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The Equivalence between Special Relativity, Newtonian Physics and Quantum Mechanics in Complex Para-Space

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Abstract

Dramatic simplification of mathematical apparatus, special relativity's hyperbolic versus circular versions, some equivalence of SR and Newton's theories, algebra of relativistic and the corresponding Galilean velocities. Complex model, say $\mathbb{C}^3$, of para-space as alternative to the real M^4 Minkowski space-time for both relativistic and classical mechanics was shortly introduced. As it turned out, the model and its theory have the power to bridge two and very likely also three physical theories in one common framework. The model, originally thought of as the model for special relativity only, exhibited the possibility also to model the classical Newtonian mechanics and, moreover, the two mechanics turned out to be equivalent in their algebraic and topological structures. As it follows from some additional analysis, placed in section 10, also the quantum mechanics (QM) may [hypothetically] be described as theory of the same $\mathbb{C}^3$ complex domain. If so, the bridging power of the introduced complex model is striking even if QM does not seem to be just equivalent to the remaining two theories. As for the SR, in the new complex Euclidean framework, it is not only totally preserved but at many important issues extended. Thus, in the new model, several hardly understood facts from the real SR such as the universality of speed of light, the Lorentz contraction or twins' paradox (especially the phenomena associated with the rocket's changing direction) find clear explanation unknown in the real M^4 version of this beautiful theory. The transition from the real to complex description (the only necessary prize) yields dramatic simplification of all the three theories which is specially striking in the case of quantum mechanics (provided that my hypothesis on QM will turn out to be the true one). Moreover, in this work, the algebraic structure of isomorphic vector spaces was imposed and analyzed on both: the set of all relativistic velocities and on the set of the corresponding to them “Galilean” velocities. In some association with that, the relativistic theory was closer analyzed. Namely, the two approaches: the hyperbolic (“classical” SR) with its four-vector formalism and Euclidean, where SR is modeled by the complex para-space $\mathbb{C}^3$, were analyzed and compared.

Keywords

Dramatic Simplification of Mathematical Apparatus, Special Relativity’s Hyperbolic versus Circular Versions, Some Equivalence of SR and Newton’s Theories, Algebra of Relativistic and the Corresponding Galilean Velocities

1. Introduction

The main goal of this work is to indicate some interesting relationship or perhaps even equivalence of two mechanics: Newtonian (actually, “semi-Newtonian” that, possibly, slightly and rather inessentially, differs from the classic Newtonian) and relativistic (SR, [1]) when both are considered as theories of the same complex (para-)space model $\mathbb{C}^3$.

As it was proposed and argued in section 10 with the hypothetical considerations, the third theory that is the quantum mechanics has also the ability to become theory of the same complex model. Therefore, the simple $\mathbb{C}^3$ para-space model may have very strong unifying properties that may possibly be also extended toward other physical theories simplifying their mathematical apparatus dramatically. According to my recent, yet unpublished, investigations the classical electrodynamics very likely may also find its model as the same $\mathbb{C}^3$. Here, however, that theory (electrodynamics) would exhibit some differences as, in a sense, being “skew-symmetric” if compared with, for example, SR.

Let us now return to the comparison of classical [Newtonian] and relativistic mechanics.

According to the range of SR we, basically, restrict both theories to kinematics while a comparison of the corresponding two dynamics (here, we will mean “relativistic dynamics” as SR is extended to a theory of accelerations as well), as performed in subsection 9.3, seems still to reveal the same results as the kinematics. Some important results on the dynamics comparison are mentioned at end of this paper.

Of course, equivalence of the mentioned two theories doesn’t take place when the classical mechanics is considered as theory of a real model and SR as the theory of the real Minkowski’s M^4 space-time.

The two real frameworks differ too much to model the essentially equivalent (or, at least, closely related) theories. What is the main device, for unifying the theories, is introducing their common complex model $\mathbb{C}^3$ (see [2] and [3]).

More detailed information on introducing the complex models, together with reasoning on the need of using complex numbers, in a proper way, as models for physical phenomena, one can find in [2], as well as in other my papers on that subject like [4], the papers [5] [6] together with other, published on internet blogs: Academia and ResearchGate within my accounts named "Jerzy K. Filus".

One of the main ideas that I tried to promote in this work is (a specific) transition from real to complex models. This approach occurs rather seldom in literature but can, nevertheless, be found in such positions like, between others, [7]-[9] and [10]. However, the methods of introducing complex models by those and other authors seem significantly different from what was done here. Other authors mostly put stress on adding imaginary parts to the reals or, very typically, on use of the idea of analytic continuations for points from the real models that can be treated as boundary subsets within the constructed complex domains. Our way of pursuing the complex extension of the real spacetime is different and relies on the recognition that motion in, say Minkowski's M^4 is equivalent to rotations in the, so created, complex domain $\mathbb{C}^4$ or in, later introduced, $\mathbb{C}^3$. Thus, the novel of our approach relies not just on use of complex models but rather on the way of their introduction and on associated physical interpretations of the underlying complex physical quantities and phenomena. Nevertheless, in every case the dimension of underlying real model was increased as imaginary directions were imposed. Some authors, however, such as Urusovsky [11] extend the dimension of real models such as $\mathbb{R}^3$ toward the, still real, six dimensional model $\mathbb{R}^6$. Notice that, topologically, this model is equivalent to the $\mathbb{C}^3$ model, which finally we adopted as the "para-space" model. However, algebraically these two models ($\mathbb{R}^6$, and $\mathbb{C}^3$) essentially differ. Besides, unlike in $\mathbb{R}^6$, the imaginary dimensions gain in the here presented $\mathbb{C}^3$ para-space model additional physical interpretations as sources of energy that are indispensable for existence of any motion as well as potential energy when physical fields [gravitational, electric, and possibly others] are present. For those reasons, and according to my best knowledge, the time free para-space $\mathbb{C}^3$ model is rather a new complex model for the (para)physical phenomena described below in this paper.

Here, in order not just to repeat the content of my previous works I only provide quite short introductory text for explanatory purposes in sections 3 - 5, and also in section 6.

Thus, in this work, we start with complex extension of the real Lorentz $M^4 \to M^4$ transformation, where M^4 is the real Minkowski space-time. The extension, which appeared to be very natural, initially implied the extension of M^4 to complex space-time, say $\mathbb{C}^4$, or rather the hyperbolic model $\mathbb{C}^{3,1}$. Next, we found that complex time, as modeled by $\mathbb{C}^1$, is not the primitive notion. As it turned out the time is definable in SR understood as theory of the complex model $\mathbb{C}^3$. The latter model we called "para-space". For some additional references to other authors who considered different complex models for physical space and time see [2] and [3].

The, here considered, complex para-space $\mathbb{C}^3$, and not "space-time", say $\mathbb{C}^{3,1}$, model turned out to have the, already mentioned, unifying properties with the ability to serve as efficient model for the mentioned above three theories. Namely, real parts of certain underlying complex physical (or as we call them "para-physical") quantities behave like relativistic quantities (distances or velocities, by example) while the (signed) absolute values of the same complex have properties like the Galilean counterparts of those relativistic. The latter recognition stands as a basis for our further considerations throughout this paper having as the main aim to show some equivalence of classical and relativistic mechanics. But, needless to say, the $\mathbb{C}^3$ para-space model would, hypothetically, also serve as a model for quantum mechanics (QM) (see, section 10) and possibly for the classical electrodynamics as well. Independently, the author met also with (a single only) opinion that this simple complex $\mathbb{C}^3$ model may, possibly, be a "new model for physics" in general!

The mentioned model is considered to be the following triple: $\left\langle \mathbb{C}^3, d(\cdot,\cdot); \mathfrak{P} \right\rangle$, usually shortened to just $\mathbb{C}^3$, where $\mathbb{C}^3$ denotes the common vector space over the field of complex numbers, $d(\cdot,\cdot)$ is the Euclidean metric on $\mathbb{C}^3$, and $\mathfrak{P}$ is the Poincaré group of all the Euclidean (not hyperbolic) isometries $\mathbb{C}^3 \to \mathbb{C}^3$.

As for the adopted $\mathbb{C}^3$ model we consider it to be more than just arithmetic space of the triples of complex numbers endowed with a mathematical structure. Unlike with usual mathematical understanding it in the complex analysis and other physical applications we will consider it dynamic as opposite to static real models. In the theory that we try to introduce, the real models including real manifolds can be endowed with a structure that determines a geometry or topology, which, by a here adopted **assumption,** deals with "static structures". Unlike static geometries the geometry in complex spaces (or manifolds) here is considered in a wider sense as containing a mechanics *i.e.*, motion of a given kind. According to the here introduced viewpoint, the motion is due to the imaginarity, considered as the source of the motion (or energy in a wider sense), being the obvious part of complex space, but here obtaining additional dynamic interpretation is not often (if at all) met in literature.

We claim the physical motion within the interior of the $\mathbb{C}^3$ is equivalent to [circular] rotations about origins (or other points as well) of the complex planes when, for example, these planes form algebraic bases for the underlying vector space $\mathbb{C}^3$. Thus, the physical content is given to the $\mathbb{C}^3$ topological vector space (obviously, here the topology is induced by the metrics $d(\cdot,\cdot)$) by the proper subgroup, say $L \subsetneq \mathfrak{P}$, of the Poincaré group consisting from the "boosts" only (In this work, however, the "boosts" instead of the hyperbolic rotations are circular rotations.). To obtain such dynamic effects with real spaces one must to resort to hyperbolicity [changing the signature of the originally Euclidean space ($\mathbb{R}^4$, for example)] such as, for example, the hyperbolic Minkowski's space-time M^4 [12] or its "deformations" toward some [real] manifolds as met in general relativity. This, however, is done for the prize of losing naturalness and significant clarity typical for the Euclidean geometry.

This may appear unfamiliar to some readers but the dynamicity within the such understood [as dynamic] complex space's interior may be visualized (or assumed to be so) as literal motion of the "points". For example, according to our assumption, all the points on the radial line OB' (see **Figure 1** in the text below) move along that line with the [Galilean] speed "$c\tan\theta$", where θ is an angle between the line OB' and the real, horizontal, axis. Thus, as the magnitude of the angle θ increases the speed of underlying points increases too.

This view, according to which points within a mathematical (but now this is rather a kind physical) model "move", is dictated by the assumption that the modeled reality which is represented by our (para)space (that we can observe the real part only) behave similarly. The moving [physical] points may eventually explain phenomena of existence of physical fields such as gravitational or electric. Whatever physical body, if placed at such moving points it will likely move along with them. If instead of straight line (such as the OB' line) we consider continuous curves, such as hyperbola or other, the curvature of that line will induce acceleration which would mean that we encounter a field of forces.

The hyperbolic analogy [12] to above-described "motion of points" in, say $\mathbb{C}^3$, is "motion" of all points of M^4 in the direction of time-axis or of some points in a direction of any other word-line. One, eventually, may choose one of the two different interpretations of dynamicity: hyperbolic or elliptic. The elliptic [circular] version has, however, this advantage over the hyperbolic that, besides providing more clarity, it sheds some additional light on nature of time reducing this notion to the notion of speed [motion] which, in turn, has the geometric roots. More on that time's genesis one can find in both [2] and [3].

Sections 3 and 4 contain a very short exposition of the basis of the created theory. In short we discuss the extension of the Lorentz transformation $M^4 \to M^4$ to its complex $\mathbb{C}^4 \to \mathbb{C}^4$ version, together with some justifications. We gave there an "old version" (unfortunately, already published in [4] in 2016) of the extended transformation (formula (3)), which, prima facie, seemed to be a kind of obvious, and then the corrected version (formula (4)) together with a short justification. Also, in section 3, we shortly discuss an illustration (**Figure 1**) of two kinds of mechanical motions, one, observable, along the real axis of the complex plane and second, the "natural one" (Galilean), across the interior of the plane, which can only be "observable mentally" by means of the mathematical model. The provided explanations are rather short and reader not familiar with that framework is advised to turn to either of the papers [2] or [3] as well as to other my papers on this subject, published in Academia and ResearchGate on the Internet.

In sections 5 and 6 we define and consider mechanical speeds whose primary definition is geometric, formula (5), with no use of time. Here, realize that the speed of light c, present in (5), can be identified with geometric notion of the orthogonality [angle $\pi/2$] whenever we assume $c = 1 = \sin\pi/2$, since the natural [Galilean] direction of the light is parallel to the imaginary axis (the considered angle θ that defines a speed equals $\pi/2$). Next, we justify and define the Galilean speeds as given by formulas (7), (8) and (9). Since those Galilean speeds turned out to have (but, actually, not to be) the same magnitudes as the well known in SR proper speeds we discuss it and compare the two [equivalent] theories (this and SR theory of the M^4) with this respect in section 6 and partially, already, in section 5.

The main subject, the actual new material that was not considered in any of my previous works, (but to explain it properly sections 3 - 6 were necessary prerequisites), is included in sections 7, 8 and 9, as well as in section 10.

Thus, in section 7 we define composition of the Galilean, initially one-dimensional, velocities (for simplicity, considered as "speeds") by means of the binary operation *i.e.*, "addition" compatible with the Lorentz-Einstein addition. Unfortunately, this, so defined, "addition" (see formula (13)) cannot, as originally expected, be the arithmetic addition for the reason the arithmetic addition is not compatible with the Lorentz-Einstein. This "inconvenient" fact seems to put into question the Galilean character of the constructed Newtonian (for arbitrarily high speeds) theory in the complex $\mathbb{C}^3$ space. With the discussion of this difficulty and eventual ways out from it we deal in section 8.

There, we call the theory containing the notion of "Galilean speeds" with non-arithmetic addition "semi-Galilean" or "semi-Newtonian". That non-arithmetic addition seems to be the only theory's property that makes it different from the classical Newtonian. On the other hand, for all relatively small velocities [or speeds], only being at play till, say, beginning of twenty century, that non-arithmetic operation "reduces" to the arithmetic. This means that physical instruments, as available that time, could possibly be unable to notice any difference which might only be present within underlying measurement errors. Besides, higher Galilean velocities that we consider in this paper can not be observed in real spaces by means of physical instruments and only can be accessed within mathematical model.

These theoretical recognitions inclined us to set the hypothesis that the semi-Newtonian theory with the defined non-arithmetic addition of the velocities is the actual Newtonian, the "fact" which, probably, was overlooked in the past.

Meanwhile, in section 7, we consider such nonarithmetic properties of this addition. It turns out that the set of all Galilean speeds endowed with the defined addition forms the Abelian group isomorphic with the group of all the relativistic speeds endowed with the Lorentz-Einstein addition. Since to each speed's magnitude corresponds exactly one angle (the speed's argument in the complex plane of the speeds [considered as points]) from the open [if we, temporarily, exclude speeds of light represented by the interval's endpoints] interval, say $A=(-\pi/2,\pi/2)$ we also define, in section 7, two-argument "addition" on the set A, compatible with the Lorentz-Einstein addition of relativistic speeds.

The, so obtained, Abelian group of the angles is isomorphic with both groups of the relativistic and Galilean speeds. Thus, all the three abelian groups are isomorphic and therefore algebraically equivalent. Moreover, in sub-sub-section 7.2.2 of sub-section 7.2 we define multiplication of the elements of each the considered three groups by real numbers so that each group became the vector space over the common field of real numbers. This multiplication by arbitrary real numbers was an extension of the defined in [12] multiplication of relativistic speeds by positive integers. It is important that the latter multiplication by the naturals is compatible with the Lorentz-Einstein addition.

Finally, the three obtained vector spaces turn out to be isomorphic *i.e.*, algebraically "identical", too.

Since, as we realized, the defined algebraic operations on the intervals $(-\pi/2,\pi/2)$, $(-1,1)$ and $(-\infty,\infty)$ are continuous with respect to their natural topologies and, moreover, their mutual isomorphisms also turn out to be homeomorphisms [and even diffeomorphisms], we finally concluded in sub-section 7.3 that the three considered sets of the angles and of the two types of the velocities, as endowed with the topological-vector spaces structures, are both algebraically and topologically equivalent. This inclines to the conclusion that the two theories of them: relativistic and semi-Newtonian kinematics as well as "geometric" theory of the space A of the angles are equivalent.

In section 8 we look closer at this thesis, giving some additional arguments for that. In that section we pursue the considerations in more general framework as "Newtonization" of relativistic physical quantities and phenomena. Since such procedure also is performed in classic SR by means of the four-vector formalism we discuss and compare the two approaches in terms of hyperbolic versus elliptic [circular, Euclidean] formalisms.

In section 9, we stated, as the main conclusion, the assertion of equivalence of the relativistic and classic kinematic theories with the suggestion of equivalence of the corresponding dynamic theories as well.

In section 10 we consider a hypothetical idea of applying $\mathbb{C}^3$ as a very simple model for some basic facts of quantum mechanics, *i.e.*, a description of a free particle and its quarks. Section 11 contains the final conclusions of this paper.

2. Short Preliminary Considerations

We start with the common real Lorentz $M^4 \to M^4$, transformation defined as:

$$\begin{aligned} x-ut &= x'\sqrt{1-\frac{u^2}{c^2}} \\ y &= y' \\ z &= z' \\ t-\frac{ux}{c^2} &= t'\sqrt{1-\frac{u^2}{c^2}} \end{aligned} \tag{1}$$

where the quadruples $(x,y,z,t)\in M^4$ describe the positions of points in coordinate system at rest while the quadruple (x',y',z',t') describes the corresponding points in M^4 whose coordinate system moves along the *x*-axis with the velocity $(u,0,0)$, so the corresponding speed equals u. We make the following known trigonometric substitutions:

$$\sqrt{1-\frac{u^2}{c^2}}=\cos\theta \text{ and } \frac{u}{c}=\sin\theta, \tag{A}$$

where θ is some circular angle. Now, transformation (1) takes on the equivalent trigonometric form:

$$\begin{aligned} x-ut &= x'\cos\theta \\ y &= y' \\ z &= z' \\ t-\frac{ux}{c^2} &= t'\cos\theta, \end{aligned} \tag{2}$$

where, initially, a geometric and an associated physical meaning of the “angle” θ are not known.

At this stage, we tried, in our previous papers, to find out what would, eventually, happen or what would be physical consequences (if any?), if we complete the coefficient $\cos\theta$, present in (2) by its, so natural from a pure mathematical viewpoint, complement “ $i\sin\theta$ “, where $i^2=-1$.

But then according to the Euler’s formula we have:

$$\cos\theta+i\sin\theta=\exp(i\theta)$$

which, geometrically, describes the circular rotation by the angle θ on the complex plane which appears as natural extension of the underlying real *x*-axis. As for the physical consequences of this extension, they turned out to be astonishing.

Look now at the so extended (complex) Lorentz transformation:

$$\begin{aligned} x-ut &= x'\exp(i\theta) \\ y &= y' \\ z &= z' \\ t-\frac{ux}{c^2} &= t'\exp(i|\theta|), \end{aligned} \tag{3}$$

where $-\pi/2<\theta<\pi/2$ and here, θ denotes the absolute value of the angle θ, needed because time cannot be reversed.

Now, first, that immediately comes to the attention is the length invariance in (3) because for the absolute value we have $|\exp(i|\theta|)|=1$ which, in turn, corresponds to geometric fact of invariance of lengths with respect to (circular) rotations. This would mean that there is no contraction of length when “anything” is in the motion and the Lorentz contraction is only due to “staying in the real space” as in such a case transformation (3) reduces back to (2). In the complex domain the Lorentz contraction as “observed” in the real subspace of the complex either space-time $\mathbb{C}^4$ or “para-space” $\mathbb{C}^3$ is the result of orthogonal projection of the invariant lengths (of a “rocket”, for example) situated in the interior of complex domain into real space. For that, see **Figure 1** where the projection of the interval OB' is the shorter Interval OA.

The big gain of naturalness, when within the complex model, strongly suggests hypothesis that, very likely, our physical space, say described by $\mathbb{R}^3$, should be considered immersed in the larger complex $\mathbb{C}^3$ space (“time” turns out to be a separate issue, see [2]) whose existence cannot be proven empirically but there arise rational (mathematical) necessities to adopt the hypothesis as the very convenient one. Notice at this point that, as it often turns out, convenient and simple theories usually are true.

As this is discussed in more detail in [2], formula (3) is only the first approach to build the correct version of the complex Lorentz transformation as derived from its real origin (1).

Here, to mention only that, at second approach, transformation (3) had to be replaced by the following $\mathbb{C}^4 \to \mathbb{C}^4$ transformation:

$$\begin{aligned} x - ut &= x' \exp(i\theta) \\ y &= y' \\ z &= z' \\ t - \frac{ux}{c^2} &= (t' \cos\theta) \exp(i|\theta|) \end{aligned} \tag{4}$$

The latter differs from (3) only by the time transformation. Reasons for that change can be found in [2]. Here, only notice that the right-hand side of the fourth row in (3) expresses not the rotation but instead the orthogonal projection of the real-time (along the horizontal time-axis) onto the radial line (of time) which makes the angle θ with the horizontal real-time measured by rest observer. Time, say $t_c = (t' \cos\theta) \exp(i|\theta|)$ is complex but its (real) absolute value $|t_c| = t' \cos\theta$, is the, so obtained, familiar proper time that is time along the radial line OB' on **Figure 1**, *i.e.*, time measured by the moving clock, say "in the rocket". For geometric illustration of this time transformation (in the complex plane of the complex time) phenomenon see Figure 3 in [2].

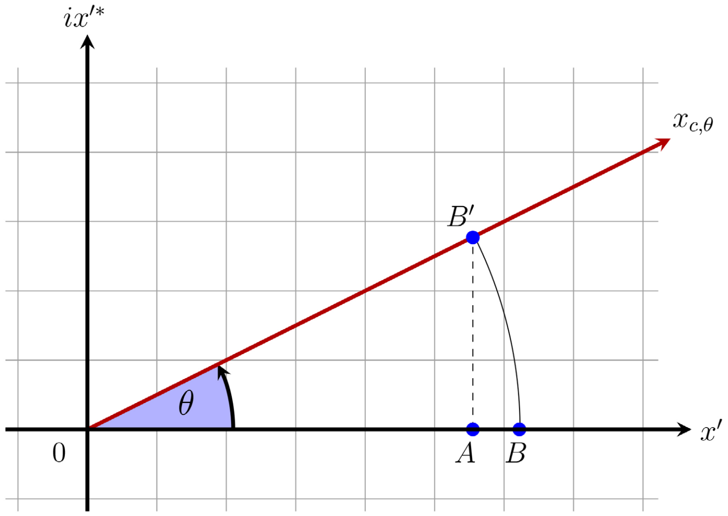

Figure 1. Example of terminological conventions being used in this article.

That figure depicts complex plane of the time. Anticipating the latter reference let me only mention that the complex time α' is the image under the projection (the fourth row of the complex Lorentz transformation (4) with $x = 0$) of real time at rest β' and the absolute value $|\cdot|$ of the complex time α' is the familiar proper time, so that $|\alpha'| = |\beta| \cos\theta$ and both β and α' are arbitrary time epochs.

This, here touched only, set of accounts can be found in [2]. In this reference, one can also find the final version of the complex Lorentz transformation in both: $\mathbb{C}^4 \to \mathbb{C}^4$ and $\mathbb{C}^3 \to \mathbb{C}^3$ versions. There were several reasons [2] for the reduction of the complex space-time $\mathbb{C}^4$ (or rather $\mathbb{C}^{(3,1)}$) to the complex para-space $\mathbb{C}^3$.

One of the important reasons was that we obtained the complex Lorentz $\mathbb{C}^4 \to \mathbb{C}^4$ transformation (formula (23) in [2]), in the form that the time transformation part did not explicitly contain any space-like variable and the space part of the transformation did not contain any time variable so time and space in the complex space-time were separated!.

Besides, within the whole SR theory as theory of the $\mathbb{C}^3$, the complex time can be defined and its theory can be viewed as a theory of single separate complex model $\mathbb{C}^1$. In other words time turns out not to be a primitive notion and can be derived from the geometry (!) of the para-space $\mathbb{C}^3$.

3. Motion in the Complex Plane of the Positions

In the, here (and in our previous works) created, theory of mechanic motion in complex spaces we, in general, do not consider the common space-time point events as basic elements of the theory's model but, instead, we only consider "positions" as points of the corresponding three-dimensional complex para-space $\mathbb{C}^3$ and, if needed, separately, "moments" [time epochs] as points of the time's complex plane $\mathbb{C}^1$.

As for the complex "position plane", which extends the real x-axis of the corresponding $\mathbb{R}^3$ or M^4 space, and for mechanical motion within it. **Figure 1**, on page 4, illustrates that.

Here, we restrict the motion to one real direction only along the real *x*-axis (within the corresponding, say $\mathbb{R}^3$) with a constant speed u. We analyze the complex counterpart of this motion in one complex plane of the positions, whose real axis is the mentioned *x*-axis from $\mathbb{R}^3$. The extension of this motion to the whole $\mathbb{C}^3$ is immediate as the rules of the motion (with velocity's coordinates, say u_x, u_y and u_z in place of one only $u_x = u$ considered here) on each of the three complex coordinate planes are the same.

The ("first") complex coordinate plane, that the considered motion takes place within, is illustrated in **Figure 1**.

Figure 1 is explained, in more detail, in [2] and [3]. Here, since we don't want, unnecessarily, repeat those paper's considerations we only need (for a basic exposition) to mention that the radial axis $x_{c,\theta}$ is thought of as the former (before the considered motion started) *x*-axis, "now" being after the rotation by the angle θ, while x' is thought of as becoming from the former radial line (not present on **Figure 1**) making with the former horizontal line an angle equal to ($-\theta$). The rotation is equivalent to the speed u [see formula (5) in following text] that the former *x*-axis [initially, at rest] gained, becoming the $x_{c,\theta}$ radial line whose points may be considered as moving in the right-up direction with the speed u. Of course, also the imaginary axis ix'^{*} is formed from a previous radial line from the first quadrant, that was passing through the same origin (as all the here considered lines), and making the angle θ with the former imaginary axis, say ix*. Thus, Figure 1 illustrates the situation after the rotation took place as the rotation (and the corresponding speed u) is considered to be the result of an action of an acceleration. Notice that a given at a time moment acceleration is, in turn, equivalent to an angular speed at the same moment. Here, a length of time of the acceleration's action is neglected.

The situation before the rotation took place is, however, not sketched on Figure 1 but was only described.

Suppose we consider a "rocket" whose back end is situated at the origin O. When the rocket is at rest it is positioned along the real horizontal axis and its front end is situated at point B. When in motion (with velocity $(u,0,0)$ so, in this case, the speed is simply u and therefore we will always call this velocity "speed") the same rocket is rotated by the angle θ toward the interior of the complex plane taking on the new position with (at the moment $t=0$) the same end-point at O and the front-point B' which is described by the complex nonreal number B'. Now, the actual rocket, especially at any time-moment $t>0$, is invisible for both human senses and physical instruments. What only can be observed is its "shadow" that (at the time moment $t=0$) spreads between the real points O and A so that the observed rocket is the orthogonal projection of the actual, but invisible rocket which is something like a "complex physical object". There is then a good reason to consider such an object (if to assume it "exists") not as strictly "physical" but rather a kind of "paraphysical" which, in this case, has a physical [observable] component *i.e.*, the rocket spread out between the points O and A and "contracted" according to the Lorentz contraction phenomenon.

Remark 1 Here, an ontological problem, doubts or, very likely, denial would occur, depending on the kind of philosophy a given reader follows. The latter doubt will certainly take place if one considers an "existence" as only "physical existence", ignoring for example the existence of numbers or any other mathematical items. Also, the view (rooted especially in Hume's philosophy, which is very typical among many physicists) according to which "to be is to be observed" with strong tendency to identify each phenomenon with its observation, will create a prone to reject the hypothesis on an existence of, say "para-physical objects" and phenomena. Fortunately (or not?), such views are not the only philosophical (because they always are "philosophical" even if one does not like this fact or any so-called "philosophy") views. □

Parallel to this Lorentz phenomenon (of the contraction), the metric of the x'-line is (according to moving observer, *i.e.*, "situated" along the OB' line) contracted by the constant (at all points of x') coefficient $\cos\theta$.

Geometrically, this contraction is due to orthogonal **projection** of the "new" complex $x_{c,\theta}$-axis *i.e.*, "radial axis" along OB' line, with invariant (according to the moving observer) metric, to the real x'-axis. This projection geometrically explains the Lorentz contraction phenomenon.

Notice, that such an explanation does not exist in the classical SR (as the theory of Minkowski's M^4 model) and notice too that "SR" considered as theory of the complex models contains, besides all the facts described by the classical SR, descriptions of some additional facts and explanations of some known facts such as, for example, the **universality of the speed of light** (see, [2] or [3]). ☐

For each single motion with speed u, consider two distinct trajectories in the complex model illustrated by **Figure 1**.

One is the observed path along the real $0A$ line and the other along $0B'$ line within the interior of the complex plane. The direction of the first we will call the "observable (real) direction" and of the second, the "natural direction" which, nevertheless, is beyond the reach of human senses and of physical instruments while still present within the easily accessible mathematical model.

If we consider motion of a rocket, our assumption is that, at the initial time epoch $t_{c,\theta}=0$, its actual, unobserved, length spreads out between the points 0 and B', whereas the observed "image" of this rocket's length spreads between 0 and A.

According to the Lorentz contraction, as illustrated by **Figure 1**, the image is shorter than the actual rocket by the proportion $|0A|/|0B'|=\cos\theta$ where, according to primary assumptions (A), $\cos\theta=\sqrt{1-u^2/c^2}$.

As one can say, the measured (if that experiment ever "happened"), by physical instruments, length $|OA|$ of the rocket is just the length of the observed **"rocket's shadow"** whereas its "true" invariant length is $|OB'|=|OB|$ and $|OB|$ is the length of the rocket when at rest.

4. On Galilean Speeds along the Natural (Complex) Directions

Let us now reinterpret a bit **Figure 1**. Instead of the "rocket" spread out between some points consider a classical particle at the moment $t=0$ situated at point B' while its real "image" has position A. The image of this particle moves along the real x'-axis with the relativistic speed u. By the first assumption of (A), which follows formula (1), we have:

$$u=c\sin\theta. \tag{5}$$

The question now is, what is the speed of the "actual particle" as it moves along the radial line $0B'$? To answer this, notice two facts, one of geometric and the other of analytic nature. First, the "two objects" (actually it is the same particle considered ["observed"] at point A and at point B' separately) are "instantaneous" in at least two meanings: geometric as both always lying on the same *vertical line* connecting them and analytic, since for the real parts we always have:

$$\operatorname{Re} B' = \operatorname{Re} A = A$$

Second, the ratio of the distances is:

$$\frac{0B'}{0A} = \sec\theta. \tag{6}$$

The conclusion from these two facts is that the [real] speed $x_{c,\theta}$ along the [complex] radial $x_{c,\theta}$-axis $0B'$ (known in literature as the proper velocity or celerity, but with no complex space framework nor geometric interpretation) is larger than u, and from (6) it follows that:

$$U_\theta = u\sec\theta, \tag{7}$$

where according to the admitted convention if $\theta = 0$ then

$$u = U_0. \tag{7*}$$

Realize that since, by (A), we have that $\sec\theta$ numerically equals the Lorentz factor, the proper velocity U_θ here was obtained geometrically with no use of (proper) time.

Remark 2 The premises for conclusion (7) are both the geometric and analytic "instantaneity" described above.

The above considerations do not yet require any primary use of the concept of time since the instantaneity is defined independently of this concept. Here notice, that this instantaneity can not be so defined in the real M^4 Minkowski model.

Nevertheless, as it is shown in [2], the times which elapse for either of the "two" objects to shift from 0 to A and from 0 to B', will turn out to be numerically the same. Thus, the instantaneity in the sense of "same time" also takes place but will be a conclusion rather than a premise. For more details on that, see [5]. ☐

Combining (7) with (5) we also obtain:

$$U_\theta = c\tan\theta. \tag{8}$$

As mentioned, the real speed $U_\theta = \left|U_{c,\theta}\right|$ is known as the "proper speed" where $U_{c,\theta} = U_\theta \exp(i\theta)$ is the corresponding "complex speed".

As for the complex speed $C_{c,\theta}$ of a light beam that is sent ahead from the considered rocket, which itself moves at speed U_θ in the $0B'$ radial direction, the same as before geometric argument yields to the conclusion that since

$$\frac{|0B'|}{|0A|} = \sec\theta, \quad \text{we have}: \left|C_{c,\theta}\right| = C_\theta = c\sec\theta, \tag{9}$$

where c is the ordinary relativistic ("small") speed of light in vacuum of the real subspace. Consequently, we have, for the complex speed of light:

$$C_{c,\theta} = C_\theta \exp(i\theta). \tag{9*}$$

In the complex quantities' framework, an overwhelming suggestion yields to the conclusion that the proper speeds U_θ as defined by (7) or (8) and C_θ defined by (9) should be considered "Galilean" or "Newtonian" since they are speeds of (para)-physical bodies that move along complex paths such as the $0B'$ path. The metrics of the paths are invariant (no Lorentz contractions), so each of the radial trajectory (as seen by observers situated on them) is a Euclidean line.

Notice, that the source of boundness of the relativistic speeds u, including speed of light c in $\mathbb{R}^3$ is the Lorentz contraction of distances, which, geometrically, are *projections* of Euclidean distances in any OB'-like radial line onto the real x'-axis. In a parallel fashion, any speed U_θ or C_θ along that radial line projected (or, analytically, multiplied by $\cos\theta$) into the real x' direction results in its bounded relativistic counterparts u or c. Realize that c as the projection of C_θ is always the same, regardless the angle (speed) θ. This and the Euclidean character of all the radial lines inclines one to treat the speeds U_θ and C_θ as Galilean (or Newtonian).

Relativistic speeds and contracted distances may roughly be considered as deformations of the Galilean and the Euclidean, respectively.

Notice that each Galilean (proper) speed U_θ is finite but unbounded. As $\theta \to \pi/2$, we have, according to (5) and (8) $u \to c$ and $U_\theta \to \infty$.

In papers [4] and [5], the last infinite limit, I considered to be the [actual] "Galilean speed of light" $C_{\pi/2} = C = \infty$. This is just in the spirit of the Newtonian theory and supports the "Newtonian version" of Einstein's universality of speed of light (Any finite speed U_θ is "infinitely smaller" than the infinite speed of light $C = C_{\pi/2}$). The "direction" of this (full) speed of light is vertical since, in this case, $\theta = \pi/2$.

Remark 3 At this point, once the infinite values (of the speed of light and of the corresponding infinitely distanced "positions" which photons "reach") enter to the creating theory, a kind of necessity arises as to complete the theory's model by incorporating "infinite objects" *i.e.*, mathematical objects "situated at infinity". This may lead to the construction of 3-D complex projective space that we propose to be denoted by $\mathfrak{P}^3$ (while the real projective space by $\mathbb{P}^3$). The main part of the well-known construction (see [13] or any other text book from the projective geometry) relies on adding to the points of the $\mathbb{C}^3$ (more generally, to the points of $\mathbb{C}^n$, $n = 1,2,\cdots$, the so called "invalid points" *i.e.*, "points at infinity", say $\{0, c_1, c_2, c_3\} \in \mathfrak{P}^3$, where $(c_1, c_2, c_3) \in \mathbb{C}^3$. The invalid points are also identified with the directions here represented by directions of the vectors, say $[c_1, c_2, c_3]$ from $\mathbb{C}^3$.

In such model light could approach an invalid point situated infinitely far from its source. According to an adopted convention, at an invalid point would also be situated the "position" of any photon after any nonzero amount of (purely imaginary) time elapsed. (This will not happen if the elapsing (complex) time has a nonzero real part, see [3]).

Ordinary "finite" points from the corresponding (open) $\mathbb{C}^3$, say $(c_1, c_2, c_3) \in \mathbb{C}^3$ are in $\mathfrak{P}^3$ identified with any quadruple proportional to the quadruple $\{1, c_1, c_2, c_3\} \in \mathfrak{P}^3$.

The whole corresponding $\mathfrak{P}^3$, as the physical model, we propose to name the "closed para-space" which topologically is compact and, possibly, represents some (para)reality with a *"transcendent part"* situated infinitely far from us? Such a *re-*

ality is out of any empiric reach but still can be, at least partially, "seen" by human minds, which use for this purpose (as a "mental instrument") an elegant and efficient mathematical model.

In my opinion, the actual need to introduce such projective extension [as the model for "closed (para)physical space"] of the "open" complex $\mathbb{C}^3$ space is motivated by the fact that such spaces $\mathfrak{P}^3$, or $\mathfrak{P}^n$ with $n=1,2,\cdots$), in general, have the best mathematical properties of all other geometric models. It is the "richest" and the most unified (see [13]) geometric space of all the finite dimensional "flat" spaces and, actually, of all the geometric spaces. For that, see for example [13].

And there is ("just") a *belief* among many, including myself, that the best, mathematically, model will, sooner or later, turn out to be the best for physics and possibly also for other applications of mathematics.

Nevertheless, there is one drawback of such the model extension. The $\mathfrak{P}^3$ model is not endowed with metrics which may be considered the serious obstacle for physicists. An easy remedy for that disadvantage is to apply a "partial metrics" as the one whose domain is restricted to the valid (finite) points only. Other, similar way out from the difficulty is to consider, as the model for physics, the Cartesian product $C^3\times\mathfrak{P}^3$ of the open and closed versions of the para-space. Also, it may be useful to extend the metrics to the infinite values whenever invalid points are considered. □

As mentioned, the Galilean speeds U_θ, are known in SR under the name "proper" but the novum of this presentation relies on different [basically geometric] derivation of them and, first of all, on their Newtonian interpretation mainly motivated by the "Euclidean consequences" of introducing the complex extensions.

Regardless their unbounded nature, the Galilean speeds never exceed the corresponding *semi-Galilean* speeds of light C_θ, for any θ such that $\theta<\pi/2$.

Recall that, unlike the (infinite) "*Galilean* speed of light" $C_{\pi/2}=\infty$, we call all finite speeds of light C_θ, for $\theta<\pi/2$, "semi-Galilean" treating them as the slant projections of the infinite Galilean into the OB'-like slant radial lines (**Figure 1**). Physically, they are the same as speeds of the light sent ahead or backwards from the rocket that moves along that slant line with the speed U_θ. Their magnitudes are defined by (9). This semi-Galilean speed of light is known in SR as the time-like coordinate of four-velocity. In this case, we always have $C_\theta>U_\theta$, and the "quadratic difference" between the two is (independently of θ) the same in the sense that we always have:

$$C_\theta^2-U_\theta=c^2. \tag{10}$$

For more on that, see [4] or [5].

Notice too that, in SR terminology, as considered in the hyperbolic settings, the left-hand side of (10) represents the [invariant] squared hyperbolic norm of four-velocity. In the Euclidean framework presented here, (10) rather relates to the universality of semi-Galilean speed of light as the squared "difference" between the two speeds is always the constant c^2. Thus, (10) represents the "generalized Einstein's universality of the velocity *c*".

5. More on Velocities

In classic SR the proper velocities (or speeds, here U_θ, and C_θ) are obtained in a quite different way than we did (geometrically), by purely analytic considerations, as the *convenient convention* with no explicit reference to Newtonian theory.

Recall, shortly, some basic facts to compare the ideas presented here, associated with the complex Euclidean $\mathbb{C}^3$ model for SR, with the common "hyperbolic" approach to SR as related to its M^4 model.

For simplicity, we will consider one real direction motions, and thus we analyze the C^1 model in place of $\mathbb{C}^3$ and M^2 (with one axis being the time axis) in place of M^4. So only one coordinate U_θ of the velocities $(U_\theta,0,0)$ will now be analyzed.

As the starting point for the comparison of both approaches to (the same) SR, consider any pair (c,u), where c is the usual relativistic speed of light and $(u,0,0)$ any relativistic (with u bounded by the speed of light c) velocity, which in this specific unidimensional case reduces to its speed u.

In SR (considered as the M^4 theory) there is adopted the following hyperbolic trigonometric speed's representation $u = c\tanh\lambda$, where the "rapidity" λ is the hyperbolic angle corresponding to speed u, see [14].

By contrast, in association with our Euclidean model (here it is the complex plane $\mathbb{C}^1$), we applied the trigonometric substitution $u = c\sin\theta$, see [15], where θ is the circular angle, the argument of a corresponding complex physical quantity. Here the complex Galilean speed $U_{c,\theta}$ so that $U_\theta = U_{c,\theta}$ and $\theta = \arg(U_{c,\theta})$.

As it is a quite common procedure in the full four-dimensional Minkowski space M^4 development of SR, the four-velocity $(C_\theta, U_{\theta 1}, U_{\theta 2}, U_{\theta 3})$ is obtained from the quadruple (c,u_1,u_2,u_3) (where the triple (u_1,u_2,u_3) denotes an ordinary relativistic velocity) by multiplying (c,u_1,u_2,u_3) by the Lorentz factor

$$\gamma(u) = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$$

With the simplified two dimensional versions of SR models we have instead (for "two-speeds") $(C_\theta, U_\theta) = \gamma(u)(c,u)$ and depending on the model (M^2 or $\mathbb{C}^1$) applied, we may substitute:

$$\gamma(u) = \cosh\lambda \quad \text{or} \quad \gamma(u) = \sec\theta.$$

The two representations of the same Lorentz factor must be equal, although the corresponding hyperbolic and circular angles λ and θ are not and are different kinds of mathematical objects.

The question may occur which of the two representations brings more information on the nature of the obtained velocities U_θ and C_θ.

It should be clear that the second representation as $U_\theta = c\tan\theta$ and $C_\theta = c\sec\theta$ (see (8) and (9)) reveals the Galilean (Newtonian) nature of U_θ and C_θ, while the first does not suggest anything like that.

In the first case, however, the invariance of the "two-speeds" with respect to its magnitude immediately follows as according to the definition of (hyperbolic) squared norm in M^2 we have

$$\left\|\left(C_\theta, U_\theta\right)\right\|_h^2 = C_\theta^2 - U_\theta^2 = c^2 = \text{constant} \tag{B}$$

where the hyperbolic trigonometric identity

$$\cosh^2\lambda - \sinh^2\lambda = 1$$

has been used. We also note that in (B) the constant c^2 is independent of θ.

In $\mathbb{C}^1$ the norm has a different definition, but equality (B) holds too, this time due to the circular trigonometric identity $\sec^2\theta - \tan^2\theta = 1$.

However, in the latter case, equality (B), instead of the squared norm, expresses the universality of the speed of light in its generalized form in any of the complex plane directions θ. For more on this see [4].

In both cases the right-hand side of equality (B) does not depend on the velocity U_θ nor C_θ, but in the second ("circular") case, (B) seems to bring some additional information (some generalization of the basic axiom of SR, between others) exhibiting, in possibly new way, the association between the relativity principle (the independence from θ *i.e.*, from the speed) and the speed of light universality.

The association is then expressed both by the hyperbolic version of SR and by the speed of light universality in the second (circular) version here being under the consideration.

The latter indicates possible existence of "logical" dependence of the universality of the speed of light on the relativity principle.

Both laws may possibly (?) be consequences of some necessity of mathematical nature (such as some trigonometric identities or extrema in other cases as met in physics).

6. On Speeds Algebraic Operations

6.1. Galilean Speeds Composition

Proper velocities are unbounded but, as mentioned in SR literature, they seem not to be understood as Galilean (Newtonian) velocities. Unlike in this paper, they are not considered as associated with the complex quantities, where real parts of these quantities are the relativistic speeds.

Recall, that in our framework proper velocities are equal to the signed absolute values of the complex velocities.

As it is known, however, composition of proper velocities is not arithmetic sums of them. Of course, the same happens with them when considered as the "Galilean" speeds, which in our complex framework can be verified as follows:

Let for the relativistic speed of light assume $c = 1$. Then we consider any two relativistic speeds as

$$u = \sin\alpha \quad \text{for } -\pi/2 < \alpha < \pi/2,$$
$$v = \sin\beta \quad \text{for } -\pi/2 < \beta < \pi/2.$$

Their Galilean counterparts (proper velocities) are

$$U = \tan\alpha,$$
$$V = \tan\beta.$$

Recall, for the composition of relativistic speeds the Lorentz-Einstein addition is given by:

$$\sin\alpha \oplus \sin\beta = \frac{\sin\alpha + \sin\beta}{1 + \sin\alpha\sin\beta} = \sin\theta \tag{11}$$

where

$$\theta = \arcsin\left(\sin\alpha \oplus \sin\beta\right). \tag{12}$$

One can easily check for any simple specific example, that:

$$\arctan\left(\tan\alpha + \tan\beta\right) \neq \theta.$$

Thus, the arithmetic addition of Galilean speeds is not compatible with the Einsteinian addition of relativistic speeds. This fact, of course, may put into doubt the interpretation of the quantities U and V as "Galilean" speeds.

In this paper the adjective "Galilean" mainly was motivated by geometric considerations in the Euclidean complex plane and motivation for this name, first of all, was based on the invariance of length of physical bodies as positioned along the radial direction. See **Figure 1** in text of this paper with the length invariance: $|OB| = |OB'|$.

A partial remedy for the problem of "non-Galilean" behavior of the composition of the "Galilean" velocities may be adoption of the following definition for "addition" of the Galilean quantities $c\tan\alpha$, $c\tan\beta$, with $c = 1$:

$$\begin{aligned}\tan\alpha \oplus^* \tan\beta &= \tan\left(\arcsin\left(\sin\alpha \oplus \sin\beta\right)\right) \\ &= \tan\theta.\end{aligned} \tag{13}$$

The later addition, denoted by $\oplus^*$, is then generated by the Lorentz-Einstein addition, denoted by $\oplus$, and it is compatible with it.

Now, it is seen that the results of both the additions, as given by (11) and (13), are related as

$$\frac{\tan\theta}{\sin\theta} = \sec\theta = \gamma,$$

where γ represents the usual Lorentz factor. Therefore, the "sum" of proper speeds defined by (13), is again a legitimate proper speed. This is the reason we call such proper speed "Galilean" again, treating the resulting $\tan\theta$ as a composition (or "sum") of the speeds $\tan\alpha$ and $\tan\beta$.

Notice, that, as it is common to know (see for example [16]), the same "addition" of proper speeds is differently defined than that by (13) and different nontrigonometric formula is used as its definition. In my opinion, however, definition (13) is simpler and somewhat more explanatory *i.e.*, (13) shows better the geometric "mechanism" of this addition, which has its source in the mathematical structure of the complex plane and, possibly, also in a relation between relativistic and Newtonian motions.

6.2. Algebraic Structure on Sets of Relativistic Speeds, Galilean Speeds and Related Circular Angles

6.2.1. Additions

Consider, first, the pair $(E, \oplus)$, where E is the set of all relativistic speeds in $\mathbb{R}^3$ *i.e.*,

$$E = \{u : -c < u < c\}, \text{ so } u = c\sin\theta, \tag{14}$$

where $c = 1$ is the (relativistic) speed of light in vacuum and the angles θ (arguments of the corresponding complex quantities) satisfy:

$$-\pi/2 < \theta < \pi/2; \tag{15}$$

$\oplus$ is the operation of Lorentz-Einstein addition defined, when trigonometric form is applied, by formula (11).

It is a well-known fact that $(E, \oplus)$ is an Abelian group. We will call it the E-group. We yet will consider two other Abelian groups which are isomorphic to the E-group.

1) First, consider the corresponding to E set A of the angles θ (the arguments) such that:

$$A = \{\theta : -\pi/2 < \theta < \pi/2\}. \tag{16}$$

In A we define the addition of angles, denoted by ${}^{**}\oplus$, as follows:

$$\begin{aligned} \alpha \,{}^{**}\oplus\, \beta &= \arcsin\left(\frac{\sin\alpha + \sin\beta}{1 + \sin\alpha\sin\beta}\right) \\ &= \arcsin(\sin\alpha \oplus \sin\beta). \end{aligned} \tag{17}$$

Since this addition of angles is generated by the relativistic addition of speeds $\oplus$, the pair $(A, {}^{**}\oplus)$ is an Abelian group isomorphic to the group $(E, \oplus)$. The isomorphism between these two groups is given by the mapping:

$$\Phi : E \to A \quad \text{defined by } \Phi(\cdot) = \arcsin(\cdot)$$

which satisfies the isomorphism condition:

$$\Phi(u \oplus v) = \Phi \,{}^{**}\oplus\, \Phi(v),$$

where, upon $c = 1$, $u = \sin\alpha$, $v = \sin\beta$. Recall that,

$$\Phi : (-1, 1) \to (-\pi/2, \pi/2),$$

is a one-to-one surjection and therefore it is an isomorphism.

2) Consider yet the set G of all the Galilean speeds, given as:

$$G = \{U : -\infty < U < \infty\}$$

where, upon $c = 1$, $U = \tan\theta$, with $\pi/2 < \theta < \pi/2$. Addition in G, denoted by $\oplus^*$, is defined by (13). It can easily be seen that the pair $\left(G, \oplus^*\right)$. is also an Abelian group isomorphic to the group $(E, \oplus)$. The mapping which establishes this isomorphism is

$$\Lambda(\cdot) = \tan\arcsin(\cdot), \tag{18}$$

which is injective and surjective mapping:

$$\Lambda : (-1,1) \rightarrow (-\infty, \infty).$$

Upon the substitution $\sin\alpha = u$, $\sin\beta = v$, from definition (18) and formula (13), it follows that:

$$\Lambda(u \oplus v) = \Lambda(u) \oplus^* \Lambda(v).$$

showing that Λ is an isomorphism. ☐

By the transitivity of isomorphisms all the three groups A, E and G are isomorphic, *i.e.* algebraically identical. Additionally realize, as interesting (though obvious) facts, that for the trigonometric representations of speeds we have:

$$\begin{aligned}\Lambda(\sin\alpha \oplus \sin\beta) &= \Lambda\left(\Lambda^{-1}(\tan\alpha) \oplus \Lambda^{-1}(\tan\beta)\right)\\ &= \tan\alpha \oplus^* \tan\beta,\end{aligned}$$

and, moreover

$$\sin\left(\alpha \ {}^{**}\!\oplus \beta\right) = \sin\alpha \oplus \sin\beta, \tag{19}$$

$$\tan\left(\alpha \ {}^{**}\!\oplus \beta\right) = \tan\alpha \oplus^* \tan\beta \tag{20}$$

The algebraical identity of the three groups and the last two identities suggest that it may be convenient to consider as basic model, for all the three "speeds", group of angles $\left(A, {}^{**}\!\oplus\right)$ as both the speed's additions may (according to (19), (20)) be reduced to the angles' addition, which we have denoted by ${}^{**}\!\oplus$.

Recall at this point, that the (geometric objects) angles are equivalent to both (physical) speeds: relativistic and Galilean (proper) speeds.

6.2.2. Scalar Multiplications

A. Even more interesting facts than the, considered above, algebraic identity of the three additive groups of the speeds and angles are:

First, the possibility of introducing vector space structure in the considered groups A, E and G and second, the fact of isomorphism of the three obtained vector spaces.

The, here introduced, multiplication of relativistic speeds by scalars from the field R of real numbers is based on multiplication by nonnegative integers as defined in [17].

Here, we also sketch some easy extensions of that multiplication to multiplication by arbitrary reals from R.

First, however, realize that any reasonably defined multiplication of, say $u \in E$ by a positive integer scalar, denoted by $\otimes$, must be compatible with the Einstein addition $\oplus$ in the sense that, for example,

$$2 \otimes u = u \oplus u. \tag{21}$$

Then, inductively, one can define,

$$n \otimes u \text{ as } \left[(n-1) \otimes u\right] \oplus u \tag{22}$$

provided one already has defined the product $(n-1) \otimes u$.

The procedure of computing in this way consecutive products $n \otimes u$ is very cumbersome and finally impossible, in spite of it being well defined.

This, fortunately becomes easy if we apply the following very useful representation of any relativistic speed u as the following algebraic quotient (see, [17]):

$$u = \frac{\frac{1+u}{1-u} - 1}{\frac{1+u}{1-u} + 1} \tag{23}$$

and

$$u \oplus v = \frac{\left(\frac{1+u}{1-u}\right)\left(\frac{1+v}{1-v}\right) - 1}{\left(\frac{1+u}{1-u}\right)\left(\frac{1+v}{1-v}\right) + 1} \tag{24}$$

Some relatively cumbersome computations will show the right-hand side of (24) equals to

$$\frac{u+v}{1+uv}.$$

Now, as it follows from (24) and the inductive definition above (conditions (21) and (22)) the product of u by any positive integer $k = 0,1,2,\cdots$ is simply given by:

$$k \otimes u = \frac{\left(\frac{1+u}{1-u}\right)^k - 1}{\left(\frac{1+u}{1-u}\right)^k + 1} \tag{25}$$

(see [17]). From (24) immediately follows that:

$$0 \otimes u = 0$$

and

$$(-1) \otimes u = (-u),$$

hence for any negative integer, say $-k$ with $k > 0$, we have

$$(-k) \otimes u = (-1) \otimes [k \otimes u].$$

From the last three equalities one easily obtains, just by setting the values 0 or (−1) in place of k in (25). Thus, multiplication of speeds u by any integer number is defined. Extend coefficient k in (25) to arbitrary rational and then to arbitrary real numbers.

First, define

$$\frac{1}{m} \oplus u = v \text{ if and only if } m \otimes v = u$$

for any integer $m \neq 0$. Then if, in (25), $k = \frac{n}{m}$

$$\frac{n}{m} \otimes u = n \otimes \left[\frac{1}{m} \oplus u\right] \tag{26}$$

The rule that

$$r \otimes (k \otimes u) = (rk) \otimes u$$

used in (26) can immediately be verified by applying the representation (25).

Hence, (26) allows to multiply any speed $u \in E$ by an arbitrary rational number q.

As for the reals, according to the standard extension, any real number $r \in \mathbb{R}$, one can represent as the limit:

$$r = \lim_{n \to \infty} q_n$$

where the sequence $\{q_n\}$ of rational numbers can be, for example, the sequence of consecutive decimal approximations of r.

Now, continuity of algebraic expression (25) (when k is considered real) with respect to k allows to define, for any real r:

$$r \otimes u = \lim_{n \to \infty} (q_n \otimes u) \text{ as } q_n \to r. \tag{27}$$

In the rest of this considerations, we will assume that k in (25) is any finite real scalar.

The immediate consequence of the above is, that

$$k \otimes u \to 1 \text{ as } k \to \infty,$$

and

$$k \otimes u \to -1 \text{ as } k \to -\infty,$$

where 1 is the (real) speed of light.

To verify this, it is enough to take the limits from right-hand side of (25).

Also, it can easily be verified that the group $(E, \oplus)$ of relativistic speeds, together with the, defined above, multiplication by scalars in $\mathbb{R}$, denoted by $\otimes$, is a vector space.

Verification of vector spaces axioms based on the representation (25) is straightforward so here we only check distributivity of this scalar multiplication for the Einstein addition.

So we will show that:

$$r \otimes (u \oplus v) = (r \otimes u) \oplus (r \otimes v). \tag{28}$$

According to formula (24) we have:

$$r \otimes (u \oplus v) = \frac{\left[\left(\frac{1+u}{1-u}\right)\left(\frac{1+v}{1-v}\right)\right]^r - 1}{\left[\left(\frac{1+u}{1-u}\right)\left(\frac{1+v}{1-v}\right)\right]^r + 1} = \frac{\left(\frac{1+u}{1-u}\right)^r \left(\frac{1+v}{1-v}\right)^r - 1}{\left(\frac{1+u}{1-u}\right)^r \left(\frac{1+v}{1-v}\right)^r + 1}$$

Realize that, according to (24), the third extreme right hand side of foregoing equality is identical to:

$$\left(r\otimes u\right)\oplus\left(r\otimes v\right)$$

so (28) holds.

Also other axioms of vector spaces theory can readily be verified. Hence, the triple $\left(E,\oplus,\otimes\right)$ is the vector space over the field $\mathbb{R}$ of finite reals. E is not an algebra since any product of any kind of speeds is not a speed.

B. Definitions of scalar multiplication of any $k\in\mathbb{R}$ by elements of G and A as generated by the scalar multiplication in E, as defined by (25), is rather straightforward.

Definition 1 *If* $k\in\mathbb{R}$*, and* $\theta\in A$*, then for their product we adopt:*

$$k\ {}^{**}\!\oplus\theta=\arcsin\left(k\otimes u\right), \tag{29}$$

where $u=\sin\theta$, and $k\otimes u$ is defined by (25).

Definition 2 *Let* $k\in\mathbb{R}$ *and* $U\in G$ *(so that* $U=u\sec\theta$*, where* $u=\sin\theta$ *). The product* $k\otimes^{*}U$ *is defined as:*

$$k\otimes^{*}U=k\otimes^{*}\tan\theta=\tan\left[\arcsin\left(k\otimes u\right)\right] \tag{30}$$

Remark 4 Realize that, both the definitions of multiplying k by elements of E and G rely, in a sense, on either taking sine or tangent of an angle: $\theta'=k\ {}^{**}\!\oplus\theta$. so that three possible theories of the models, say A, E and G can, likely, be reduced to one that investigates the vector space $\left(A,{}^{**}\oplus,{}^{**}\otimes\right)$ *i.e.*, to a "geometric" theory.

This methodology is, however, of rather theoretical value as indicates the relationship between pure "geometry" and the two mechanics. In practice, when computations matter, we rather consider the Lorentz-Einstein addition and related scalar multiplication in E as generating two remaining algebraic structures of G and A.

It could be said that one may choose among the two possibilities as for the of basis (basic language) for the three theories to unify them. The "geometric" approach that choses the theory of the model $\left(A,{}^{**}\oplus,{}^{**}\otimes\right)$ as a primary theory is probably the way the Nature "acts" (possibly, through a Creation Process?) but it is not a convenient method from any human epistemological as well as computational viewpoint.

The way of getting knowledge (not that of creating a reality) rather requires considering as a basic model the triple $\left(E,\oplus,\otimes\right)$ since all the operations in the remaining models are induced by the operations $\oplus$ and $\otimes$. Creating a model $\left(A,{}^{**}\oplus,{}^{**}\otimes\right)$ without knowing any model from the remaining two seems to be extremely difficult if not impossible for humans (?). □

B1. It can easily be verified that multiplication by scalars in the sets A and in G, as defined by (29) and (30) respectively, together with the corresponding additions, impose on A and on G algebraic structures of vector spaces. For example, let us check the distributivity of scalar multiplications over the corresponding additions, for both triples $\left(A,{}^{**}\oplus,{}^{**}\otimes\right)$ and $\left(G,\oplus^{*},\otimes^{*}\right)$.

Both rules are based on the rule (28) as (primarily) satisfied in $(E,\oplus,\otimes)$.

So, now let us substitute $r=k$, $u=\sin\alpha$, and $v=\sin\beta$ (so, $c=1$) and let k be any real number. As $\alpha,\beta\in A$ we proceed in A as follows:

$$\begin{aligned} k\,{}^{**}\!\otimes\left(\alpha\,{}^{**}\!\oplus\beta\right) &= \arcsin\left[k\oplus\left(\sin\alpha\oplus\sin\beta\right)\right] \\ &= \arcsin\left[\left(k\otimes\sin\alpha\right)\oplus\left(k\otimes\sin\beta\right)\right] \\ &= \arcsin\left(k\otimes\sin\alpha\right){}^{**}\!\oplus\arcsin\left(k\otimes\sin\beta\right) \\ &= \left(k\,{}^{**}\!\otimes\alpha\right){}^{**}\!\oplus\left(k\,{}^{**}\!\otimes\beta\right). \end{aligned} \tag{31}$$

where the last equality is based on definition (29). Concluding, we have shown that in A:

$$k\,{}^{**}\!\otimes\left(\alpha\,{}^{**}\!\oplus\beta\right)=\left(k\,{}^{**}\!\otimes\alpha\right){}^{**}\!\oplus\left(k\,{}^{**}\!\otimes\beta\right). \tag{32}$$

B2. As for the distributivity rule to be satisfied in $\left(G,\oplus^*,\otimes^*\right)$ we proceed similarly as latter: Set for $U,V\in G$: $U=\tan\alpha$, $V=\tan\beta$, ($c=1$). Then, to show that

$$k\otimes^*\left(\tan\alpha\oplus^*\tan\beta\right)=\left(k\otimes^*\tan\alpha\right)\oplus^*\left(k\otimes^*\tan\beta\right) \tag{33}$$

we proceed as follows:

$$\begin{aligned} & k\otimes^*\left(\tan\alpha\oplus^*\tan\beta\right) \\ &= \tan\arcsin\left(k\otimes\left(\sin\alpha\oplus\sin\beta\right)\right) \\ &= \tan\arcsin\left(\left(k\otimes\sin\alpha\right)\oplus\left(k\otimes\sin\beta\right)\right) \\ &= \tan\left[\arcsin\left(k\otimes\sin\alpha\right){}^{**}\!\oplus\arcsin\left(k\oplus\sin\beta\right)\right] \\ &= \tan\left[\left(k\,{}^{**}\!\oplus\alpha\right)\oplus^*\left(k\,{}^{**}\!\oplus\beta\right)\right] \\ &= \left(k\otimes^*\tan\alpha\right)\oplus^*\left(k\otimes^*\tan\beta\right), \end{aligned}$$

what was to be shown.

In the latter sequence of equalities we were using in turn rules (30) and (13). Then, for the next equality, (25) and (28). Then (31). Then the first two equalities in (31). And finally we applied (32).

6.3. The Algebraic Structures Equivalence

In light of the above this is clear that the three vector spaces we encountered are isomorphic with the following isomorphisms:

$$\arcsin(\cdot):E\to A,\ \text{ for } c=1$$

and

$$\tan\arcsin(\cdot):E\to G.$$

Besides the identical algebraic structure, the sets A, E, G bear the same topological structure as the three open intervals $(-\pi/2,\pi/2)$, $(-1,1)$ and $(-\infty,\infty)$ are homeomorphic [and also diffeomorphic] and that the defined above algebraic operations are continuous (as defined by the compositions of real continuous functions) with respect to the natural topologies on the intervals.

That means, the three sets A, E and G have all imposed identical structure as topological (metric) linear spaces and therefore it can be said that, to a degree, the three corresponding theories reduce to the same one that is, for example, to SR theory of the model, say $\left(\mathbb{C}^3, \mathfrak{P}, A^3\right)$. Here, $\mathfrak{P}$ denotes the Poincare group of all isometries of $\mathbb{C}^3$, and A^3 is the cartesian product $A \times A \times A$ of the above defined vector spaces $\left(A, {}^{**}\oplus, {}^{**}\otimes\right)$ with the coordinate-wise defined algebraic operations ${}^{**}\oplus$, ${}^{**}\otimes$ (as for the scalar multiplication it is used that all the three coordinates of A^3 are ${}^{**}\oplus$-multiplied by the same real number). The elements of A^3 are the "three-arguments" of the underlying complex vectors that belong to $\mathbb{C}^3$.

Recall at this point, that the endpoints of the closures of the considered intervals represent speeds of light which, geometrically, correspond to the orthogonality ($\pi/2$ angle).

These endpoints (including $\pm\infty$ [the Galilean speed of light]) are very natural as parts of the three corresponding topological (and not always metric) spaces but, when one tries to endow them with the algebraic structures of vector spaces, difficulties with continuity may be met, so that, for simplicity I limited the considerations to the open (finite and one infinite) intervals as to the topological (metric) vector spaces.

As already mentioned, the unidimensional spaces can be extended to the full three-dimensional, as their third cartesian power, and together with the whole considered $\mathbb{C}^3$ para-space [plus the Euclidean metrics on it and the "Poincaré group" of all the $\mathbb{C}^3 \to \mathbb{C}^3$ isometries that generate the geometry of the model] may be considered as equivalent models for both SR and a theory that might be named "semi-Newtonian theory" (SN) at least when reduced to kinematics only. That theory differs from, say "legitimate Newtonian" (possibly only) by the rule of adding (high) speeds which do not rely on the usual arithmetic addition of velocities' coordinates.

7. On the "Newtonization" of Relativistic Physical Quantities

The procedure mentioned in title of this section has, to a measure, its counterpart in standard SR known as the theory of four-vectors invariant within the hyperbolic geometry, see [18] and [19]. However, in SR, the four-vector's concept, has rather no direct reference to the Newtonian concepts which, in turn, follow the Euclidean rules.

In the language of this and other my papers on the subject, the procedure of Newtonization relies, between others, on replacing (mainly for conceptual purposes) relativistic velocities, say $u, v, \cdots$ and other relativistic quantities, by their corresponding Galilean counterparts, say $U, V, \cdots$, which, when reduced to one-dimensional cases, relies on transferring the motion along the real x-axis to the (radial) r-axis that form corresponding angles, say $\alpha, \beta, \cdots$ with the x-axis (recall that, in this case, $u = c\sin\alpha, v = c\sin\beta, \cdots$ and the Newtonization, when speeds are considered on the [different] r-axes, relies on replacing the speeds relativistic magnitudes $c\sin\alpha, c\sin\beta,\cdots$ by the Galilean magnitudes $c\tan\alpha, c\tan\beta,\cdots$). As mentioned before, in SR numerically the same action has, typically, been taken but within different (four-vectors), possibly equivalent (numerically, but not conceptually), framework that we do not apply in the $\mathbb{C}^3$ theory.

The main motivation for the classical SR construction is changing the coordinate system, say from *x*-axis (when the considered case is unidimensional) to another, also real, *x′*-axis, which moves with respect to the original one with speed u. We, instead, change *x*-axis for the corresponding (to u]) *r*-axis lying in the interior of the complex plane.

In classical SR, this change is forced by the fact of shortening time t to the "proper time", say τ, related to the "previous time t" as $t = (\sec\alpha)\tau$. This forces one to differentiate various quantities over $\mathrm{d}\tau$ rather than over $\mathrm{d}t$.

Anyway, both approaches, the "classic" SR and our approach, yield the same increment of physical quantities by

$$\gamma(u) = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = \cosh\lambda = \sec\alpha,$$

(see [19]). (The second, equally important motivation of SR theory in M^4 is the necessity of working with the invariant four-vectors (here, four-velocities) *i.e.*, as the magnitudes of all the four-vectors are invariant while changing coordinate systems by applying the Lorentz boosts.).

It looks like both the theories work with the same physical quantities but their understanding and the motivations for turning to the four-velocities or to the (semi)-Galilean speeds are different.

In both, the key to interpreting increased or decreased (comparing to results of direct physical measurements) quantities, (mostly by the Lorentz factor $\gamma(u) = \sec\alpha$, whenever $u = c\sin\alpha$) is *invariance* of the norms (hyperbolic or Euclidean correspondingly) of some of them when one coordinates system is replaced by other.

In this work the basic quantity that is invariant with respect to the motion is Euclidean **length** to which in SR corresponds, not as natural, space-time interval with its invariant hyperbolic "length". As usually the Euclidean length is invariant upon the (circular and not hyperbolic) rotation (by the, related to speed $u = c\sin\alpha$, angle α).

The length preservation under circular rotation in (complex) Euclidean space appeared to me the very natural background to the Newtonian version of mechanics especially that, parallelly, the speeds along the rotated axis ("proper speeds" in SR) seem also be natural (Galilean).

"The same" length, when only considered by SR on the real line, is shortened according to the Lorentz contraction, so here the two theories (or rather the two *different models of the same theory*) "disagree".

At this point, within standard SR, the connection with Newtonian mechanics as based on Euclidean ideas had, probably, been lost. This is, probably the reason the Newtonian concepts or Newtonian (Galilean) interpretations of the quantities such as, for example, "proper velocity" have been overlooked. As it was often articulated in my works, the physical results of both the considered theories (SR and ours) are basically *the same.* The differences rely on different models applied and consequently (often) on different interpretations of the notions.

Why we consider the "proper velocities" as Galilean? Let me compare the two approaches that yield these quantities.

In SR one of the *primary* realizations was shortening the time in moving coordinate system (as observed from a stationary coordinate system) which made it necessary for defining velocities as the invariant four-vectors. This (upon differentiation of invariant (at rest) distances over "shorter" (proper) time) resulted in higher velocities whose Newtonian sense was, probably, overlooked.

Our claim then is that speeds and times along various radial lines (that in SR correspond to the moving real coordinate systems with various speeds but all in the same direction) are natural and, possibly, Newtonian quantities.

Unlike these quantities, some SR quantities, such as for example lengths and speeds that are measured along the (one) real line are "artificially" contracted while other, such as time, are, in turn, "artificially" extended.

The latter facts are the subject of SR (in M^4) as this theory deals with real spaces (space-times) only. The requirement of considering this part of physics in *real* domains implies the necessity of dealing with hyperbolic geometry with a little artificial (non-Euclidean) metrics, which is not a "metric" in the usual topological sense as met in mathematics. My impression is that whenever the SR theory is considered as theory of M^4 model, with the hyperbolic rotations instead of the circular, talking about some quantities like for example proper speeds being four-velocities as at least closely related to their Newtonian counterparts appears be inappropriate and, consequently, their Newtonian character is lost.

On the other hand, in the complex $\mathbb{C}^3$ model one meets an overwhelming impression that the, here mentioned, quantities are Newtonian (as [signed] absolute values of the corresponding complex quantities) while their relativistic counterparts are simply their real parts so that relativistic mechanics is kind of "distorted" classical.

This is, however, not as simple. For example, as it was elaborated in previous section, the "Newtonian" speeds or velocities whenever composed do not arithmetically add up.

Moreover, they do "add up" but the addition is generated by the Lorentz-Einstein addition as defined by (13). Now, it looks like the SR generates (at least partially) the classical theory.

Resuming, at least as for the velocities, Newtonian (Galilean) quantities are subjected to some "relativistic" structure, or conversely.

What then to say about the theory of these complex quantities (actually, their [signed] absolute values only) in $\mathbb{C}^3$.

This question provokes another question.

Why to assume that arithmetic addition of the Galilean velocities is the necessary condition for preservation true classical character of the theory of Newtonian quantities?

Of course, this assumption, probably, had to be seen by 18-th and 19-th centuries' scientists as the only possibility, and maybe they were wrong. Maybe for the really legitimate classical theory the proper addition of the velocities is not the arithmetic? Maybe its relation to SR is more complex than it prima facie seemed to be.

Taking these questions into account we have proposed above to call the above theory of all the mentioned Newtonian "items" in the interior of the $\mathbb{C}^3$, (whose structure is, however, not necessarily *Newtonian* [or, possibly, not the "naïve-Newtonian"???]) the "semi-Newtonian theory". And, perhaps, this semi-Newtonian theory is the actual Newtonian (or rather just "classical"), possibly overlooked so far?

(Realize too, that for regular, not very, high velocities, the arithmetic and the semi-Newtonian addition (13) almost does not differ so that, practically in the past history, the difference was indiscernible.)

Now, one possibly can say, that SR is the "projection of the semi-Newtonian theory" (as valid in $\mathbb{C}^3$) into the real subspace $\mathbb{R}^3$. Consequently, the considered quantities could be called (semi) Galilean or (semi) Newtonian but, perhaps, they still should be understood as a legitimate *classical?*

As for the possible reason, why the Galilean speeds (when collinear one-dimensional motion in real subspace is considered) do not add arithmetically when composed, realize that, even if on the real x-axis all the relativistic speeds have the same direction, all the corresponding (not equal to each other) complex speeds in the corresponding complex plane have the directions different!

The latter fact was not realized in 18 and 19 centuries and the assumption of arithmetic additivity of velocities, as the only reasonable possibility, was then very natural.

Thus, after deeper analysis, the two theories SR and the semi-Newtonian seem to be mutually dependent on each other.

On the other hand, one also may consider speed addition in $\mathbb{C}^3$ determined in the semi-Newtonian theory as regular but simply overlooked fact.

And suppose that such the nonarithmetic addition say, $\oplus^*$, of, say here, "semi-Galilean" speeds U, V was the primarily known addition. Then from (13) one can "determine back" the Lorentz-Einstein addition $\oplus$ by:

$$u \oplus v = \sin\left(\arctan\left(U \oplus^* V\right)\right), \tag{13*}$$

where $c = 1$, which is given by the inverse transformation to the one applied in (13). In other words, the two additions can be considered equivalent.

8. Further Development of the Newtonization Procedure

So far we considered the "Newtonization" in the frameworks of kinematics (as in

SR velocities are constant in time), both relativistic and semi-Newtonian ("classical"). Within this set of problems one may conclude that the two theories ("classical" and relativistic), if considered *as theories* of the $\mathbb{C}^3$ model, are *equivalent.* This equivalence is supported by isomorphism and homeomorphism (with continuity of the underlying algebraic operations) of the linear-topological spaces of relativistic and Galilean velocities considered in sections 4, 5 and 6.

Some questions, however, may arise on how to consider the whole SR [both versions] as the theory of $\mathbb{C}^3$ instead of M^4.

For example, the length or norm in $\mathbb{C}^3$ is invariant while in SR modeled by M^4 we rather talk about the invariance of space-time intervals. The latter belongs to M^4 with the hyperbolic norm while the "regular" length in M^4 is contracted. Besides, M^4 is not immersible in the $\mathbb{C}^3$. A way out of this difficulty is to transport the classic SR quantities to the $\mathbb{C}^3$ in a proper manner. For example the space-time interval from M^4 is "replaced" by the complex quantity, where instead of its time component in M^4 the imaginary part of such "complex interval" (the invariant Euclidean "length" of it is its absolute value) will be present in the new ($\mathbb{C}^3$) model's setting.

Also, some real SR quantities like proper speeds (or velocities) the "new SR" should consider as the absolute values of underlying complex values representing underlying physical quantities. Other, such as "observed lengths", are the real parts of the given complex. With this approach, the real (observable) part of a complex length is subjected to the Lorentz contraction while its absolute value is invariant.

Here realize that the classical SR theory is preserved. On one hand the Lorentz contraction is described in the very natural way and on the other the invariance of the "space-time interval" under the hyperbolic rotation is replaced by the invariance of regular length under usual circular rotation.

Consequently, we also resigned from the four-vector formalism as irreverent in the new complex settings.

As one can see, at this point, we could omit the description of the mechanics by the Minkowski model with no loss of the corresponding physical content.

What happens at this point, mathematically, is that we replaced the usual Minkowski's plane, say M^2 with its vertical time coordinate, by $\mathbb{C}^1$ plane with, instead, vertical imaginary (length) coordinate. Under these conventions, all the underlying physical phenomena, as considered by SR, are described within the $\mathbb{C}^3$, possibly even more clearly. Recall yet, that within the theory of the here considered $\mathbb{C}^1$ plane, time can be defined as the product of geometry (angles of the rotations) of the complex plane and thus one, if she likes, can return to the M^2 model once imposing back the hyperbolic norm. We, may talk about hyperbolic and circular (Euclidean) versions of the same SR considered as two distinct descriptions of the same physical reality.

Now, the above assertion on *equivalence* of the considered two kinematic theories (by Newton and by Einstein) seems to be more convincing.

Now, let me shortly delineate an idea of the two considered theories' (*i.e.*, SR as the theory of the Minkowski real model M^4, and as the theory of the complex para-space $\mathbb{C}^3$) equivalence, (also in their dynamic parts *i.e.*, when accelerations and forces are of concern) with the classic Newtonian.

One can realize the equivalence at least in the sense of systematic obtaining the same numerical results for the same specific physical problems. But possibly, it is not only that.

Look then at some formulas that will be obtained, to compare them with the formulas known in classic (extended) SR. Those new are now given below without proofs.

First, consider the Einstein's version [18] of Newton's motion equations in terms of the four-vectors:

$$f_\mu = \frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}, \tag{34}$$

where f_μ is the four-force and p_μ is the four-momentum given by:

$$p_\mu = m_o \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}, \tag{35}$$

where $x_\mu \in M^4$ and τ is the proper time. From above two formulas it follows:

$$f_\mu = m_o \frac{\mathrm{d}^2 x_\mu}{\mathrm{d}\tau^2}, \tag{36}$$

Formulas (34) and (36) represent the Newtonian equations Einstein formulated within the four-vector formalism.

Compare them with "the same" Newtonian equations as formulated in the complex C^1 model's formalism since we consider the one-dimensional motion, originally along the real x-axis.

Recall again, the "Newtonian motion" in the complex model is considered along the complex line OB' *i.e.*, along the $x_{c,\theta}$ axis, see **Figure 1**. To set up the "Newtonian problem" (an initial value problem for the Newtonian differential equation) in the real framework, we consider instead of the complex line $x_{c,\theta}$ the real line $r_\theta = x_{c,\theta}$ along which the unobserved Newtonian motion is to be considered. Here, by $x_{c,\theta}$ we mean the (signed) absolute values of the complex numbers $x_{c,\theta}$.

Notice also, that as we consider the "dynamic version of SR", which is a bit beyond the regular "kinematic SR", the speeds and velocities before considered constant, now became non-constant functions of times (either the proper time τ, ("along" the radial time-axis (see Figure 3 in [2]) or the time t' "along" the horizontal time-axis) and this is essential over which time (τ or t') the differentiation is performed.

Compare the following two, say, "semi-Newtonian" equations, in the complex model's formalism, with the known formulas (34) and (36).

$$F(\tau) = m_o \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}, \tag{37}$$

Realize, that in (37) the product $P_r(\tau) = m_o U(\tau)$, where the rest mass m_o is a constant, is the momentum of the considered body at time epoch τ in the radial direction so that P_r is exactly the same as the spatial part of the four-vector p_μ present in (34) and (35).

Moreover, the following formula:

$$F(\tau) = m_o \frac{\mathrm{d}^2 r_\theta(\tau)}{\mathrm{d}\tau^2}, \tag{38}$$

numerically, is the same as the spatial part of (36) for the motion along the real x-axis.

Also, $F(\tau)$ is the same as the spatial part of the four-force f_μ.

So, basically, the operations on right hand sides of the known in (extended) SR formulas (34) and (36) are equivalent to the (slightly different) operations on right-hand sides of (37) and (38), respectively. According to my best knowledge, those last two differential equations, are given, *in this context,* possibly, the first time.

Also, the numerical results *i.e.*, the solutions (for the same, in the spatial parts of (34), (36), initial value problems) of the underlying differential Equations (34), (36) yield, finally, the same as Equations (37), (38) solutions to the same physical problems.

At this point yet notice that the solutions, say $U(\tau)$ and $r_\theta(\tau)$ of an initial value problem for Equations (37) and (38) are solutions for the (semi)Newtonian motion which is unobserved as taking place within the complex plane outside the horizontal real line. The same can be said about the solution of the Equation (34), say $u_\mu(\tau) = p_\mu(\tau)/m_o$ in its spatial part which is exactly equal to $U(\tau)$.

Nevertheless, the spatial part of the solution $x_\mu(\tau)$ one has to obtain from the solution $r_\theta(\tau)$ of the same initial value problem for equation (38) by multiplying the latter by $\cos\theta$.

Geometrically, the final solution $x(\tau) = r_\theta(\tau)\cos\theta$ is the orthogonal projection of the Galilean position, of the considered moving physical body, to the horizontal real axis. This value $x(\tau)$ is exactly equal to the spatial part of the four-vector $x_\mu(\tau)$ that, in turn, belongs to M^4. Thus, both the dynamics' theories yield the same numerical solutions.

Resuming, the "Newtonization" of SR considered as the theory of the $\mathbb{C}^3$ model, or adopting the "four-vector formalism" for SR in M^4, turn out to yield equivalent theories of the same physical phenomena also for the dynamic version of SR. The existing differences are only caused by the different mathematical models applied. The similarities relay on the fact that in the two different frameworks the same (para)physical quantities *i.e.*, the "Newtonian versions" (in the spatial parts of underlying four-vectors) of relativistic velocities, momenta, accelerations, and forces are under consideration.

At end, as an example, notify the simple relationship between the (semi-)Newtonian and relativistic (observable) forces:

$$F(\tau) = f(\tau)\sec\theta,$$

where $f(\tau)$ is the real (observable) force acting along the horizontal direction (in one complex plane model) and $F(\tau)$ is its Newtonization.

An easy justification of this fact is left as an exercise.

Also, recall once more that the quantity $\sec\theta$ numerically equals the Lorentz factor.

Resuming subsection 9.3 the following conclusion can be stated.

Comparing the known (classic) relativistic versions of the Newton equations (34) or (36) with that of Newton equations (37) or (38) following the complex model, one can realize that the latter, actually, are identical to the same equations as those in the original Newtonian framework.

Unlike that, "the same" well known equations expressed in formulas (34) and (36) seem to be somewhat different from their Newtonian origin. This follows from the fact that the four-vector formalism is a bit foreign to the classical Newtonian settings. Thus, when talking about the equivalence of SR and the Newtonian mechanics it is rather better to consider the complex Euclidean version of (the same) SR.

9. Para-Space as a Possible Model for Quantum Mechanics

9.1. Preliminary Considerations

A third physical theory that should also be mentioned in the complex framework is quantum mechanics (QM), [20].

This theory does not seem to be equivalent to the two considered above but, nevertheless, it seems to be possible that QM can be modeled by the same complex model $\mathbb{C}^3$. If this hypothetical fact would really take place, with the approach described below, the theory, whose current form looks fuzzy and for many unclear, if not "weird", may achieve dramatically simpler and more understandable form with preservation of all the underlying physical facts. The simplicity, as well as a significant extension, of SR can possibly be achieved by transition from real to complex models. Then one might expect a similar simplification and clarification of quantum *mechanics* if we extend the real framework of QM, which already contains some complex descriptions, to the complex one.

Such transition from the real to complex formalism for QM, may (as a first approach) parallel the transition from both the standard Hilbert space model $L^2(\mathbb{R}^3)$ to the Hilbert $L^2(\mathbb{C}^3)$ and, even more importantly, from the underlying Hermitian operators with their real eigenvalues to the normal by composing the Hermitians with unitary operators [21]. This procedure resembles formula (3) in this paper where real quantities were rotated (like now by the use of unitary operators) toward the complex plane opening the way to the complex domain. The so obtained normal operators as models for the quantum measurements have complex eigenvalues which, in turn, may represent some measured physical quantities such as complex positions in $\mathbb{C}^3$ or momenta of particles.

But then, say after applying a proper normal operator to the wave function, we may obtain as a result a complex eigenvalue which corresponds to a particle position of the form, say $r\exp(i\theta)$.

This quantity will vary from one measurement to another so we may expect the reason is that it depends on (a hidden) time t or, in other words, we should instead consider that eigenvalue of the form $r\exp(i\omega t)$ (so $\theta = \omega t$), and in this form set it into the exponent of the wave function.

At this stage, the possibility arises as to dismiss an existing model in the form of the class of wave functions just to their complex exponents that depend on time and to consider as the primary modeling mathematical entities, functions that look like $r\exp(i\omega t)$.

Thus, what now seems to be possible and quite useful is to resign at all from the functional analysis, (see [21] or [22]), apparatus with its Hilbert spaces and the operators on them toward much simpler description by the ordinary complex plane of the complex values to which the numbers (the former eigenvalues) $r\exp(i\omega t)$ belong.

The possibility that now arises is a direct physical interpretation of the mathematical value $r\exp(i\omega t)$, as, for example, swirling one-dimensional "stick". Such a "stick" we, initially, will identify with a single free particle like an electron for example. Realize yet, at this point, that the corresponding to that particle mathematical entity $r\exp(i\omega t)$ does not, as a matter of fact, much differ from the corresponding former wave function and the actual difference more rely on the *physical interpretation*.

9.2. Particles and Quarks as Swirling "Sticks"

In association with the mathematical entities, that is with the complex time' functions $r\exp(i\omega t)$ we will try to adopt the hypothesis that (some at least) elementary free particles can be imagined as the, above mentioned, unidimensional (para)physical entities such as fast swirling "sticks" within a complex plane. Obviously, they cannot be observed in such a ("particle's") form, because all we can see (here, in the sense of "seeing" by physical instruments) is the real line (axis) across the plane of the rotation. The obtained by measuring instruments experimental data only corresponds to the projection ("shade") of the vibrating stick into the, say, real axis. This projection is described by the real part of the "stick" *i.e.*, by

$$\operatorname{Re}\left(r\exp(i\omega t)\right) = r\cos(\omega t)$$

which, by the observer, is correctly understood as the wave.

Actually, one observes not the whole waving shadow of the stick but only a single (in one observation) trace of it, fixed at the very moment of the observation (at a particular time epoch t). Since the time function ωt is rather not known (neither ω nor t) and the choice of the moment of the measurement is unplanned, and so random, one obtains as a result of making many such "photos" of the dynamic stick's shadow, relatively chaotic set of the observations which suggests the *"random nature"* of the "particle" or its behavior.

This conclusion is natural in the situation when only the real reality is under the consideration. My supposition is that under better understanding of the nature of the data one might, at least theoretically, foresee [deterministically] the "state" of the particle (wave) if the time of the "making photo" would be carefully enough chosen? Even if, practically, such a high precision would not be possible, the conclusion on deterministic nature of such particles would be near.

To mention one difficulty that might occur on the way to make the above statement and the underlying measurement precision achieved, one, additionally, will need to know at the given time moment of the observation, the (possibly, unknown) value of the parameter ω which also may vary depending on the level of energy of, say, the particle. But even so, the way to make the conclusion on deterministic (even if unknown) nature of some particles would be open.

The value of such a statement (if enough proven) on deterministic character of the theory and the underlying reality would be at least epistemological. As for the possibilities of improving this way the underlying calculations, this is not clear at this stage.

Another issue that stands out is the question on the general nature of our "swirling stick". What we partially observe has the properties of the wave and on the other hand the whole "stick" should rather be identified as a "particle".

This "conflict" between what is observable and what is not may, hypothetically, be the source of the assertion on the dualism between wave and particle natures of the same object. The hypothesis of the stick swirling within the complex [paraphysical] plane seems to be promising but that is only the beginning. There, probably, is more questions than the answers. The first question is that, as it is known, particles in general are composed and are not as simple as just one swirling stick.

This inclines one to state the hypothesis that the swirling sticks are not elementary particles but the quarks from which they are composed. Now, suppose that a given particle is composed from three interconnected swirling sticks (quarks) each rotating on a separate complex plane such that no two planes of the rotation are parallel (This may effect with an impression that such a particle has a ball-like "shape" rather than a unidimensional piece of line.). If the sticks are rotating around their medium points the resulted particle is a fermion but if around their endpoints it is a boson.

The question how to explain variety of properties of different particles can, probably, be answered as follows. Various types of the particles differ by two main types of features. First, each type of particles differs from other types by, possibly, different but fixed angles between the planes of rotations of their quarks. Second characteristic may be explained as follows: Suppose the three quarks of the given particle are mathematically described by 3 complex functions of the time t, $r_k \exp(i\omega_k t)$ for $k = 1, 2, 3$. The individual features of such a particle may, additionally, rely on proportion of sizes of, say, "energy levels" ω_1, ω_2 and ω_3.

Now, one can see that the proportions of the angles between the planes of rotations and between the values ω_1, ω_2 and ω_3, as well as relations between these proportions characterizing a given particle may, possibly, explain all the variety of the so many distinct species of the particles.

As an example, let us consider the comparison of electrons with the muons. As this is well known, they have the same properties with the only exception for mass. In the framework here presented, the phenomenon of such a far going similarity, with actually the one exception only, one might try to explain as follows. The two kinds of the particles are both characterized by the same angles between planes of their quarks rotation, the same lengths of the underlying quarks' as the lengths of the corresponding "sticks", the same proportion between the numbers ω_1, ω_2 and ω_3, but the magnitudes of these numbers are significantly higher for the muons (with, of course, preservation of their proportions). These big (possibly, "too big to survive") magnitude of ω_1, ω_2 and ω_3, may be responsible for the short lifespan of the muons comparing it with long lifespan of electrons.

Now, as for the values ω_1, ω_2 and ω_3, characterizing any given particle they obviously are responsible for levels of energy of the particles or only of specific quarks. Suppose we consider a single quark. Its parameter, say ω_k, determines a frequency of the corresponding wave which, in turn, is proportional to the energy contribution by that quark. As this is well known, the energy is proportional to the mass (here, the mass at rest). As for the mass phenomenon let me try to explain it by the analogy with gyroscope and the principle of the "angular momentum preservation". Namely, if a force acts on (macroscopic too) a physical body to which a given particle belongs it usually acts (at a given time moment) in some *direction* which, usually, disagree with the direction of a given by a rotating stick's angular momentum (of a quark in one of billions contributing to the given body particles). That angular momentum exerts resistance against the "attempt" of the force to change its direction which is well known **gyroscope effect**. That resistance one experiences as the body's inertia (mass). Fact that such resistance does not depend on the direction the force acts relies on statistical effect as billions of the particles takes part in the phenomenon. The situation may be different if one or small number of the particles (and their quarks) takes part in an experiment. This difference, of the statistical nature, may be the reason of the differences between classical and quantum phenomena.

Notice too that the particle's (or rather quark's) angular momentum need not necessarily be identified with its spin.

The phenomena above described are undetectable in the real space or space-time such as M^4. If above mass phenomenon's exposition is right [that should yet be discussed] then the strong argument for usefulness of the complex spaces [which are not only as the mathematical but also (para)physical reality] would be obtained.

Other, related to this elementary particle description's device of interest, is the imaginary part of the swirling stick which mathematically is identified with the expression:

$$\operatorname{Im}\left(r\exp\left(i\omega t\right)\right) = r\sin\left(\omega t\right)$$

that represents another wave.

The latter imaginary part is the kind of complementary to the observable real part. As the real part is $r\cos(\omega t)$ the imaginary, apparently, differs from time derivative of that real part of the complex value only by the constant $-\omega$. If the particle is free (no progressive motion), the real part may be considered as describing the particle's position. If so, then the imaginary part of the considered complex value may be understood as the "particle's speed" (To obtain the actual speed we, probably, only need to multiply the imaginary part $r\sin(\omega t)$ by the constant $-\omega$.).

All this suggests, that within the complex framework the **Heisenberg rule** need not to be valid, even if this rule is necessary when the real model is applied. Moreover, it seems that a measurement of the position, here automatically determines (the common angle ωt) the time dependent speed or the momentum which, again, differs from the speed by the multiplicative constant only.

Above complex framework for the quantum mechanics seems to be a promising one, although somewhat risky (too good to be true?). If that would work, the simplification of the QM theory would be very dramatic, and this also would suggest kind of a proof for the existence of para-physical complex reality and not only the well-known fact of usefulness of the underlying complex mathematics.

With the approach above presented, quantum mechanics doesn't look to be reducible to Newtonian mechanics of the classical particles in such a direct like SR way (just by taking the absolute values of underlying complex quantities). So, it is risky to say that QM is equivalent to the previously considered two theories the way they were equivalent to each other.

However, if to consider a particle as a system of three interconnected quarks that are subjected to the rotations, an idea may come to mind that instead of classical mechanics of the material point one may apply the classical rigid body's mechanics???

This possibility likewise others here presented must, of course, be discussed by other researchers to be adopted or rejected by the scientific community.

10. Conclusions

1) The introduced in the paper complex para-space $\mathbb{C}^3$ model for special relativity may serve as an alternative to the real Minkowski space-time model.

2) The introduced new model for special relativity, besides preserving all the facts and theorems of SR, also allows for some extension of that theory as well as for better clarification of many facts formerly often considered as the "paradoxes".

3) The SR theory, as the theory of the new complex model, sheds a new light on the nature of time. The notion of time turned out not to be the primitive notion anymore but "became" definable within the extended SR theory as the theory of the new complex model.

4) The complex model, originally thought of as the model for SR, turned out to have strong unifying properties bridging fundamental physical theories such as

SR, Newtonian mechanics and very likely also the quantum mechanics. At the moment, one does not exclude the possibility that other physical theories can also be modeled by the same complex para-space.

5) The introduced complex formalism unexpectedly yields a dramatic simplification of the underlying physical theories. The only price for that benefit is the transition from real to the complex formalism.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Schutz, B.F. (1985) A First Course in General Relativity. Cambridge University Press.

[2] Filus, J.K. (2024) Motion and Special Relativity in Complex Spaces. *Journal of Applied Mathematics and Physics*, **12**, 330-361. https://doi.org/10.4236/jamp.2024.121025

[3] Filus, J.K. (2023) Motion, Time and Special Relativity in Complex Spaces. *Academia Blog*.

[4] Filus, J.K. (2019) Newtonian Mechanics Extension (to Arbitrary Speeds) and Its Relation to Special Relativity in Complex Model (An Improved Version). ResearchGate Blog.

[5] Filus, J.K. (2019) A Complex “Time Free” Model for Both Classical and Relativistic Motion. A Shorter Slightly Modified Version of a 2015 Version at Research Gate.

[6] Filus, J.K. (2016) Complex “Para-Space” Model for Special Relativity Theory. Scholar’s Press.

[7] Kaiser, G. (2009) Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis.

[8] Kaiser, G. (2003) Physical Wavelets and Their Sources: Real Physics in Complex Spacetime. *Journal of Physics A*: *Mathematical and General*, **36**, R291-R338. https://doi.org/10.1088/0305-4470/36/30/201

[9] Rauscher, E.A. and Targ, R. (2006) Investigation of a Complex Space-Time Metric to Describe Precognition of the Future. *AIP Conference Proceedings*, **283**, 121-146.

[10] Rauscher, E. and Targ, R. (2006) Complex Space-Time Metrics. *Journal of Scientific Exploration*, **15**, 331-354.

[11] Urusovsky, I.A. (1996) Six Dimensional Treatment of *Relativistic Mechanics* and Spin, Metric Gravitation Theory and Expanding Universe. In: *Works of Contemporary Radioelectronics*, 3-22.

[12] Naber, G. (1992) The Geometry of Minkowski Spacetime. Springer.

[13] Borsuk, K. (1969) Multidimensional Analytic Geometry. Polish Scientific Publishers.

[14] Dattoli, G. and Del Franco, M. (2010) Hyperbolic and Circular Trigonometry and Application to Special Relativity. ResearchGate Blog.

[15] Majerník, V. (1986) Representation of Relativistic Quantities by Trigonometric Functions. *American Journal of Physics*, **54**, 536-538. https://doi.org/10.1119/1.14557

[16] Wikipedia (2024) The Free Encyclopedia. Proper Velocity.

[17] Talukder, M.O.G. and Ahmad, M. (2012) Relativistic Rule of Multiplication of Velocities Consistent with Lorentz-Einstein Law of Addition and Derivation of the

Missing Equations of Special Relativity. *Asian Journal of Applied Science and Engineering*, **1**, 70-83.

[18] Feynman, R.P., Leighton, R.B. and Sands, M.L. (1963) The Feynman Lectures on Physics, Vol. 2. Adsison Wesley Publishing.

[19] Ungar, A.A. (2008) Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity. World Scientific Publishing.

[20] Griffiths, D.J. and Schoroeter, D.F. (2018) Introduction to Quantum Mechanics. 3rd Edition, Wiley.

[21] Kreyszig, E. (2006) Introductory Functional Analysis with Applications. Wiley.

[22] Rudin, W. (1973) Functional Analysis. Mir Publishers.