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A Least Energy Sign-Changing Solution for Nonlocal Schrödinger Equations with Asymptotically Linear Nonlinearities

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Abstract

In this paper, we are concerned with the following nonlocal Schrödinger equations

$$-\mathcal{L}_K u + V(x)u = f(x,u), \ \ x \in \mathbb{R}^N,$$

where $-\mathcal{L}_K$ is an integro-differential operator of fractional Laplacian type and V is coercive at infinity. Combining the Nehari manifold and the quantitative deformation lemma, a least energy sign-changing solution was obtained, and the energy doubling phenomenon was also found.

Keywords

Nonlocal Schrödinger Equations, Least Energy Sign-Changing Solution, Integro-Differential Operator, Asymptotically Linear, Energy Doubling

1. Introduction

Sign-changing solutions of nonlinear elliptic equations have received increasing attention in recent years. Such a topic can be traced back to some earlier works concerning the existence of sign-changing solutions of the semilinear elliptic equations

$$-\Delta u + V(x)u = f(x,u), \ \ x \in \Omega, \tag{1}$$

where Ω is a domain of $\mathbb{R}^N$. See, for example, [1]-[6] and the references therein. In Particular, in [1], a least energy nodal solution of (1) was obtained, and the authors also showed that the energy of any sign-changing solution is larger than two times the ground state energy, this property is called energy doubling by Weth in [7]. Chang and Wang [8] considered a fractional Laplacian equation in a bounded domain and obtained the multiplicity of sign-changing solutions by applying the Caffarelli-Silvestre extension method. In [9] [10], a least energy sign-changing solution and infinitely many sign-changing solutions were obtained for nonlocal elliptic equations set on a bounded domain.

It should be point out that, almost all of the above-mentioned papers considered the case that $f(x,u)$ is superlinear with respect to u at infinity, that is

$$\lim_{u\to\infty}\frac{f(x,u)}{u}=+\infty,$$

and the (AR) superlinear condition ([11]) was imposed. However, the study of many practical problems, such as the self-trapping of an electromagnetic wave, leads to some problems related to (1), in which $f(x,u)$ is not superlinear but asymptotically linear with respect to u at infinity, such typical models can be found in [12] [13].

In this paper, we consider the following nonlocal Schrödinger equation

$$-\mathcal{L}_K u+V(x)u=f(x,u),\ x\in\mathbb{R}^N, \tag{2}$$

where f is asymptotically linear at infinity, $\mathcal{L}_K$ is an integro-differential operator with the kernel K which is a measurable function with the properties that

(K_1) there is $\theta>0$ and $s\in(0,1)$ such that $K(x)\ge\theta|x|^{-(N+2s)}$ for any $x\in\mathbb{R}^N\setminus\{0\}$;

(K_2) $mK\in L^1(\mathbb{R}^N)$, where $m(x):=\min\{|x|^2,1\}$.

Different from the above results, we shall study the existence of the least energy sign-changing solution of (2), which has the least energy among all sign-changing solutions. To the best of our knowledge, only few works concerning this case up to now. Chang [14] obtained a ground state solution for a fractional Laplacian equation on a bounded domain. Based on the inspiration from the aforementioned articles, we now consider the study of the least energy sign-changing solution within the context of nonlocal Schrödinger equations, and investigate the energy doubling property, namely, the energy of the least energy sign-changing solution is strictly greater than twice the energy of the ground state solution. Another aim of this paper is to show the energy doubling property, which was not found in [9].

The above equation appears widely in describing several different physical phenomena. For instance, in the context of materials science, it models the dynamics of the dislocation of atoms in crystals [15]. Here, the nonlocal nature of the operator $-\mathcal{L}_K$ captures the long-range interactions between dislocations, which are crucial for understanding the material's mechanical properties and deformation mechanisms. The existence of a least energy sign-changing solution may correspond to stable or metastable states in the dislocation system, providing insights into the material's stability and deformation behavior. Furthermore, the equation is also relevant to the study of anomalous diffusion [16]. In such processes, the nonlocal interactions described by the integral-differential operator $-\mathcal{L}_K$ play a key role in capturing the anomalous spreading of particles or energy. The least energy solution can be interpreted as an optimal distribution in the context of anomalous diffusion, offering a theoretical framework for understanding and predicting such non-classical diffusion behaviors. When $K(x)=|x|^{-N-2s}$, then $-\mathcal{L}_K$ reduce to the fractional Laplacian operator $(-\Delta)^s$, for more details we refer to the readers to [17] [18]. Different from the operator $-\Delta$, the integro-differential operator $\mathcal{L}_K$ is nonlocal, which brings us some difficulties in applying variational methods. For the variational setting and the existence of nontrivial solutions of such problems, we refer the reader to see [19] and [20].

To state our main results, we need the following assumptions on $V(x)$ and f.

(V_1) $V\in C(\mathbb{R}^N,R)$ satisfies $V_0:=\inf_{x\in\mathbb{R}^N}V(x)>0$;

(V_2) For any $M>0$, there exists $r>0$ such that

$$mes\left(\{x\in B_r(y):V(x)\le M\}\right)\to 0 \text{ as } |y|\to\infty,$$

where mes denotes for the Lebesgue measure and $B_R(x)$ denotes any open ball of $\mathbb{R}^N$ centered at x and of radius $R>0$.

(f_1) $f\in C(\mathbb{R}^N\times R,R)$ and $f(x,u)=o(|u|)$ as $u\to 0$ uniformly in x;

(f_2) There is a constant $a\in(0,+\infty)$ such that $f(x,u)u^{-1}\to a$ as $u\to\infty$ uniformly in x and

$$a>\inf\sigma\left(-\mathcal{L}_K+V(x)\right),$$

where $\sigma\left(-\mathcal{L}_K+V(x)\right)$ denotes the spectrum of the operator $-\mathcal{L}_K+V(x)$;

(f_3) $\dfrac{f(x,u)}{|u|}$ is a increasing function of $u\ne 0$.

Recall the Nehari manifold $\mathcal{N}$ associated with (2) is

$$\mathcal{N}=\left\{u\in E\setminus\{0\}:\left\langle\Psi'(u),u\right\rangle=0\right\},$$

and denote the energy of ground state by

$$c_0=\inf_{u\in\mathcal{N}}\Psi(u), \tag{3}$$

where Ψ is the energy functional corresponding to (2)(see §2). Now, the main results in this paper can be stated as follows.

Theorem 1. If $(V_1),(V_2)$ and (f_1)-(f_3) hold, then (2) possesses a least energy sign-changing solution u_0. In addition, c_0 can be achieved either by a positive or a negative function and $\Psi(u_0)>2c_0$.

Remark. Condition (V_2), which is weaker than the coercive assumption: $V(x)\to\infty$ as $|x|\to\infty$, was firstly introduced by Bartsch and Wang in [2] to overcome the lack of compactness.

Throughout this paper, we denote $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^N)$, $1\le p<\infty$, $u_n\rightharpoonup u$ and $u_n\to u$ mean the weak and strong convergence, respectively, as $n\to\infty$. C or $C_i\,(i=1,2,\cdots)$ denote some positive constants many change from line to line.

2. Proof of Main Results

To state our main results, we first recall the variational setting corresponding to (2). For any $s \in (0,1)$, we define $X^s(\mathbb{R}^N)$ as

$$X^s(\mathbb{R}^N) = \Big\{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable, } u \in L^2(\mathbb{R}^N) \text{ and the mapping } (x,y) \mapsto (u(x)-u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\Big\},$$

where the kernel K satisfies (K_1) and (K_2). The more properties of $X^s(\mathbb{R}^N)$ can be found in [17] [20]. With the presence of potential function V, we will work in the following subspace E of $X^s(\mathbb{R}^N)$

$$E := \Big\{u \in X^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < +\infty\Big\},$$

which is a Hilbert space equipped with the inner product

$$(u,v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x.$$

The norm on E induced by the above inner product denoted by $\|u\|$, and the energy functional associated with (2) is

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))^2 K(x-y)\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x, u \in E.$$

where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$. Under our assumptions, it is standard to check that $\Psi \in C^1(E,\mathbb{R})$ and for $v \in E$, there holds

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(x)-u(y))(v(x)-v(y))K(x-y)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x.$$

To obtain the least energy sign-changing solution in Theorem 1, we will follow [4] [7] and seek the minimizer of the energy functional Ψ restricted to the set

$$\mathcal{M} = \Big\{u \in E : u^{\pm} \neq 0, \langle \Psi'(u), u^+\rangle = \langle \Psi'(u), u^-\rangle = 0\Big\}, \quad (4)$$

which contains all sign-changing solutions. Recall that

$$u^+(x) = \max\{u(x),0\} \text{ and } u^-(x) = \min\{u(x),0\}.$$

We will show that the minimizer is actually a sign-changing solution of (2).

By direct computations, one has

$$\Psi(u) = \Psi(u^+) + \Psi(u^-) + (u^+,u^-), \langle \Psi'(u), u^{\pm}\rangle = \langle \Psi'(u^{\pm}), u^{\pm}\rangle + (u^+,u^-), \quad (5)$$

where

$$(u^+,u^-) = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u^+(x)u^-(y) + u^-(x)u^+(y)\right]K(x-y)\mathrm{d}x\mathrm{d}y \geq 0. \quad (6)$$

Firstly, to overcome the difficulties brought by the nonlocal feature of operator $\mathcal{L}_K$, we need the following embedding result.

Theorem 2. If (V_1) and (V_2) hold, then the embeddings $E \hookrightarrow L^p(\mathbb{R}^N)$ are continuous for $p \in [2, 2_s^*]$ and compact for $p \in [2, 2_s^*)$, where $2_s^* = \dfrac{2N}{N-2s}$ is the fractional Sobolev critical exponent.

Proof. By (V_1) we know that $E \hookrightarrow X^s(\mathbb{R}^N)$ is continuous, from the [17] and (K_1) we obtain $X^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$, then $E \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$.

Next we show that $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in [2, 2_s^*)$. In fact, let $\{u_n\} \subset E$ be a bounded sequence of E, going if necessary to a subsequence we have $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$, $p \in [2, 2_s^*)$ and $\|u_n(x)\| + \|u(x)\| \le \bar{C}$, where $\bar{C}$ is a positive constant. We first prove that $u_n \to u$ in $L^2(\mathbb{R}^N)$, it suffices to prove $|u_n|_2 \to |u|_2$.

Fix $M > 0$ and set $A_M(y) := \{x \in \mathbb{R}^N : V(x) \le M\} \cap B_r(y)$, where $r > 0$ is given by (V_2). When $p = 2$, $u_n \to u$ in $L^2_{loc}(\mathbb{R}^N)$ implies that $u_n \to u$ in $L^2(B_R)$ for any $R > 0$. Now, we choose $\{y_n\} \subset \mathbb{R}^N$ such that $\mathbb{R}^N \subset \bigcup_{i=1}^{\infty} B_r(y_i)$ and each $x \in \mathbb{R}^N$ is covered by at most 2^N balls. Denote the set $C_M(y_i) := \{x \in \mathbb{R}^N : V(x) > M\} \cap B_r(y_i)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_R} |u_n(x) - u(x)|^2 \,\mathrm{d}x \\
&\le \sum_{|y_i| \ge R-r}^{\infty} \int_{B_r(y_i)} |u_n(x) - u(x)|^2 \,\mathrm{d}x \\
&= \sum_{|y_i| \ge R-r}^{\infty} \left(\int_{C_M(y_i)} |u_n(x) - u(x)|^2 \,\mathrm{d}x + \int_{A_M(y_i)} |u_n(x) - u(x)|^2 \,\mathrm{d}x \right).
\end{aligned}$$

Using the definition of $C_M(y_i)$ and Hölder inequality we obtain

$$\int_{C_M(y_i)} |u_n(x) - u(x)|^2 \,\mathrm{d}x \le \frac{1}{M} \int_{B_r(y_i)} V(x) |u_n(x) - u(x)|^2 \,\mathrm{d}x,$$

$$\begin{aligned}
\int_{A_M(y_i)} |u_n(x) - u(x)|^2 \,\mathrm{d}x &\le \left(\int_{A_M(y_i)} |u_n(x) - u(x)|^{2t} \,\mathrm{d}x \right)^{\frac{1}{t}} \left(\int_{A_M(y_i)} 1^{t'} \right)^{\frac{1}{t'}} \\
&= |u_n(x) - u(x)|^2_{L^{2t}(A_M(y_i))} \left(mesA_M(y_i) \right)^{\frac{1}{t'}},
\end{aligned}$$

where $t \in \left(1, \frac{N}{N-2s}\right)$ and $\frac{1}{t} + \frac{1}{t'} = 1$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_R} |u_n(x) - u(x)|^2 \,\mathrm{d}x &\le \sum_{|y_i| > R-r} \left(\frac{1}{M} \int_{B_r(y_i)} V(x) |u_n(x) - u(x)|^2 \,\mathrm{d}x \right. \\
&\quad \left. + \sup_{|y_i| > R-r} \left(mesA_M(y_i) \right)^{\frac{1}{t'}} |u_n(x) - u(x)|^2_{L^{2t}(A_M(y_i))} \right) \\
&\le \frac{2^N}{M} \int_{\mathbb{R}^N \setminus B_{R-2r}} V(x) |u_n(x) - u(x)|^2 \,\mathrm{d}x \\
&\quad + 2^N C_2^2 \sup_{|y_i| > R-r} \left(mesA_M(y_i) \right)^{\frac{1}{t'}} \|u_n(x) - u(x)\|^2 \\
&\le \frac{2^N \bar{C}^2}{M} + 2^N \left(C_2 \bar{C} \right)^2 \sup_{|y_i| > R-r} \left(mesA_M(y_i) \right)^{\frac{1}{t'}},
\end{aligned}$$

where $C_2 > 0$ is the embedding constant. Now, for any $\varepsilon > 0$ we choose $M > 0$ so large that

$$\frac{2^{N+1}\overline{C}^2}{M} < \varepsilon. \tag{7}$$

For fixed $M > 0$, there exists $R_M > 0$ such that

$$2^{N+1}\left(C_2\overline{C}\right)^2 \sup_{|y_i|>R_M-r}\left(mesA_M\left(y_i\right)\right)^{\frac{1}{t'}} < \varepsilon, \tag{8}$$

since

$$\sup_{|y_i|\geq R-r}\left(mesA_M\left(y_i\right)\right)^{\frac{1}{t'}} \to 0 \text{ as } R\to\infty.$$

For such R_M, by (7) and (8) we have

$$\int_{\mathbb{R}^N\setminus B_{R_M}}\left|u_n\left(x\right)-u\left(x\right)\right|^2 \mathrm{d}x \leq \varepsilon.$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|u_n\left(x\right)-u\left(x\right)\right|^2 \mathrm{d}x\\&=\int_{B(0,R_M)}\left|u_n\left(x\right)-u\left(x\right)\right|^2 \mathrm{d}x+\int_{\mathbb{R}^N\setminus B_{R_M}}\left|u_n\left(x\right)-u\left(x\right)\right|^2 \mathrm{d}x \leq 2\varepsilon,\end{aligned} \tag{9}$$

this prove $|u_n|_2 \to |u|_2$ in $L^2\left(\mathbb{R}^N\right)$. Finally, by the Interpolation inequality we have (up to renaming C)

$$\begin{aligned}|u_n-u|_p &\leq C|u_n-u|_2^{\theta}|u_n-u|_{2_s^*}^{1-\theta}\\&\leq C|u_n-u|_2^{\theta}\|u_n-u\|^{1-\theta}\\&\leq C|u_n-u|_2^{\theta}\left(\|u_n\|+\|u\|\right)^{1-\theta},\end{aligned} \tag{10}$$

where $\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{2_s^*}$ and $\theta\in(0,1)$. Hence the right hand of (10) is small enough, therefore, $u_n\to u$ in $L^p\left(\mathbb{R}^N\right)$ for $p\in\left(2,2_s^*\right)$.

Since f satisfies asymptotically linear growth condition, the order of the nonlinearity term is the same as the nonlocal term, it is rather difficult to show the sign-changing Nehari manifold is nonempty. Therefore, we need to rely on the following set

$$\Theta := \left\{u\in E : u^{\pm}\neq 0, \left\|u^{\pm}\right\|^2 - a\left|u^{\pm}\right|_2^2 < 0\right\}.$$

Lemma 3. Suppose (f_2) and (f_3) hold, then $\mathcal{M}\subset\Theta$.

Proof. It is derived from (f_2) and (f_3), for any $u\in E\setminus\{0\}$ and $\varepsilon>0$, we know that

$$\frac{f(x,u)}{u} < a+\varepsilon. \tag{11}$$

Thus, when $u\in\mathcal{M}$, we can deduce from (8) and (11) that

$$\left\|u^{\pm}\right\|^2+\left(u^+,u^-\right)=\int_{\mathbb{R}^N} f\left(x,u^{\pm}\right)u^{\pm}\mathrm{d}x < (a+\varepsilon)\left|u^{\pm}\right|_2^2,$$

which implying $\|u^{\pm}\|^2 - a|u^{\pm}|_2^2 < 0$ for ε small enough.

Remark. Due to the inequality (11), when $u \in E$ with $u^{\pm} \neq 0$ and $\langle \Psi'(u), u^{\pm} \rangle \leq 0$, we have $u \in \Theta$.

With the help of Θ, we have successfully proved the following Lemma.

Lemma 4. Suppose $(V_1),(V_2)$ and (f_1)-(f_3) hold, if $u \in \Theta$, then there exists a unique pair of positive numbers (s_u, t_u) such that $s_u u^+ + t_u u^- \in \mathcal{M}$.

Proof. It is derived from assumptions (f_1)-(f_3) that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon|u| + C_\varepsilon |u|^{p-1}. \tag{12}$$

And so, we have

$$|F(x,u)| \leq \varepsilon u^2 + C_\varepsilon |u|^p, \tag{13}$$

where $p \in (2, 2_s^*)$. For fixed $u \in E$ with $u^{\pm} \neq 0$, $(s,t) \in^+ \times^+$ and $^+ := (0, +\infty)$, we denote

$$h_1(s,t) = \langle \Psi'(su^+ + tu^-), su^+ \rangle,\ h_2(s,t) = \langle \Psi'(su^+ + tu^-), tu^- \rangle. \tag{14}$$

Then $su^+ + tu^- \in \mathcal{M}$ if and only if $h_1(s,t) = 0$ and $h_2(s,t) = 0$. It is not difficult to see that

$$h_1(s,t) \geq s^2 \left(\|u^+\|^2 - \int_{\mathbb{R}^N} \frac{f(x, su^+)}{su^+} (u^+)^2 \,\mathrm{d}x \right), \tag{15}$$

$$h_2(s,t) \geq t^2 \left(\|u^-\|^2 - \int_{\mathbb{R}^N} \frac{f(x, tu^-)}{tu^-} (u^-)^2 \,\mathrm{d}x \right). \tag{16}$$

(12), (15) and (16) imply $h_1(s,t) > 0$ for $s > 0$ small enough and $h_2(s,t) > 0$ for $t > 0$ small enough. In addition, it is easy to see that $h_1(0,t) = h_2(s,0) = 0$. Thus there exists $r > 0$ small enough such that $h_1(r,r) > 0$, $h_2(r,r) > 0$. For any fixed $t < s$, (11) allows us to apply Lebesgue dominated convergence theorem to get

$$\begin{aligned} \limsup_{s \to +\infty} \frac{h_1(s,t)}{s^2} &= \limsup_{s \to +\infty} \left(\|u^+\|^2 + \frac{t}{s}(u^+, u^-) - \int_{\mathbb{R}^N} \frac{f(x, su^+)}{su^+} |u^+|^2 \,\mathrm{d}x \right) \\ &= \|u^+\|^2 - a|u^+|_2^2 \\ &< 0, \end{aligned}$$

which implying that $h_1(R, R-1) < 0$ for $R > 0$ large enough. Similarly, we know that $h_2(R-1, R) < 0$. Because the functions $h_1(s,t)$ is increasing with respect to t and $h_2(s,t)$ is increasing with respect to s, where $s,t \in \mathbb{R}^+$, we know for all $s,t \in [r, R-1]$, the following hold

$$h_1(r,t) > 0,\ h_1(R,t) < 0,\ h_2(s,r) > 0,\ h_2(s,R) < 0. \tag{17}$$

It has been shown in Miranda's theorem [21] that there exist two positive constants $s_u, t_u \in [r, R-1]$ such that $h_1(s_u, t_u) = 0$ and $h_2(s_u, t_u) = 0$, which implies that $s_u u^+ + t_u u^- \in \mathcal{M}$.

Next, we will show that the uniqueness of (s_u, t_u), and we divide the argument into two cases.

Case 1. $u \in \mathcal{M}$. By the definition of $\mathcal{M}$ we know that $\langle \Psi'(u), u^+ \rangle = \langle \Psi'(u), u^- \rangle = 0$, *i.e.*,

$$\|u^\pm\|^2 + (u^+, u^-) = \int_{\mathbb{R}^N} f(x, u^\pm) u^\pm \mathrm{d}x. \tag{18}$$

To complete the proof, we only have to show that $(s_u, t_u) = (1,1)$ is the unique pair of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$. Suppose, for contradiction,that $(\bar{s}, \bar{t}) \neq (1,1)$ such that $\bar{s}u^+ + \bar{t}u^- \in \mathcal{M}$. Without loss of generality, we suppose $\bar{s} \geq \bar{t} > 0$, then

$$\begin{cases} \int_{\mathbb{R}^N} f(x, \bar{s}u^+) \bar{s}u^+ = \bar{s}^2 \|u^+\|^2 + \bar{s}\bar{t}(u^+, u^-) \leq \bar{s}^2 \|u^+\|^2 + \bar{s}^2 (u^+, u^-), \\ \int_{\mathbb{R}^N} f(x, \bar{t}u^-) \bar{t}u^- = \bar{t}^2 \|u^-\|^2 + \bar{s}\bar{t}(u^+, u^-) \geq \bar{t}^2 \|u^-\|^2 + \bar{t}^2 (u^+, u^-). \end{cases} \tag{19}$$

By (18) and (19), we have

$$0 \leq \int_{\mathbb{R}^N} \left(\frac{f(x, u^+)}{u^+} - \frac{f(x, \bar{s}u^+)}{\bar{s}u^+} \right) (u^+)^2 \mathrm{d}x,$$

this implies $\bar{s} \leq 1$. Similarly, we have $\bar{t} \geq 1$. Consequently, $\bar{s} = \bar{t} = 1$.

Case 2. $u \notin \mathcal{M}$. We know that there exists $(s_1, t_1) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $s_1 u^+ + t_1 u^- \in \mathcal{M}$. Suppose that there exists another pair of numbers $(s_2, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfies $s_2 u^+ + t_2 u^- \in \mathcal{M}$. Denote $u_1 := s_1 u^+ + t_1 u^-$ and $u_2 := s_2 u^+ + t_2 u^-$, then

$$u_2 = s_2 u^+ + t_2 u^- = \left(\frac{s_2}{s_1} \right) s_1 u^+ + \left(\frac{t_2}{t_1} \right) t_1 u^- = \left(\frac{s_2}{s_1} \right) u_1^+ + \left(\frac{t_2}{t_1} \right) u_1^- \in \mathcal{M}.$$

Due to the fact that $u_1 \in \mathcal{M}$, repeating the proof of the case 1 we have $\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1$. Thus, we assert that (s_u, t_u) is the unique pair of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}$.

Lemma 5. If (f_1) - (f_3) hold, assume that $u \in E$ with $u^\pm \neq 0$ such that $h_1(1,1) \leq 0$ and $h_2(1,1) \leq 0$, where $h_1(s,t), h_2(s,t)$ are given in (14). Then the unique pair (s_u, t_u) of positive numbers obtained in Lemma 4 satisfies $0 < s_u, t_u \leq 1$.

Proof. Since $s_u u^+ + t_u u^- \in \mathcal{M}$, without loss of generality, we suppose that $s_u \geq t_u > 0$ and obtain the following inequality

$$\int_{\mathbb{R}^N} f(x, s_u u^+) s_u u^+ \mathrm{d}x = s_u^2 \|u^+\|^2 + s_u t_u (u^+, u^-) \leq s_u^2 \|u^+\|^2 + s_u^2 (u^+, u^-). \tag{20}$$

By $h_1(1,1) \leq 0$, we have

$$\left\|u^+\right\|^2 + \left(u^+, u^-\right) \le \int_{\mathbb{R}^N} f\left(x, u^+\right) u^+ \mathrm{d}x. \tag{21}$$

We know from (20) and (21) that

$$\int_{\mathbb{R}^N} \left(\frac{f\left(x, s_u u^+\right)}{s_u u^+} - \frac{f\left(x, u^+\right)}{u^+} \right) \left(u^+\right)^2 \mathrm{d}x \le 0,$$

which implies that $s_u \le 1$. Then, $0 < s_u, t_u \le 1$.

In the sequel, we consider the following minimization problem

$$m = \inf\left\{\Psi(u) : u \in \mathcal{M}\right\}. \tag{22}$$

Lemma 6. If $(V_1),(V_2)$ and (f_1)-(f_3) hold, then $m > 0$ can be achieved.

Proof. We first show that $m > 0$. By the definition of $\mathcal{M}$ and $\mathcal{N}$, we obtain $\mathcal{M} \subset \mathcal{N}$, hence we only need to show $c_0 > 0$ which is given in (3). Let $d = d(u) = \|u\|$. By (13) we know there exists $C > 0$ such that

$$\begin{aligned} \Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(\varepsilon |u|^2 + C_\varepsilon |u|^p\right)\mathrm{d}x \\ &\ge d^2\left(\frac{1}{2} - C_2\varepsilon - C_p C_\varepsilon d^{p-2}\right), \end{aligned}$$

where $2 \le p < 2_s^*$ and C_2, C_p are the embedding constants. Now, we choose d and ε small enough such that

$$\frac{1}{2} - C_2\varepsilon - C_p C_\varepsilon d^{p-2} \ge \frac{1}{8}. \tag{23}$$

We take $u \in \mathcal{N}$ and $\lambda > 0$ small enough such that $d(\lambda u)$ satisfies (23). We can easily to know $\Psi(u) = \max_{s,t \ge 0} \Psi\left(su^+ + tu^-\right)$ from [22], then $\Psi(u) \ge \Psi(\lambda u) \ge \frac{1}{8} d^2 > 0$, hence $c_0 > 0$, and $m > 0$.

Next, we prove that m can be achieved. By Lemma 4, there exists a minimizing sequence $\{u_n\} \subset \mathcal{M}$ such that $\Psi(u_n) \to m > 0$. Based on the work of Lions [23] on the concentration compactness principle, as shown in [24], we can prove that $\{u_n\}$ is bounded in E. Hence there exists $u_0 \in E$ such that $u_n^\pm \rightharpoonup u_0^\pm$ in E and $u_n^\pm \to u_0^\pm$ in $L^p\left(\mathbb{R}^N\right)$ for $p \in \left(2, 2_s^*\right)$. In addition, $\{u_n\} \subset \mathcal{M}$ implies that $u_n^\pm \ne 0$ and

$$\left\langle \Psi'(u_n), u_n^+ \right\rangle = \left\langle \Psi'(u_n), u_n^- \right\rangle = 0. \tag{24}$$

By (12), (24) and the boundedness of u_n, there exists $C_0 > 0$ such that

$$C_0 \le \left\|u_n^\pm\right\|^2 \le \left\|u_n^\pm\right\|^2 + \left(u_n^+, u_n^-\right) = \int_{\mathbb{R}^N} f\left(x, u_n^\pm\right) u_n^\pm \mathrm{d}x \le \int_{\mathbb{R}^N} \left(\varepsilon\left|u_n^\pm\right|^2 + C_\varepsilon \left|u_n^\pm\right|^p\right)\mathrm{d}x.$$

When $\varepsilon > 0$ small enough, we get

$$\int_{\mathbb{R}^N} \left|u_n^\pm\right|^p \mathrm{d}x \ge \frac{C_0}{2C_\varepsilon} > 0 \text{ and } \int_{\mathbb{R}^N} \left|u_0^\pm\right|^p \mathrm{d}x \ge \frac{C_0}{2C_\varepsilon} > 0,$$

which implies that $u_0^{\pm} \neq 0$. By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathbb{R}^N} f\left(x,u_n^{\pm}\right)u_n^{\pm}\mathrm{d}x = \int_{\mathbb{R}^N} f\left(x,u_0^{\pm}\right)u_0^{\pm}\mathrm{d}x, \\ &\lim_{n\to\infty}\int_{\mathbb{R}^N} F\left(x,u_n^{\pm}\right)\mathrm{d}x = \int_{\mathbb{R}^N} F\left(x,u_0^{\pm}\right)\mathrm{d}x, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \lim_{n\to\infty}\left(u_n^+,u_n^-\right) &= \lim_{n\to\infty}\left\{-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_n^+(x)u_n^-(y)+u_n^-(x)u_n^+(y)\right]K(x-y)\mathrm{d}x\mathrm{d}y\right\} \\ &= \left(u_0^+,u_0^-\right). \end{aligned} \tag{26}$$

By the weak semicontinuity of norm and (24) - (26), we obtain

$$\begin{aligned} \left\|u_0^{\pm}\right\|^2 + \left(u_0^+,u_0^-\right) &\le \liminf_{n\to\infty}\left(\left\|u_n^{\pm}\right\|^2 + \left(u_n^+,u_n^-\right)\right) \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^N} f\left(x,u_n^{\pm}\right)u_n^{\pm}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} f\left(x,u_0^{\pm}\right)u_0^{\pm}\mathrm{d}x, \end{aligned} \tag{27}$$

which implies $h_1(1,1) \le 0$ and $h_2(1,1) \le 0$. Thus, $u_0 \in \Theta$. By Lemma 4 and 5, there exists a unique pair of $\left(s_{u_0},t_{u_0}\right) \in (0,1]\times(0,1]$ such that

$$\tilde{u}_0 := s_{u_0}u_0^+ + t_{u_0}u_0^- \in \mathcal{M}.$$

It from (f_3) that $f(x,t)t - 2F(x,t)$ is increasing in $t>0$ and decreasing in $t<0$, then we have

$$\begin{aligned} m &\le \Psi\left(\tilde{u}_0\right) = \Psi\left(\tilde{u}_0\right) - \frac{1}{2}\left\langle \Psi'\left(\tilde{u}_0\right),\tilde{u}_0\right\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left(f\left(x,s_{u_0}u_0^+\right)s_{u_0}u_0^+ - 2F\left(x,s_{u_0}u_0^+\right)\right)\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left(f\left(x,t_{u_0}u_0^-\right)t_{u_0}u_0^- - 2F\left(x,t_{u_0}u_0^-\right)\right)\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{R}^N}\left(f\left(x,u_0\right)u_0 - 2F\left(x,u_0\right)\right)\mathrm{d}x \\ &\le \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}\left(f\left(x,u_n\right)u_n - 2F\left(x,u_n\right)\right)\mathrm{d}x\right] \\ &= \liminf_{n\to\infty}\left[\Psi\left(u_n\right) - \frac{1}{2}\left\langle \Psi'\left(u_n\right),u_n\right\rangle\right] \\ &= m. \end{aligned} \tag{28}$$

Therefore, $\Psi\left(\tilde{u}_0\right) = m$, and from the proof of (28), we have $s_{u_0} = t_{u_0} = 1$. Hence, $\Psi\left(u_0\right) = m$.

Lemma 7. Assume that $(V_1),(V_2)$ and (f_1) - (f_3) hold, if $u_0 \in \mathcal{M}$ and $\Psi\left(u_0\right) = m$, then u_0 is a critical point of Ψ.

Proof. Suppose by contradiction that $\Psi'\left(u_0\right) \neq 0$, then there exist $\delta > 0$ and $\alpha > 0$ such that

$$\Psi'(u) \ge \alpha \text{ for all } u \in \left\{u \in E : \left\|u - u_0\right\| \le 3\delta\right\}.$$

Define the map $w(s,t) := su_0^+ + tu_0^-$, $(s,t) \in D$, where

$D:=(1-\delta,1+\delta)\times(1-\delta,1+\delta)$. By the similar argument as [25], we know that

$$\Psi(u)=\max_{s,t\geq 0}\Psi\left(su^{+}+tu^{-}\right),\ \forall u\in\mathcal{M}, \tag{29}$$

which implies that

$$m_0:=\max_{(s,t)\in\partial D}\Psi\circ w<m. \tag{30}$$

We take $\varepsilon:=\min\{(m-m_0)/3,\alpha\delta/8\}$ and $S_\delta:=\{v\in E:\|v-u_0\|\leq\delta\}$, it follows from [26] that there exists a deformation $\eta\in C([0,1]\times E,E)$ such that

- $\eta(1,u)=u$ if $u\notin\Psi^{-1}([m-2\varepsilon,m+2\varepsilon])\cap S_{2\delta}$;
- $\eta\left(1,\Psi^{m+\varepsilon}\cap S_\delta\right)\subset\Psi^{m-\varepsilon}$;
- $\Psi(\eta(.,u))$ is nonincreasing for all $u\in E$.

It follows from (iii) and (29) that

$\Psi(\eta(1,w(s,t)))\leq\Psi(w(s,t))<\Psi(u_0)=m$ for all $(s,t)\in\bar{D}$ with $(s,t)\neq(1,1)$.

When $(s,t)=(1,1)$, by $u_0\in S_\delta\cap\Psi^{m+\varepsilon}$ and (ii), we know $\Psi(\eta(1,w(1,1)))\leq m-\varepsilon<m$. So,

$$\max_{(s,t)\in\bar{D}}\Psi(\eta(1,w(s,t)))<m. \tag{31}$$

Next, we claim that $\eta(1,w(s,t))\cap\mathcal{M}\neq\varnothing$ by the degree theory. Let us define $W(s,t):=\eta(1,w(s,t))$ and the following vectors

$$\Psi_0(s,t):=\left(\Psi'(w(s,t))u_0^{+},\Psi'(w(s,t))u_0^{-}\right),$$

$$\Psi_1(s,t):=\left(\frac{1}{s}\Psi'(W(s,t))W(s,t)^{+},\frac{1}{t}\Psi'(W(s,t))W(s,t)^{-}\right).$$

Obviously, we have

$$f'(x,t)t^2-f(x,t)t>0\ \text{ for } t\neq 0, \tag{32}$$

where f' denotes the partial derivatives with respect to t. By (32) and $u_0\in\mathcal{M}$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\Psi'(w(s,t))u_0^{+}\right)\Big|_{(1,1)}&=\|u_0^{+}\|^2-\int_{\mathbb{R}^N}f'(x,u_0^{+})(u_0^{+})^2\,\mathrm{d}x\\&<\|u_0^{+}\|^2-\int_{\mathbb{R}^N}f(x,u_0^{+})u_0^{+}\mathrm{d}x\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^{+}(x)u_0^{-}(y)+u_0^{-}(x)u_0^{+}(y)\right]K(x-y)\,\mathrm{d}x\mathrm{d}y\\&<0,\end{aligned}$$

$$\frac{\partial}{\partial t}\left(\Psi'(w(s,t))u_0^{+}\right)\Big|_{(1,1)}=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^{+}(x)u_0^{-}(y)+u_0^{-}(x)u_0^{+}(y)\right]K(x-y)\,\mathrm{d}x\mathrm{d}y.$$

Similarly

$$\frac{\partial}{\partial s}\left(\Psi'(w(s,t))u_0^{-}\right)\Big|_{(1,1)}=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^{+}(x)u_0^{-}(y)+u_0^{-}(x)u_0^{+}(y)\right]K(x-y)\,\mathrm{d}x\mathrm{d}y,$$

$$\frac{\partial}{\partial t}\left(\Psi'(w(s,t))u_0^{-}\right)\Big|_{(1,1)}<\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^{+}(x)u_0^{-}(y)+u_0^{-}(x)u_0^{+}(y)\right]K(x-y)\,\mathrm{d}x\mathrm{d}y<0.$$

Therefore

$$J_{\Psi_0}(1,1)=\begin{vmatrix}\frac{\partial}{\partial s}\left(\Psi'(w(s,t))u_0^+\right)\Big|_{(1,1)} & \frac{\partial}{\partial t}\left(\Psi'(w(s,t))u_0^+\right)\Big|_{(1,1)}\\ \frac{\partial}{\partial s}\left(\Psi'(w(s,t))u_0^-\right)\Big|_{(1,1)} & \frac{\partial}{\partial t}\left(\Psi'(w(s,t))u_0^-\right)\Big|_{(1,1)}\end{vmatrix}$$
$$>\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^+(x)u_0^-(y)+u_0^-(x)u_0^+(y)\right]K(x-y)\mathrm{d}x\mathrm{d}y\right)^2$$
$$-\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[u_0^+(x)u_0^-(y)+u_0^-(x)u_0^+(y)\right]K(x-y)\mathrm{d}x\mathrm{d}y\right)^2$$
$$=0.$$

It follows from the degree theory that $\deg(\Psi_0,D,0)=1$. On the other hand, we obtain from (30) and (i) that $w=W$ on ∂U. Thus, we have $\deg(\Psi_0,D,0)=\deg(\Psi_1,D,0)=1$. There exists a pair of $(s_0,t_0)\in D$ such that $\Psi_1(s_0,t_0)=0$, therefore $\eta(1,w(s_0,t_0))=W(s_0,t_0)\in\mathcal{M}$, which contradicts (31), hence, $\Psi'(u_0)=0$. From above, we know that (2) has a least energy sign-changing solution u_0.

Proof of Theorem 1. In the sequel, we just need to establish that its energy is strictly larger than twice that of least energy solutions. We are going to prove that for any $u\in\mathcal{M}$, there exists a unique pair of $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$ such that $su^+\in\mathcal{N}$ and $tu^-\in\mathcal{N}$. In fact, $su^+\in\mathcal{N}$ if and only if $\left\langle\Psi'(su^+),su^+\right\rangle=0$. For the sake of convenience, we let $\varphi(s):=\left\langle\Psi'(su^+),su^+\right\rangle$, by (12), we have

$$\begin{aligned}\varphi(s)&=s^2\left\|u^+\right\|^2-\int_{\mathbb{R}^N}f(x,su^+)su^+\mathrm{d}x\\ &\ge s^2\left(\left\|u^+\right\|^2-\varepsilon\int_{\mathbb{R}^N}\left|u^+\right|^2\mathrm{d}x-C_\varepsilon s^{p-2}\int_{\mathbb{R}^N}\left|u^+\right|^p\mathrm{d}x\right)\\ &\ge s^2\left(\left\|u^+\right\|^2-\varepsilon C_2\left\|u^+\right\|^2-C_pC_\varepsilon s^{p-2}\left\|u^+\right\|^p\right).\end{aligned}$$

On the other hand, for any $u\in\mathcal{M}$, it follows from Lemma 3 that

$$\limsup_{s\to+\infty}\frac{\varphi(s)}{s^2}=\left\|u^+\right\|^2-a\left|u^+\right|_2^2<0.$$

Choosing ε small enough, there exists $s_1>0$ small enough such that $\varphi(s_1)>0$ and $s_2>1$ large enough such that $\varphi(s_2)<0$. According to the continuity of $\varphi(s)$, we can obtain there exists a $s_u>0$ such that $\varphi(s_u)=0$. Next, let us prove the uniqueness of s_u. Suppose there exists another positive number $\overline{s}_u\neq s_u$ such that $\varphi(\overline{s}_u)=0$, then

$$\varphi(s_u)=s_u^2\left\|u^+\right\|^2-\int_{\mathbb{R}^N}f(x,s_uu^+)s_uu^+\mathrm{d}x=0, \tag{33}$$

$$\varphi(\overline{s}_u)=\overline{s}_u^2\left\|u^+\right\|^2-\int_{\mathbb{R}^N}f(x,\overline{s}_uu^+)\overline{s}_uu^+\mathrm{d}x=0. \tag{34}$$

Combining (33) and (34), we have

$$0=\int_{\mathbb{R}^N}\left(\frac{f(x,\overline{s}_uu^+)}{\overline{s}_uu^+}-\frac{f(x,s_uu^+)}{s_uu^+}\right)(u^+)^2\mathrm{d}x. \tag{35}$$

Suppose $\overline{s}_u > s_u$, then the right of the equality (35) is positive, this is impossible. Similarly, there is a unique $t_u > 0$ such that $t_u u^- \in \mathcal{N}$. From (29), we have

$$\begin{aligned} m = \Psi(u_0) &= \max_{s,t\geq 0} \Psi\left(su_0^+ + tu_0^-\right) \\ &\geq \Psi\left(s_{u_0}u_0^+ + t_{u_0}u_0^-\right) \\ &= \Psi\left(s_{u_0}u_0^+\right) + \Psi\left(t_{u_0}u_0^-\right) + \left(s_{u_0}u_0^+, t_{u_0}u_0^-\right) \\ &> \Psi\left(s_{u_0}u_0^+\right) + \Psi\left(t_{u_0}u_0^-\right) \\ &\geq 2c_0. \end{aligned}$$

This completes the proof.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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