Journal of Applied Mathematics and Physics, 2025, 13(6), 2097-2112
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Computing Octonions Roots by Newton's Method

W. Eltayeb Ahmed[1], Nahla Mohamed Ahmed[2]

[1]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
[2]Department of Mathematics, College of Mathematical Sciences, University of Khartoum, Khartoum, Sudan
Email: waahmed@imamu.edu.sa, nahlaibrahim@uofk.edu.su

How to cite this paper: Ahmed, W.E. and Ahmed, N.M. (2025) Computing Octonions Roots by Newton's Method. *Journal of Applied Mathematics and Physics*, **13**, 2097-2112.
https://doi.org/10.4236/jamp.2025.136117

Received: May 1, 2025
Accepted: June 27, 2025
Published: June 30, 2025

Abstract

When the task is to find the roots of octonions or quaternions numbers, we immediately think of Euler's and De Moivre's formulas. In this work, we show that the task can be accomplished numerically using Newton-Raphson method.

Keywords

Octonion, Quaternion, Matrix Algebra, Newton-Raphson Method

1. Introduction

The mathematical structure represented by algebra of quaternions [1] [2] and octonions [3] [4] has become of great importance in some physical and engineering applications, so it is useful to develop mathematical methods or formulas to simplify the algebraic operations on these numbers such as computing the powers and roots.

It is well known that Euler's and De Moivre's formulas are used to compute powers and roots of quaternions and octonions numbers [5] [6] as they avoid the complicated multiplication of these numbers. Our work presents a new and unconventional method to find powers and roots of these numbers; the method involves constructing specific matrices for representing octonions and quaternions and deriving algebraic formulas for their powers and roots.

To find the roots, our algebraic formulas give equations for real and imaginary parts, the real part equation can be solved numerically using any of numerical methods used to find polynomials roots such as the Newton-Raphson method, and the imaginary part is subsequently determined.

For a quaternion number $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, where x_0, x_1, x_2, x_3 are real numbers and i_1, i_2, i_3 are imaginary units satisfy the multiplication rule shown in **Table 1**, we can create $4(4!)$ distinct real matrices to represent x [7], one of these matrices is:

$$X = \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix} \quad (1)$$

Table 1. Multiplication rule for imaginary units of quaternions.

$\times$	i_1	i_2	i_3
i_1	-1	i_3	$-i_2$
i_2	$-i_3$	-1	i_1
i_3	i_2	$-i_1$	-1

For an octonion number $y = y_0 + y_1 j_1 + y_2 j_2 + y_3 j_3 + y_4 j_4 + y_5 j_5 + y_6 j_6 + y_7 j_7$, where $y_0, y_1, \cdots, y_7$ are real numbers and $j_1, j_2, \cdots, j_7$ are imaginary units satisfy the multiplication rule shown in **Table 2**, we can create $4(8!)$ distinct real matrices to represent y, one of these matrices [8] is:

$$Y = \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & -y_3 & y_2 & -y_5 & y_4 & y_7 & -y_6 \\ y_2 & y_3 & y_0 & -y_1 & -y_6 & -y_7 & y_4 & y_5 \\ y_3 & -y_2 & y_1 & y_0 & -y_7 & y_6 & -y_5 & y_4 \\ y_4 & y_5 & y_6 & y_7 & y_0 & -y_1 & -y_2 & -y_3 \\ y_5 & -y_4 & y_7 & -y_6 & y_1 & y_0 & y_3 & -y_2 \\ y_6 & -y_7 & -y_4 & y_5 & y_2 & -y_3 & y_0 & y_1 \\ y_7 & y_6 & -y_5 & -y_4 & y_3 & y_2 & -y_1 & y_0 \end{bmatrix} \quad (2)$$

Table 2. Multiplication rule for imaginary units of octonions.

$\times$	j_1	j_2	j_3	j_4	j_5	j_6	j_7
j_1	-1	j_3	$-j_2$	j_5	$-j_4$	$-j_7$	j_6
j_2	$-j_3$	-1	j_1	j_6	j_7	$-j_4$	$-j_5$
j_3	j_2	$-j_1$	-1	j_7	$-j_6$	j_5	$-j_4$
j_4	$-j_5$	$-j_6$	$-j_7$	-1	j_1	j_2	j_3
j_5	j_4	$-j_7$	j_6	$-j_1$	-1	$-j_3$	j_2
j_6	j_7	j_4	$-j_5$	$-j_2$	j_3	-1	$-j_1$
j_7	$-j_6$	j_5	j_4	$-j_3$	$-j_2$	j_1	-1

To obtain our goal, we will take the following steps: First, we derive an algebraic formula for computing y^n, (where n is a positive integer). Then, we use it to form equations for real and imaginary parts of $y^{1/n}$. Next, we solve the equation of the real part using Newton-Raphson method. Finally, we substitute this solution into imaginary part equation, and thus we can obtain $y^{1/n}$.

2. Computing Powers

In this section and the next, we will derive algebraic formulas to comput y^n, y^{-n} and $y^{1/n}$.

For x, we can use the same formulas with the following slight change:

$$y_k = \begin{cases} x_k, & \text{for } 0 \le k \le 3 \\ 0, & \text{for } 4 \le k \le 7 \end{cases} \tag{3}$$

To find y^n, we can multiply Y by itself n times, and since

$$\begin{aligned} Y^2 &= \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & -y_3 & y_2 & -y_5 & y_4 & y_7 & -y_6 \\ y_2 & y_3 & y_0 & -y_1 & -y_6 & -y_7 & y_4 & y_5 \\ y_3 & -y_2 & y_1 & y_0 & -y_7 & y_6 & -y_5 & y_4 \\ y_4 & y_5 & y_6 & 0 & y_0 & -y_1 & -y_2 & -y_3 \\ y_5 & -y_4 & y_7 & 0 & 0 & y_0 & y_3 & -y_2 \\ y_6 & -y_7 & -y_4 & 0 & 0 & 0 & y_0 & y_1 \\ y_7 & y_6 & -y_5 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} \\ &= \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \left(Y^*\right)Y \end{aligned} \tag{4}$$

we use Y^* instead of Y.

To find y^2,

$$\begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right) \\ 2y_0y_1 \\ 2y_0y_2 \\ 2y_0y_3 \\ 2y_0y_4 \\ 2y_0y_5 \\ 2y_0y_6 \\ 2y_0y_7 \end{bmatrix} \tag{5}$$

So,

$$y^2 = \left[y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] + 2y_0\left(y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7\right) \tag{6}$$

To find y^3,

$$Y^3 = \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right) \\ 2y_0 y_1 \\ 2y_0 y_2 \\ 2y_0 y_3 \\ 2y_0 y_4 \\ 2y_0 y_5 \\ 2y_0 y_6 \\ 2y_0 y_7 \end{bmatrix}$$

$$= \begin{bmatrix} y_0^3 - 3y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right) \\ y_1\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_2\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_3\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_4\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_5\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_6\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_7\left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \end{bmatrix} \tag{7}$$

So,

$$\begin{aligned} y^3 &= \left[y_0^3 - 3y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ &\quad + \left[3y_0^2 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]\left(y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7\right) \end{aligned} \tag{8}$$

To find y^4,

$$Y^3 = \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0^3 - 3y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right) \\ y_1\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_2\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_3\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_4\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_5\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_6\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_7\left[3y_0 - \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} y_0^4 - 6y_0^2\left(y_1^2 + y_2^2 + \cdots + y_7^2\right) + \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)^2 \\ y_1\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_2\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_3\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_4\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_5\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_6\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \\ y_7\left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right] \end{bmatrix}$$

So,

$$\begin{aligned} y^4 = &\left[y_0^4 - 6y_0^2\left(y_1^2 + y_2^2 + \cdots + y_7^2\right) + \left(y_1^2 + y_2^2 + \cdots + y_7^2\right)^2\right] \\ &+ \left[4y_0^3 - 4y_0\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]\left(y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7\right) \end{aligned} \tag{10}$$

By this way we will reach to the following general formulas:

$$\begin{aligned} y^n = &\left[\sum_{k=0}^{n/2} \binom{n}{n-2k} y_0^{n-2k}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^k\right] \\ &+ \left[\sum_{k=0}^{n/2-1} \binom{n}{n-2k-1} y_0^{n-2k-1}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^k\right]\left(y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7\right) \end{aligned} \tag{11}$$

for n is an even number, and

$$\begin{aligned} y^n = &\left[\sum_{k=0}^{(n-1)/2} \binom{n}{n-2k} y_0^{n-2k}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^k\right] \\ &+ \left[\sum_{k=0}^{(n-1)/2} \binom{n}{n-2k-1} y_0^{n-2k-1}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^k\right]\left(y_1 j_1 + y_2 j_2 + \cdots + j_7 j_7\right) \end{aligned} \tag{12}$$

for n is an odd number.

It is obvious that the real number y_0 has a major role in the result, let's see the result when we have a pure octonion number $(y_0 = 0)$.

The matrix Y^* will be:

$$Y^* = \begin{bmatrix} 0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

Then the formulas (11) and (12) become (14) and (15) respectively.

$$y^n = \left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^{n/2} \tag{14}$$

$$y^n = \left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^{(n-1)/2} \left(y_1 j_1 + y_2 j_2 + \cdots + j_7 j_7\right) \tag{15}$$

To find y^{-n}, since

$$y^{-n} = \frac{1}{y^n} = \frac{1}{y^n}\frac{\left(\bar{y}\right)^n}{\left(\bar{y}\right)^n} = \frac{\left(\bar{y}\right)^n}{\left(y\bar{y}\right)^n} \tag{16}$$

we need to find $y\bar{y}$,

$$\begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \begin{bmatrix} y_0 \\ -y_1 \\ -y_2 \\ -y_3 \\ -y_4 \\ -y_5 \\ -y_6 \\ -y_7 \end{bmatrix} = \begin{bmatrix} y_0^2 \\ y_1^2 \\ y_2^2 \\ y_3^2 \\ y_4^2 \\ y_5^2 \\ y_6^2 \\ y_7^2 \end{bmatrix} \tag{17}$$

So,

$$y\bar{y} = y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2 \tag{18}$$

and to find $\left(\bar{y}\right)^n$, the matrix Y^* will be:

$$Y^* = \begin{bmatrix} y_0 & y_1 & y & y_3 & y_4 & y_5 & y_6 & y_7 \\ -y_1 & y_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -y_2 & 0 & y_0 & 0 & 0 & 0 & 0 & 0 \\ -y_3 & 0 & 0 & y_0 & 0 & 0 & 0 & 0 \\ -y_4 & 0 & 0 & 0 & y_0 & 0 & 0 & 0 \\ -y_5 & 0 & 0 & 0 & 0 & y_0 & 0 & 0 \\ -y_6 & 0 & 0 & 0 & 0 & 0 & y_0 & 0 \\ -y_7 & 0 & 0 & 0 & 0 & 0 & 0 & y_0 \end{bmatrix} \tag{19}$$

Similarly, we will find that, when n is an even number,

$$\begin{aligned} y^{-n} = \left[\frac{1}{\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^n}\right] & \left[\left[\sum_{i=0}^{n/2}\binom{n}{n-2i} y_0^{n-2i}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^i\right]\right. \\ & \left. -\left[\sum_{i=0}^{n/2-1}\binom{n}{n-2i-1} y_0^{n-2i-1}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^i\right]\left(y_1 j_1 + y_2 j_2 + \cdots + j_7 j_7\right)\right] \end{aligned} \tag{20}$$

and when n is an odd number,

$$\begin{aligned} y^{-n} = \left[\frac{1}{\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^n}\right] & \left[\left[\sum_{i=0}^{(n-1)/2}\binom{n}{n-2i} y_0^{n-2i}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^i\right]\right. \\ & \left. -\left[\sum_{i=0}^{(n-1)/2}\binom{n}{n-2i-1} y_0^{n-2i-1}\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^i\right]\left(y_1 j_1 + y_2 j_2 + \cdots + j_7 j_7\right)\right] \end{aligned} \tag{21}$$

For a pure octonion, when n is an even number,

$$y^{-n} = \left[\frac{1}{\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^n}\right]\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^{n/2} \tag{22}$$

and when n is an odd number,

$$y^{-n} = \left[\frac{-1}{\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^n}\right]\left[-\left(y_1^2 + y_2^2 + \cdots + y_7^2\right)\right]^{(n-1)/2}\left(y_1 j_1 + y_2 j_2 + \cdots + j_7 j_7\right) \tag{23}$$

3. Computing Roots

For computing octonions roots, suppose that $y_1, y_2, \cdots, y_7$ are not all zero, and set

$$\left(y_0 + y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7\right)^{1/n} = b_0 + b_1 j_1 + b_2 j + \cdots + b_7 j_7 \tag{24}$$

For $y^{1/2}$, from (24) and (6) we find

$$\begin{aligned} y_0 + y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7 &= \left(b_0 + b_1 j_1 + b_2 j_2 + \cdots + b_7 j_7\right)^2 \\ &= \left[b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right] + 2b_0\left(b_1 j_1 + b_2 j + \cdots + b_7 j_7\right) \end{aligned} \tag{25}$$

which gives

$$y_0 = b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \text{ and } y_k = 2b_0 b_k \text{, for } 1 \le k \le 7 . \tag{26}$$

and therefore

$$\begin{aligned} & y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2 \\ &= \left[b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]^2 + 4b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= \left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \end{aligned} \tag{27}$$

From (26) and (27) we determine y_0 as

$$\begin{aligned} y_0 &= b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + b_0^2 - b_0^2 \\ &= 2b_0^2 - \left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= 2b_0^2 - \left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/2} \end{aligned} \tag{28}$$

So, the equation of the real part b_0 is:

$$2b_0^2 - \left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/2} - y_0 = 0 \tag{29}$$

and the equation of the imaginary part b_k, for $k = 1, 2, \cdots, 7$ is:

$$b_k = \frac{y_k}{2b_0}\text{, for } 1 \le k \le 7 \tag{30}$$

For $y^{1/3}$, from (24) and (8) we find

$$\begin{aligned} y_0 + y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7 &= \left(b_0 + b_1 j_1 + b_2 j + \cdots + b_7 j_7\right)^3 \\ &= \left[b_0^3 - 3b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right] \\ &\quad + \left[3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]\left(b_1 j_1 + b_2 j_2 + \cdots + b_7 j_7\right) \end{aligned} \tag{31}$$

which gives

$$y_0 = b_0^3 - 3b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \quad \text{and} \quad y_k = \left[3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right] b_k \text{, for } 1 \le k \le 7 \tag{32}$$

and therefore

$$\begin{aligned} & y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2 \\ &= \left[b_0^3 - 3b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]^2 \\ &\quad + \left[3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]^2 \left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= \left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right)^3 \end{aligned} \tag{33}$$

From (32) and (33) we determine y_0 as

$$\begin{aligned} y_0 &= b_0^3 - 3b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= b_0^3 - 3b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + 3b_0^3 - 3b_0^3 \\ &= 4b_0^3 - 3b_0\left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right) \\ &= 4b_0^3 - 3b_0\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/3} \end{aligned} \tag{34}$$

So, the equation of the real part b_0 is:

$$4b_0^3 - 3b_0\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/3} - y_0 = 0 \tag{35}$$

To find b_k, for $1 \le k \le 7$

$$\begin{aligned} y_k &= \left[3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right] b_k \\ &= \left[3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + b_0^2 - b_0^2\right] b_k \\ &= \left[4b_0^2 - \left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right)\right] b_k \\ &= \left[4b_0^2 - \left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/3}\right] b_k \end{aligned} \tag{36}$$

So, the equation of the imaginary part b_k is:

$$b_k = \frac{y_k}{3b_0^2 - \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)} \text{, for } 1 \le k \le 7 \tag{37}$$

For $y^{1/4}$, from (24) into (10) we find

$$\begin{aligned} y_0 + y_1 j_1 + y_2 j_2 + \cdots + y_7 j_7 &= \left(b_0 + b_1 j_1 + b_2 j + \cdots + b_7 j_7\right)^4 \\ &= \left[b_0^4 - 6b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2\right] \\ &\quad + \left[4b_0^3 - 4b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]\left(b_1 j_1 + b_2 j_2 + \cdots + b_7 j_7\right) \end{aligned} \tag{38}$$

which gives

$$y_0 = b_0^4 - 6b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \text{ and}$$
$$y_k = \left[4b_0^3 - 4b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]b_k, \text{ for } 1 \le k \le 7 \tag{39}$$

and therefore

$$\begin{aligned}
&y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2 \\
&= \left[b_0^4 - 6b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2\right]^2 \\
&\quad + \left[4b_0^3 - 4b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\
&= b_0^8 - 12b_0^6\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + 36b_0^4\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \\
&\quad + 2b_0^4\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 - 12b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^3 \\
&\quad + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^4 + 16b_0^6\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\
&\quad - 32b_0^4\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 + 16b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^3 \\
&= b_0^8 + 4b_0^6\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + 6b_0^4\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \\
&\quad + 4b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^3 + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^4 \\
&= \left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right)^4
\end{aligned} \tag{40}$$

From (39) and (40) we determine y_0 as

$$\begin{aligned}
y_0 &= b_0^4 - 6b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \\
&= b_0^4 - 6b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)^2 \\
&\quad + 8b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) - 8b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) \\
&= \left[b_0^2 + \left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]^2 - 8b_0^2\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + 8b_0^4 - 8b_0^4 \\
&= \left[b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right]^2 - 8b_0^2\left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right) + 8b_0^4 \\
&= 8b_0^4 - 8b_0^2\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/4} + \left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/2}
\end{aligned} \tag{41}$$

So, the equation of the real part b_0 is:

$$8b_0^4 - 8b_0^2\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/4} + \left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/2} - y_0 = 0 \tag{42}$$

To find b_k, for $1 \le k \le 7$

$$\begin{aligned}
y_k &= \left[4b_0^3 - 4b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]b_k \\
&= \left[4b_0^3 - 4b_0\left(b_1^2 + b_2^2 + \cdots + b_7^2\right) + 4b_0^3 - 4b_0^3\right]b_k \\
&= \left[8b_0^3 - 4b_0\left(b_0^2 + b_1^2 + b_2^2 + \cdots + b_7^2\right)\right]b_k \\
&= \left[8b_0^3 - 4b_0\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/4}\right]b_k
\end{aligned} \tag{43}$$

So, the equation of the imaginary part b_k is:

$$b_k = \frac{y_k}{8b_0^3 - 4b_0\left(y_0^2 + y_1^2 + y_2^2 + \cdots + y_7^2\right)^{1/4}}, \text{ for } 1 \le k \le 7 \tag{44}$$

Proceeding as before, the equation of the real part b_0 will be:

$$\begin{aligned}
&2^{n-1}b_0^n - n2^{n-3}b_0^{n-2}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{1/n} \\
&+\frac{n(n-3)}{2!}2^{n-5}b_0^{n-4}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{2/n} \\
&-\frac{n(n-4)(n-5)}{3!}2^{n-7}b_0^{n-6}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{3/n} \\
&+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-9}b_0^{n-8}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{4/n} \\
&-\frac{n(n-6)(n-7)(n-8)(n-9)}{5!}2^{n-11}b_0^{n-10}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{5/n} \\
&+\frac{n(n-7)(n-8)(n-9)(n-10)(n-11)}{6!}2^{n-13}b_0^{n-12}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{6/n} \\
&-\frac{n(n-8)(n-9)(n-10)(n-11)(n-12)}{7!}2^{n-15}b_0^{n-14}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{7/n}+\cdots-y_0=0
\end{aligned} \tag{45}$$

and the equation of the imaginary part b_k will be:

$$\begin{aligned}
b_k = y_k\Bigg[&(2b_0)^{n-1}-(n-2)(2b_0)^{n-3}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{1/n} \\
&+\frac{(n-3)(n-4)}{2!}(2b_0)^{n-5}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{2/n} \\
&-\frac{(n-4)(n-5)(n-6)}{3!}(2b_0)^{n-7}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{3/n} \\
&+\frac{(n-5)(n-6)(n-7)(n-8)}{4!}(2b_0)^{n-9}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{4/n} \\
&-\frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{5!}(2b_0)^{n-11}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{5/n} \\
&+\frac{(n-7)(n-8)(n-9)(n-10)(n-11)(n-12)}{6!}(2b_0)^{n-13}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{6/n} \\
&-\frac{(n-8)(n-9)(n-10)(n-11)(n-12)(n-13)}{7!}(2b_0)^{n-15}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{7/n}+\cdots\Bigg]^{-1}
\end{aligned} \tag{46}$$

Let's write (45) and (46) in another form.

If n is an even number greater than 2, then the real part equation will be:

$$\begin{aligned}
&(2b_0)^n - n(2b_0)^{n-2}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{1/n} \\
&+\sum_{i=2}^{n/2}\left[\left[\prod_{j=i+1}^{2i-1}(n-j)\right]\frac{(-1)^i n}{i!}(2b_0)^{n-2i}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{i/n}\right]=2y_0
\end{aligned} \tag{47}$$

and the imaginary part equation will be:

$$b_k = y_k\left[(2b_0)^{n-1}+\sum_{i=1}^{n/2}\left[\left[\prod_{j=i+1}^{2i}(n-j)\right]\frac{(-1)^i}{i!}(2b_0)^{n-2i-1}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{i/n}\right]\right]^{-1} \tag{48}$$

If n is an odd number greater than 3, then the real part equation will be:

$$(2b_0)^n - n(2b_0)^{n-2}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{1/n} + \sum_{i=2}^{(n-1)/2}\left[\left[\prod_{j=i+1}^{2i-1}(n-j)\right]\frac{(-1)^i n}{i!}(2b_0)^{n-2i}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{i/n}\right] = 2y_0 \quad (49)$$

and the equation of the imaginary part will be:

$$b_k = a_k\left[(2b_0)^{n-1} + \sum_{i=1}^{(n-1)/2}\left[\left[\prod_{j=i+1}^{2i}(n-j)\right]\frac{(-1)^i}{i!}(2b_0)^{n-2i-1}\left(y_0^2+y_1^2+y_2^2+\cdots+y_7^2\right)^{i/n}\right]\right]^{-1} \quad (50)$$

4. Newton-Raphson Method

In this section, we apply one of the more popular methods for finding roots of non-linear algebraic equations, this method is called Newton-Raphson, named after Isaac Newton (1643-1727) and Joseph Raphson (1648-1715).

Starting with initial guess z_0, the iterated formula of Newton-Raphson to determine approximation to an exact root is:

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)},\quad k = 0,1,2,\cdots \quad (51)$$

From the previous algebraic formulas (47) and (49), we can form the function $f(z)$ by setting $z = b_0$, $A = y_0^2+y_1^2+y_2^2+\cdots+y_7^2$, and $B = y_0$, (see Table 3).

Table 3. Form functions from the real part equations.

n	$f(z)$	n	$f(z)$
2	$2z^2 - A^{1/2} - B$	3	$4z^3 - 3(A^{1/3})z - B$
4	$8z^4 - 8(A^{1/4})z^2 + A^{2/4} - B$	5	$16z^5 - 20(A^{1/5})z^3 + 5(A^{2/5})z - B$
6	$32z^6 - 48(A^{1/6})z^4 + 18(A^{2/6})z^2 - A^{3/6} - B$	7	$64z^7 - 112(A^{1/7})z^5 + 56(A^{2/7})z^3 - 7(A^{3/7})z - B$
…	…	…	…

Now, how do we select z_0?

Of course, a proper initial guess is critical to approximate an exact root, as a general way, we begin from a value between α and β, such $f(\alpha)\cdot f(\beta) < 0$, this definitely ensures that there is a root between α and β, then we can take $z_0 = \frac{\alpha+\beta}{2}$.

To get the approximate root z^*, we use (51) with the following constraints:

- $|z - z^*| < \epsilon$, where ϵ is the error tolerance; that means, we want to obtain the root with an error of at most of ϵ.
- Or, the iterated reaches the maximum iterated we want.

Once we found z^*, we move on to find another root by replacing $f(z)$ with $f(z)/(z-z^*)$, and $f'(z)$ with $\left[(z-z^*)f'(z) - f(z)\right]/(z-z^*)^2$, and choosing an appropriate value for z_0, and so on until we find all n roots.

5. Examples

To verify the practical effectiveness of our technique in computing powers and roots of quaternions and octonions numbers, we will take examples solved using Euler's and De Moivre's formulas by some authors in [9] and [10] respectively.

Example 5.1

Given the quaternion $x=\frac{1}{2}(1+i_1+i_2+i_3)$, we want to find $x^n, \sqrt{x}$, and $\sqrt[5]{x}$.

For x^n, use (11) for an even number, and (12) for an odd number,

$$x^n=\left[\sum_{k=0}^{n/2}(-1)^k\binom{n}{n-2k}\left(\frac{1}{2}\right)^{n-2k}\left(\frac{3}{4}\right)^k\right]+\left[\sum_{k=0}^{n/2-1}(-1)^k\binom{n}{n-2k-1}\left(\frac{1}{2}\right)^{n-2k-1}\left(\frac{3}{4}\right)^k\right]\left[\frac{1}{2}(i_1+i_2+i_3)\right]$$

$$x^n=\left[\sum_{k=0}^{(n-1)/2}(-1)^k\binom{n}{n-2k}\left(\frac{1}{2}\right)^{n-2k}\left(\frac{3}{4}\right)^k\right]+\left[\sum_{k=0}^{(n-1)/2}(-1)^k\binom{n}{n-2k-1}\left(\frac{1}{2}\right)^{n-2k-1}\left(\frac{3}{4}\right)^k\right]\left[\frac{1}{2}(i_1+i_2+i_3)\right]$$

Let's find the pattern of the powers.

$$\begin{aligned}x^2&=\left[\sum_{k=0}^{1}(-1)^k\binom{2}{2-2k}\left(\frac{1}{2}\right)^{2-2k}\left(\frac{3}{4}\right)^k\right]+\left[\frac{1}{2}(i_1+i_2+i_3)\right]\\&=-\frac{1}{2}+\frac{1}{2}(i_1+i_2+i_3)\\&=\frac{1}{2}(-1+i_1+i_2+i_3)\end{aligned}$$

$$\begin{aligned}x^3&=\left[\sum_{k=0}^{1}(-1)^k\binom{3}{3-2k}\left(\frac{1}{2}\right)^{3-2k}\left(\frac{3}{4}\right)^k\right]+\left[\sum_{k=0}^{1}(-1)^k\binom{3}{2-2k}\left(\frac{1}{2}\right)^{2-2k}\left(\frac{3}{4}\right)^k\right]\left[\frac{1}{2}(i_1+i_2+i_3)\right]\\&=-1\end{aligned}$$

$$\begin{aligned}x^4&=\left[\sum_{k=0}^{2}(-1)^k\binom{4}{4-2k}\left(\frac{1}{2}\right)^{4-2k}\left(\frac{3}{4}\right)^k\right]+\left[\sum_{k=0}^{1}(-1)^k\binom{4}{3-2k}\left(\frac{1}{2}\right)^{3-2k}\left(\frac{3}{4}\right)^k\right]\left[\frac{1}{2}(i_1+i_2+i_3)\right]\\&=-\frac{1}{2}-\frac{1}{2}(i_1+i_2+i_3)\\&=-\frac{1}{2}(1+i_1+i_2+i_3)\end{aligned}$$

We notice that

$$x = -x^4 = x^7 = -x^{10} = \cdots = \frac{1}{2}(1 + i_1 + i_2 + i_3)$$

$$x^2 = -x^5 = x^8 = -x^{11} = \cdots = \frac{1}{2}(-1 + i_1 + i_2 + i_3)$$

$$x^3 = -x^6 = x^9 = -x^{12} = \cdots = -1$$

To find $\sqrt{x}$, use (29) and (30),

$$2b_0^2 - (1)^{1/2} - \frac{1}{2} = 0 \to b_0 = \pm\frac{\sqrt{3}}{2}$$

$b_k = \pm\frac{x_k}{\sqrt{3}}$, for $1 \le k \le 3$

Therefore, $\sqrt{x} = \pm\left[\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}}(i_1 + i_2 + i_3)\right]$.

To find $\sqrt[5]{x}$, first, determine the real part equation from (49),

$$(2b_0)^5 - 5(2b_0)^3 (1)^{1/5} + \sum_{i=2}^{2}\left[\left[\prod_{j=i+1}^{2i-1}(5-j)\right]\frac{(-1)^i 5}{i!}(2b_0)^{5-2i}(1)^5\right] = 1$$

$$(2b_0)^5 - 5(2b_0)^3 + 5(2b_0) = 1$$

Then, take $f(z) = 32z^5 - 40z^3 + 10z - 1$, and find its roots by using (51), (see Table 4).

Table 4. Newton-Raphson method to determine the roots of $f(z) = 32z^5 - 40z^3 + 10z - 1$.

Function	(α, β)	z_0	z
$f_1(z) = f(z)$	(−1, −0.9)	−0.95	−0.913545 at 4 iterations
$f_2(z) = \frac{f_1(z)}{z + 0.913545}$	(−0.8, −0.6)	−0.7	−0.669131 at 3 iterations
$f_3(z) = \frac{f_2(z)}{z + 0.669131}$	(0, 0.1)	0.05	0.104528 at 21 iterations
$f_4(z) = \frac{f_3(z)}{z - 0.104528}$	(0.4, 0.6)	0.5	0.5 at 0 iteration
$f_5(z) = \frac{f_4(z)}{z - 0.5}$	(0.9, 1)	0.95	0.978148 at 82 iterations

Next, determine the imaginary part from (50).

$$b_k = a_k\left[(2z)^4 + \sum_{i=1}^{2}\left[\left[\prod_{j=i+1}^{2i}(5-j)\right]\frac{(-1)^i}{i!}(2z)^{4-2i}(1)^{i/5}\right]\right]^{-1}, \text{ for } 1 \le k \le 3$$

$$= a_k\left(16z^4 - 12z^2 + 1\right)^{-1}$$

Finally, determine the roots of $\sqrt[5]{x}$ from:

$$z + \left(16z^4 - 12z^2 + 1\right)^{-1}\left[\frac{1}{2}(i_1 + i_2 + i_3)\right]$$

The first root is $0.104528+0.574187(i_1+i_2+i_3)$.

The second root is $0.104528+0.574187(i_1+i_2+i_3)$.

The third root is $0.104528+0.574187(i_1+i_2+i_3)$.

The fourth root is $0.104528+0.574187(i_1+i_2+i_3)$.

The fifth root is $0.104528+0.574187(i_1+i_2+i_3)$.

Example 5.2

Given the octonion $y=1+j_2+j_4+j_6$, we want to find $\sqrt[3]{y}$.

First, determine the real part equation from (49),

$$(2b_0)^3-3(4)^{1/3}b_0=2$$

Then, take $f(z)=4z^3-3(4)^{1/3}z-1$, and find its roots by using (51), (see Table 5).

Table 5. Newton-Raphson method to determine the roots of $f(z)=4z^3-3(4)^{1/3}z-1$.

Function	(α,β)	z_0	z
$f_1(z)=f(z)$	(−1, −0.9)	−0.95	−0.965156 at 3 iterations
$f_2(z)=\dfrac{f_1(z)}{z+0.965156}$	(−0.3, −0.2)	−0.25	−0.218783 at 3 iterations
$f_3(z)=\dfrac{f_2(z)}{z+0.218783}$	(1.1, 1.2)	1.15	1.183939 at 543 iteration

Next, determine the imaginary part from (50),

$$b_k=a_k\left[(2b_0)^2-(4)^{1/3}\right]^{-1},\ \text{for } 1\le k\le 7$$
$$=a_k\left[(2z)^2-(4)^{1/3}\right]^{-1}$$

Finally, determine the roots of $\sqrt[3]{y}$ from

$$z+\left[(2z)^2-(4)^{1/3}\right]^{-1}(j_2+j_4+j_6).$$

The first root is: $-0.965156+0.467573(j_2+j_4+j_6)$.

The second root is: $-0.218783-0.716365(j_2+j_4+j_6)$.

The third root is: $1.183939-0.248791(j_2+j_4+j_6)$.

6. Conclusion

The foregoing algebraic formulas give powers and roots of octonions and quaternions numbers. For the roots, the equation of the real part can be solved numerically using a numerical method such as Newton-Raphson method, therefore, we can find the powers and roots of these numbers without using Euler's or De Moivre's formulas.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Hamilton, W.R. (2000) On Quaternions or a New System of Imaginaries in Algebra. *Philosophical Magazine*.

[2] Lewis, D.W. (2006) Quaternion Algebras and the Algebraic Legacy of Hamilton's Quaternions. *Irish Mathematical Society Bulletin*, **57**, 41-64. https://doi.org/10.33232/bims.0057.41.64

[3] Conway, J.H. and Smith, D.A. (2003) On Quaternions and Octonions. *AMC*, **10**, 12.

[4] Baez, J. (2001) The Octonions. *Bulletin of the American Mathematical Society*, **39**, 145-205. https://doi.org/10.1090/s0273-0979-01-00934-x

[5] Cho, E. (1999) Euler's Formula and De Moivre's Formula for Quaternions. *Missouri Journal of Mathematical Sciences*, **11**, 80-83. https://doi.org/10.35834/1999/1102080

[6] Flaut, C. and Shpakivskyi, V. (2014) De Moivre's Formula and Euler's Formula for Octonions.

[7] Ahmed, W.E. (2020) Ninety-Six Distinct Real Matrices for Representing a Quaternion Number. *Journal of Mathematics*, **2020**, Article ID: 2412491. https://doi.org/10.1155/2020/2412491

[8] Tian, Y. (2000) Matrix Representations of Octonions and Their Applications. *Advances in Applied Clifford Algebras*, **10**, 61-90. https://doi.org/10.1007/bf03042010

[9] Jafari, M. and Yayli, Y. (2011) Demoivre's Formula for Matrices of Quaternions. *JP Journal of Algebra*, *Number Theory and Application*, **21**, 57-67.

[10] Bektaş, Ö. and Yüce, S. (2017) De Moivre's and Euler's Formulas for the Matrices of Octonions. *Proceedings of the National Academy of Sciences*, *India Section A*: *Physical Sciences*, **89**, 113-127. https://doi.org/10.1007/s40010-017-0388-z

Appendix

Python code 1: To find powers for an octonion/quaternion number.

```
import math
a,b,c,d,e,f,g,h=input("Please insert the number as y(0) y(1) y(2) y(3) y(4) y(5) y(6) y(7): ").split()
a=float(a); b=float(b); c=float(c); d=float(d); c=float(c); d=float(d); e=float(e); f=float(f); g=float(g); h=float(h)
A=b**2+c**2+d**2+e**2+f**2+g**2+h**2
n=int(input("Please insert n: "))
if n<0:
    m=-n
else:
    m=n
realpart=0; imaginarypart=0
if m%2==0:
    for i in range(0,m//2+1):
        realpart=realpart+((-A)**i)*(a**(m-2*i))*(math.factorial(m))/((math.factorial(m-2*i))*(math.factorial(2*i)))
    for j in range(0,m//2):
        imaginarypart=imaginarypart+((-A)**j)*(a**(m-2*j-1))*(math.factorial(m))/((math.factorial(m-2*j-1))*(math.facto-
rial(2*j+1)))
else:
    for i in range(0,(m-1)//2+1):
        realpart=realpart+((-A)**i)*(a**(m-2*i))*(math.factorial(m))/((math.factorial(m-2*i))*(math.factorial(2*i)))
    for j in range(0,(m-1)//2+1):
        imaginarypart=imaginarypart+((-A)**j)*(a**(m-2*j-1))*(math.factorial(m))/((math.factorial(m-2*j-1))*(math.facto-
rial(2*j+1)))
if n<0:
    print("The power is: %f -(%f)(%fi(1)+%fi(2)+%fi(3)+%fi(4)+%fi(5)+%fi(6)+%fi(7))" %(realpart/(a**2+A)**m,imagi-
narypart/(a**2+A)**m,b,c,d,e,f,g,h))
else:
    print("The power is: %f +(%f)(%fi(1)+%fi(2)+%fi(3)+%fi(4)+%fi(5)+%fi(6)+%fi(7))" %(realpart,imagi-
narypart,b,c,d,e,f,g,h))
```

Python code 2: To find the real part of an octonion/quaternion root by using Newton-Raphson method.

```
f=lambda z:    …
fprime=lambda z:    …
z=float(input("Insert an initial guess: "))
def newtonraphson(f1,fprime1,z,tol=0.000000001,maxitereation=10001):
    for i in range(maxitereation):
        znew=z-f1(z)/fprime1(z)
        if abs(znew-z)<tol:break
        z=znew
    return znew,i
z,m=newtonraphson(f,fprime,z)
print("The real part is: %f at %d iterations. "%(z,m))
```