1 + \frac{2\bar{v}}{v_{\mathrm{eqR}}}\coth\left(\frac{\bar{v}g}{2v_{2\mathrm{RT}}^2}t\right)}, \tag{A10}$$

which is Equation (141) in the main text.

With only linear air resistance $v_{2\mathrm{RT}} \to \infty$ and $\bar{v} \to v_{2\mathrm{RT}}$. Using that

$$\coth x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{\mathrm{e}^{x} - \mathrm{e}^{-x}}, \tag{A11}$$

Equation (A1) then takes the form

$$v_{1\mathrm{R}} = v_{1\mathrm{RT}} \left(1 - \mathrm{e}^{-\frac{g}{v_{1\mathrm{RT}}} t} \right), \tag{A12}$$

which is Equation (48) in the main text.

With only quadratic air resistance $v_{1\mathrm{RT}} \to \infty$. Then $\bar{v} \to v_{\mathrm{eq}}/2$. In this case $\coth\left[\left(\bar{v} g / 2 v_{2\mathrm{RT}}^{2}\right) t\right] \to \infty$, so the number 1 in the denominator of Equation (A10) can be neglected. Equation (A10) then takes the form

$$v_{2\mathrm{R}} = v_{2\mathrm{RT}} \tanh\left(\frac{g}{v_{2\mathrm{RT}}} t \right), \tag{A13}$$

which is Equation (135) in the main text.

Appendix B. Alternative Method for Integrating Equation (141)

Since this review is meant to be of pedagogical value I also present a useful method for integrating Equation (114) without using integral tables or integrators on internet. Equation (140) can be written

$$\int \frac{\mathrm{d}v_{12\mathrm{R}}}{v_{12\mathrm{R}}^{2} + v_{\mathrm{eqR}} v_{12\mathrm{R}} - v_{2\mathrm{RT}}^{2}} = C - \frac{g}{v_{2\mathrm{RT}}^{2}} t, \tag{B1}$$

where v_{eqR} is given in Equation (107). With the formula

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{artanh} x = \frac{1}{1 - x^{2}} \tag{B2}$$

in the mind we prepare for introducing a new variable by writing Equation (B1) in the form

$$\frac{1}{\bar{v}} \int \frac{\mathrm{d}v_{12\mathrm{R}}}{1 - \left[\left(v_{12\mathrm{R}} + \frac{v_{\mathrm{eqR}}}{2} \right) \Big/ \bar{v} \right]^{2}} = \bar{v} \left(\frac{g}{v_{\mathrm{tRT}}^{2}} t - C \right), \tag{B3}$$

where $\bar{v}$ is given in Equation (141). Introducing the variable

$$x = \frac{v_{12\mathrm{R}} + \frac{v_{\mathrm{eqR}}}{2}}{\bar{v}}, \tag{B4}$$

Equation (B3) takes the form

$$\int \frac{\mathrm{d}x}{1 - x^{2}} = \bar{v} \left(\frac{g}{v_{2\mathrm{RT}}^{2}} t - C \right). \tag{B5}$$

Integration by use of Equations (B2) and (B4) gives

$$\operatorname{artanh}\frac{v_{12\mathrm{R}}+\frac{v_{\mathrm{eqR}}}{2}}{\overline{v}}=\overline{v}\left(\frac{g}{v_{2\mathrm{RT}}^2}t-C\right). \tag{B6}$$

Hence

$$\frac{v_{12\mathrm{R}}+\frac{v_{\mathrm{eqR}}}{2}}{\overline{v}}=\tanh\left[\overline{v}\left(\frac{g}{v_{2\mathrm{RT}}^2}t-C\right)\right], \tag{B7}$$

or

$$v_{12\mathrm{R}}=\overline{v}\tanh\left[\overline{v}\left(\frac{g}{v_{2\mathrm{RT}}^2}t-C\right)\right]-\frac{v_{\mathrm{eqR}}}{2}. \tag{B8}$$

The initial condition $v(0)=0$ gives

$$\overline{v}\tanh\left[\overline{v}\cdot(-C)\right]=\frac{v_{\mathrm{eqR}}}{2}. \tag{B9}$$

Hence

$$C=-\frac{1}{\overline{v}}\operatorname{artanh}\frac{1}{\sqrt{1+4\frac{v_{2\mathrm{RT}}^2}{v_{\mathrm{eqR}}^2}}}. \tag{B10}$$

Inserting this into Equation (B8) gives

$$v_{12\mathrm{R}}=\overline{v}\tanh\left(\frac{g\overline{v}}{v_{2\mathrm{RT}}^2}t+\operatorname{artanh}\frac{1}{\sqrt{1+4\frac{v_{2\mathrm{RT}}^2}{v_{\mathrm{eqR}}^2}}}\right)-\frac{v_{\mathrm{eqR}}}{2}. \tag{B11}$$

Using that

$$\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\cdot\tanh y}, \tag{B12}$$

inserting the expression for $\overline{v}$ from Equation (B3) and using Equation (82), we get

$$v_{12\mathrm{R}}=\frac{2v_{1\mathrm{RT}}}{1+k_{\mathrm{R}}\coth\left(\frac{k_{\mathrm{R}}g}{2v_{1\mathrm{RT}}}t\right)}, \tag{B13}$$

where k_{R} is defined in Equation (A4). This is identical to Equation (A10).

Appendix C. Solving Equation (156)

With the notation $v_{2\mathrm{iso}}'=\mathrm{d}v_{2\mathrm{iso}}/\mathrm{d}z$ and introducing $y=v_{2\mathrm{iso}}^2$ Equation (154) can be written in the form

$$y'+\frac{2g}{v_{2\mathrm{RT}}^2}\mathrm{e}^{-\frac{z}{z_{\mathrm{T}}}}y=2g. \tag{C1}$$

This is a linear first order differential equation of the form

$$y' + P(z)y = Q(z)\,, \tag{C2}$$

with

$$P(z) = \frac{2g}{v_{2\mathrm{RT}}^2}\mathrm{e}^{-\frac{z}{z_\mathrm{T}}}\,, \quad Q(z) = 2g\,. \tag{C3}$$

The general solution is

$$y(z) = \mathrm{e}^{-\int P(z)\mathrm{d}z}\left[\int Q(z)\mathrm{e}^{\int P(z)\mathrm{d}z}\mathrm{d}z + C\right]. \tag{C4}$$

Inserting the functions (C3) gives

$$\int P(z)\mathrm{d}z = -\frac{2gz_\mathrm{T}}{v_{2\mathrm{RT}}^2}\mathrm{e}^{-z/z_\mathrm{T}}\,. \tag{C5}$$

It is useful here to introduce the dimensionless constant

$$p = \frac{2gz_\mathrm{T}}{v_{2\mathrm{RT}}^2}\,. \tag{C6}$$

The integral inside the bracket takes the form

$$\int Q(z)\mathrm{e}^{\int P(z)\mathrm{d}z}\mathrm{d}z = 2g\int \mathrm{e}^{-p\mathrm{e}^{-z/z_\mathrm{T}}}\mathrm{d}z\,. \tag{C7}$$

We here introduce a new variable u as follows

$$u = p\mathrm{e}^{-z/z_\mathrm{T}}\,, \quad z = -z_\mathrm{T}\ln(u/p), \quad \mathrm{d}z = -\frac{z_\mathrm{T}}{u}\mathrm{d}u \tag{C8}$$

Hence

$$\int \mathrm{e}^{-p\mathrm{e}^{-z/z_\mathrm{T}}}\mathrm{d}z = -z_\mathrm{T}\int\frac{\mathrm{e}^{-u}}{u}\mathrm{d}u = -z_\mathrm{T}Ei(-u) = -z_\mathrm{T}Ei\left(-p\mathrm{e}^{-z/z_\mathrm{T}}\right), \tag{C9}$$

where Ei is the exponential integral function. Inserting Equation (C9) into Equation (C7), and then Equation (C7) into Equation (C4) gives

$$y = \mathrm{e}^{p\mathrm{e}^{-z/z_\mathrm{T}}}\left[-pv_{2\mathrm{RT}}^2 Ei\left(-p\mathrm{e}^{-z/z_\mathrm{T}}\right) + C\right], \tag{C10}$$

where C is a constant of integration. The initial condition that the droplet falls from rest, $v_{2\mathrm{iso}}(0) = 0$, leads to

$$C = pv_{2\mathrm{RT}}^2 Ei(p)\,, \tag{C11}$$

and hence,

$$y(z) = pv_{2\mathrm{RT}}^2\mathrm{e}^{p\mathrm{e}^{-z/z_\mathrm{T}}}\left[Ei(-p) - Ei\left(-p\mathrm{e}^{-z/z_\mathrm{T}}\right)\right]. \tag{C12}$$

Using that $y = v_{2\mathrm{iso}}^2$ finally gives the solution of Equation (C1) with the initial condition $v_{2\mathrm{iso}}(0) = 0$,

$$v_{2\mathrm{iso}} = v_{2\mathrm{RT}}\sqrt{p\mathrm{e}^{p\mathrm{e}^{z/z_\mathrm{T}}}\left[Ei(-p) - Ei\left(-p\mathrm{e}^{-z/z_\mathrm{T}}\right)\right]}\,. \tag{C13}$$

Appendix D. Deduction of Equation (178)

It follows from Equations (143) and (173) and $v = \dot{h}$ that the vertical distance a droplet has moved is

$$
\begin{aligned}
&h_{12\mathrm{Rvs}} = 2v_{1\mathrm{RTvs}} \int_0^t \frac{\mathrm{d}t}{k_{\mathrm{air}} + k_{\mathrm{R}} \coth \hat{t}}, \; v_{1\mathrm{RTvs}} = v_{1\mathrm{RT}} - v_{\mathrm{air}} - v_{\mathrm{air}}^2 \big/ v_{\mathrm{eqR}}, \\
&\hat{t} = \frac{t}{t_1}, \; t_1 = \frac{2v_{1\mathrm{RT}}}{k_{\mathrm{R}} g},
\end{aligned}
\tag{D1}
$$

here k_{air} is given in Equation (172). Introducing

$$
x = \coth \hat{t}, \tag{D2}
$$

Equation (D1) takes the form

$$
h_{12\mathrm{Rvs}} = \frac{2v_{1\mathrm{RT}} v_{1\mathrm{RTvs}}}{k_{\mathrm{R}} g} \int_0^x \frac{\mathrm{d}x}{\left(1-x^2\right)\left(k_{\mathrm{R}} x + k_{\mathrm{air}}\right)}. \tag{D3}
$$

In order to perform the integration we can make a partial fraction decomposition, for example in the form

$$
\frac{1}{\left(1-x^2\right)\left(k_{\mathrm{R}} x + k_{\mathrm{air}}\right)} = \frac{1}{2\left(k_{\mathrm{R}}^2 - k_{\mathrm{air}}^2\right)} \left(\frac{2k_{\mathrm{R}}}{x + k_{\mathrm{air}}/k_{\mathrm{R}}} - \frac{k_{\mathrm{R}} + k_{\mathrm{air}}}{x+1} - \frac{k_{\mathrm{R}} - k_{\mathrm{air}}}{x-1} \right). \tag{D4}
$$

Using Equation (144) for k_{R}, Equation (172) for k_{air} and Equation (173) for $v_{1\mathrm{RTvs}}$ we get

$$
k_{\mathrm{R}}^2 - k_{\mathrm{air}}^2 = \frac{4v_{1\mathrm{RTvs}}}{v_{\mathrm{eqR}}}. \tag{D5}
$$

Inserting this into Equation (D4), integrating, inserting $x = \coth \hat{t}$ and using that $v_{\mathrm{eqR}} = v_{2\mathrm{RT}}^2 \big/ v_{1\mathrm{RT}}$ gives the position of the position as a function of time as given in Equations (D1) and (D3),

$$
h_{12\mathrm{RTvs}} = h_{\mathrm{RT}} \ln \frac{1 + \dfrac{k_{\mathrm{air}}}{k_{\mathrm{R}}} \tanh \hat{t}}{\left(1 + \tanh \hat{t}\right)^{\frac{k_{\mathrm{R}} + k_{\mathrm{air}}}{2k_{\mathrm{R}}}} \left(1 - \tanh \hat{t}\right)^{\frac{k_{\mathrm{R}} - k_{\mathrm{air}}}{2k_{\mathrm{R}}}}}. \tag{D6}
$$

where $h_{\mathrm{RT}} = v_{\mathrm{RT}}^2 \big/ g$. Using that

$$
\left(1 + \tanh \hat{t}\right)^{\frac{k_{\mathrm{R}} + k_{\mathrm{air}}}{2k_{\mathrm{R}}}} \left(1 - \tanh \hat{t}\right)^{\frac{k_{\mathrm{R}} - k_{\mathrm{air}}}{2k_{\mathrm{R}}}} = \sqrt{1 - \tanh^2 \hat{t}} \, \frac{1 - \tanh \hat{t}}{1 + \tanh \hat{t}}, \tag{D7}
$$

together with the identities

$$
\sqrt{1 - \tanh^2 \hat{t}} = \frac{1}{\cosh \hat{t}} \quad \text{and} \quad \frac{1}{2} \ln \frac{1 + \tanh \hat{t}}{1 - \tanh \hat{t}} = \hat{t}, \tag{D8}
$$

the expression for the falling distance as a function of time can be written in the form

$$
h_{12\mathrm{RTvs}} = h_{\mathrm{RT}} \left[\ln \left(\cosh \hat{t} + \frac{k_{\mathrm{air}}}{k_{\mathrm{R}}} \sinh \hat{t} \right) - \frac{k_{\mathrm{air}}}{k_{\mathrm{R}}} \hat{t} \right]. \tag{D9}
$$

The case with vanishing vertical air velocity, $v_{\mathrm{air}} = 0$, is obtained by putting $k_{\mathrm{air}} = 1$.

It turns out to be advantageous to go back to the expression (D6) in order to obtain a simplest possible velocity-falling distance relationship. From Equation (173) we have

$$1+\frac{k_{\mathrm{air}}}{k_{\mathrm{R}}}\tanh\hat{t}=\frac{1}{1-\frac{k_{\mathrm{air}}v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}},\quad 1+\tanh\hat{t}=\frac{1+\left(k_{\mathrm{R}}-k_{\mathrm{air}}\right)}{1-\frac{k_{\mathrm{air}}v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}},$$
$$1-\tanh\hat{t}=\frac{1-\left(k_{\mathrm{R}}+k_{\mathrm{air}}\right)}{1-\frac{k_{\mathrm{air}}v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}}. \tag{D10}$$

Since the sum of the exponents in Equation (D6) is $\frac{k_{\mathrm{R}}+k_{\mathrm{air}}}{2k_{\mathrm{R}}}+\frac{k_{\mathrm{R}}-k_{\mathrm{air}}}{2k_{\mathrm{R}}}=1$, the factor $1-\frac{k_{\mathrm{air}}v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}$ cancels. Hence inserting the expressions (D10) into Equation (D6) finally gives

$$h_{12\mathrm{Rvs}}=\frac{v_{2\mathrm{RT}}^{2}}{g}\ln\left\{\frac{\left[1-\left(k_{\mathrm{R}}+k_{\mathrm{air}}\right)\frac{v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}\right]^{\frac{k_{\mathrm{air}}-k_{\mathrm{R}}}{2k_{\mathrm{R}}}}}{\left[1+\left(k_{\mathrm{R}}-k_{\mathrm{air}}\right)\frac{v_{12\mathrm{Rvs}}}{2v_{1\mathrm{RTvs}}}\right]^{\frac{k_{\mathrm{air}}+k_{\mathrm{R}}}{2k_{\mathrm{R}}}}}\right\}, \tag{D11}$$

which is Equation (178).