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A New Extension of the Riemann-Lebesgue Lemma

Na Li[1], Yongguo Shi[2*], Kelin Li[1]

[1]College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, China
[2]Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang, China
Email: *scumat@163.com

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Abstract

The asymptotic behavior for a sequence of integrals

$$\int_a^b f(x)g(nx)\mathrm{d}x,\ a,b\in\mathbb{R},\ \text{as } n\to\infty,$$

is investigated, where f is continuous differentiable function and g is a continuous and periodic function. The first two terms asymptotic estimates have been given by [D. Andrica, M. Piticari. Intern. Conf. on Theory and Appl. of Math. and Inf. Thessaloniki, Acta Universitatis Apulensis, 2004(8), 26-39.] with some different assumptions. This paper obtains the complete asymptotic expansions. Some examples are provided.

Keywords

Riemann-Lebesgue Lemma, Integral Sequence, Asymptotic Expansions

1. Introduction

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. If for each integer $k\geq 0$,

$$a_n=\sum_{j=0}^{k}\frac{b_j}{n^j}+o\left(\frac{1}{n^k}\right),\ \text{as } n\to\infty,$$

then the series $\sum_{j=0}^{\infty}\frac{b_j}{n^j}$ is called Poincaré asymptotic expansion (or asymptotic power series) of the sequence $(a_n)_{n\geq 1}$, which is convergent or divergent [1].

One of the well-known results of asymptotic analysis of integrals is the Riemann-Lebesque lemma [2], which is widely used, it can be used to prove the effectiveness of integral asymptotic approximations, and to study the asymptotic expansion of the Fourier integral [3]. The classic Riemann-Lebesgue lemma [4] states that

$$\lim_{|n|\to\infty}\int_a^b f(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t=0,\quad f\in L^1[a,b],$$

i.e.,

$$\lim_{n\to\infty}\int_a^b f(x)\sin(nx)\mathrm{d}x=0 \text{ and } \lim_{n\to\infty}\int_a^b f(x)\cos(nx)\mathrm{d}x=0.$$

A finer result is given by Andrica and Piticari ([5], Corollary 3.2), who showed that for $k\in\mathbb{Z}$

$$\int_{2k\pi}^{2(k+1)\pi} f(x)\sin nx\mathrm{d}x=\frac{f(2k\pi)-f(2(k+1)\pi)}{n}+o\left(\frac{1}{n}\right),\quad n\to\infty \tag{1.1}$$

where $f\in C^1[0,\infty)$. A generalization of this lemma is given by Kahane [6], who showed that

$$\lim_{n\to\infty}\int_0^\infty f(t)g(nt)\mathrm{d}t=\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T g(t)\mathrm{d}t\right)\int_0^\infty f(t)\mathrm{d}t,\quad f\in L^1[0,\infty),\, g\in L^\infty[0,\infty).$$

When $f\in C[a,b]$ and $g\in C[0,\infty)$ is a T-periodic function, the generalization above is rewritten by Siretki [7], who proved

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)\mathrm{d}x=\frac{1}{T}\int_0^T g(x)\mathrm{d}x\int_a^b f(x)\mathrm{d}x.$$

In particular, if $T=1$, then it can be traced back to [8]. The further generalization is obtained by Andrica and Piticari [5], who gave the first two terms of the asymptotic expansion of the sequence of integrals $\int_a^b f(x)g(nx)\mathrm{d}x$. When g has a continuous derivative on $[0,T]$, then

$$\begin{aligned}&\int_a^b f(x)g(nx)\mathrm{d}x\\&=\frac{1}{T}\int_0^T g(x)\mathrm{d}x\int_a^b f(x)\mathrm{d}x+\frac{1}{nT}\big(f(T)-f(0)\big)\left(G(T)-\int_0^T G(x)\mathrm{d}x\right)\\&\quad+o\left(\frac{1}{n}\right),\ \text{as } n\to\infty,\end{aligned}$$

where $G(x)=\int_0^x g(t)\mathrm{d}t$.

For the sequence of integrals $\int_a^b f(x)g(nx)\mathrm{d}x$, the classic Riemann-Lebesque lemma gives the first asymptotic estimate, and the paper [5] gives the first two asymptotic estimates. Thus, when $g(nx)=\sin(nx)$ or $\cos(nx)$, it is very common in signal processing. For example, such integrals are needed in the calculation of Fourier coefficients to decompose the frequency components of the signal. To the best of our knowledge, no one has yet given a complete asymptotic expansion of this sequence of integrals. However, the efficacy of the asymptotic expansion in estimating the integral is contingent upon the number of terms in the expansion, *i.e.* the order of the asymptotic analysis. It is evident that the higher the order, the more effective the expansion. Inspired by this, the aim of this paper is to obtain the complete asymptotic expansion of $\int_a^b f(x)\sin(nx)\mathrm{d}x$ and

$\int_a^b f(x)\cos(nx)\mathrm{d}x$, then extend it to a more general case whereby the integral is defined as follows: $\int_a^b f(x)g(nx)\mathrm{d}x$.

2. Main Results

In this section, the asymptotic expansions of $\int_a^b f(x)\sin(nx)\mathrm{d}x$ and $\int_a^b f(x)\cos(nx)\mathrm{d}x$ are given, respectively. Then we obtain the asymptotic expansion of $\int_a^b f(x)g(nx)\mathrm{d}x$, which is a new extension of the Riemann-Lebesgue lemma.

Denote the *r*th order derivative of $f(x)$ by $f^{(r)}(x)$ for $r=1,2,\cdots$, and $f^{(0)}(x)=f(x)$.

First, we give a generalisation of Equation (1.1).

Theorem 2.1 *Let* $k\in\mathbb{Z}$ *and* $r\in\mathbb{N}$ *be given. Suppose* $f\in C^{2r+1}\left[2k\pi,2(k+1)\pi\right]$. *Then, for* $m=0,1,\cdots,r$, *the following holds*:

$$\int_{2k\pi}^{2(k+1)\pi} f(x)\sin(nx)\mathrm{d}x=\sum_{i=0}^{m}\frac{a_i}{n^{2i+1}}+\frac{(-1)^m}{n^{2m+1}}\int_{2k\pi}^{2(k+1)\pi} f^{(2m+1)}(x)\cos(nx)\mathrm{d}x,$$

where $a_i=(-1)^{i+1}\left(f^{(2i)}(2(k+1)\pi)-f^{(2i)}(2k\pi)\right)$, $i=0,1,\cdots,m$.

Proof. We use induction on m. Using the integration by parts, the integral is expressed as

$$\begin{aligned}\int_{2k\pi}^{2(k+1)\pi} f(x)\sin(nx)\mathrm{d}x&=-\frac{1}{n}\int_{2k\pi}^{2(k+1)\pi} f(x)\mathrm{d}(\cos(nx))\\&=-\frac{1}{n}\left(f(2(k+1)\pi)-f(2k\pi)\right)+\frac{1}{n}\int_{2k\pi}^{2(k+1)\pi} f'(x)\cos(nx)\mathrm{d}x.\end{aligned}$$

It shows that the theorem is true for $m=0$.

Assume that Theorem 2.1 holds for $m=s$, where $0\le s<r$. We shall prove it is also true for $m=s+1$. Then

$$\int_{2k\pi}^{2(k+1)\pi} f(x)\sin(nx)\mathrm{d}x=\sum_{i=0}^{s}\frac{a_i}{n^{2i+1}}+\frac{(-1)^s}{n^{2s+1}}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+1)}(x)\cos(nx)\mathrm{d}x. \tag{2.1}$$

Thus, using the integration by parts twice, we have

$$\begin{aligned}&\int_{2k\pi}^{2(k+1)\pi} f^{(2s+1)}(x)\cos(nx)\mathrm{d}x\\&=\frac{1}{n}f^{(2s+1)}(x)\sin(nx)\Big|_{2k\pi}^{2(k+1)\pi}-\frac{1}{n}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+2)}(x)\sin(nx)\mathrm{d}x\\&=\frac{1}{n^2}f^{(2s+2)}(x)\cos(nx)\Big|_{2k\pi}^{2(k+1)\pi}-\frac{1}{n^2}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+3)}(x)\cos(nx)\mathrm{d}x\\&=\frac{1}{n^2}\left(f^{(2s+2)}(2(k+1)\pi)-f^{(2s+2)}(2k\pi)\right)-\frac{1}{n^2}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+3)}(x)\cos(nx)\mathrm{d}x,\end{aligned}$$

Substituting the above result into Equation (2.1), we obtain

$$\int_{2k\pi}^{2(k+1)\pi} f(x)\sin(nx)\,\mathrm{d}x$$
$$=\sum_{i=0}^{s}\frac{a_i}{n^{2i+1}}+\frac{(-1)^s}{n^{2s+3}}\left(f^{(2s+2)}\left(2(k+1)\pi\right)-f^{(2s+2)}(2k\pi)\right)$$
$$+\frac{(-1)^{s+1}}{n^{2s+3}}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+3)}(x)\cos(nx)\,\mathrm{d}x$$
$$=\sum_{i=0}^{s+1}\frac{a_i}{n^{2i+1}}+\frac{(-1)^{s+1}}{n^{2s+3}}\int_{2k\pi}^{2(k+1)\pi} f^{(2s+3)}(x)\cos(nx)\,\mathrm{d}x.$$

Therefore the theorem is true for $m=s+1$. This completes the proof.

□

Theorem 2.2 *Let* $k\in\mathbb{N}$ *and* $r\in\mathbb{N}$ *be given. Let* $f\in C^{2r+1}\left[2k\pi,2(k+1)\pi\right]$. *Then, for* $m=0,1,\cdots,r$, *the following holds*:

$$\int_{2k\pi}^{2(k+1)\pi} f(x)\cos(nx)\,\mathrm{d}x=\sum_{i=1}^{m}\frac{a_i}{n^{2i}}+\frac{(-1)^{m+1}}{n^{2m+1}}\int_{2k\pi}^{2(k+1)\pi} f^{(2m+1)}(x)\sin(nx)\,\mathrm{d}x,$$

where $a_i=(-1)^{i+1}\left(f^{(2i-1)}\left(2(k+1)\pi\right)-f^{(2i-1)}(2k\pi)\right)$, $i=1,2,\cdots,m$.

Proof. The proof is similar to Theorem 2.1.

□

And then, we recall the following lemma, which states that the antiderivative of a periodic function can be expressed as the sum of a periodic function and a linear function.

Lemma 2.1 ([5], Lemma 2.2) Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous, non-constant, and T-periodic function. If F is an antiderivative of f, then

$$F(x)=\left(\frac{1}{T}\int_0^T f(t)\,\mathrm{d}t\right)x+g(x),$$

where $g:\mathbb{R}\to\mathbb{R}$ is some T-periodic function.

For $g\in C[0,\infty)$ and every $k\in\mathbb{Z}$, define the following recurrence relations.

$$\begin{cases} M_1=\dfrac{1}{T}\displaystyle\int_{kT}^{(k+1)T} g(t)\,\mathrm{d}t, & \quad (2.2a)\\ G_1(x)=\displaystyle\int_{kT}^{x} g(t)\,\mathrm{d}t,\quad x\in\left(kT,(k+1)T\right], & \quad (2.2b)\\ h_{j-1}(x)=G_{j-1}(x)-M_{j-1}\cdot x, & \text{for } j=2,3,\cdots \quad (2.2c)\\ G_j(x)=\displaystyle\int_{kT}^{x} h_{j-1}(t)\,\mathrm{d}t, & \text{for } j=2,3,\cdots \quad (2.2d)\\ M_j=\dfrac{1}{T}G_j\left((k+1)T\right), & \text{for } j=2,3,\cdots \quad (2.2e)\end{cases}$$

We see that h_j is continuous, T-periodic, and

$$h_j\left((k+1)T\right)=h_j(kT)=-kTM_j,\ \text{ for } j=1,2,3,\cdots.$$

The following is our main result, which gives the asymptotic expansion of $\int_a^b f(x)g(nx)\,\mathrm{d}x$.

Theorem 2.3 *Let* $k\in\mathbb{Z}$ *and* $r\in\mathbb{N}$ *be given. Suppose* $f\in C^{r+1}\left[kT,(k+1)T\right]$, *and* $g\in C[0,\infty)$ *be* T*-periodic. Then, for*

$m = 0,1,\cdots,r$, *the following holds*:

$$\int_{kT}^{(k+1)T} f(x) g(nx) \mathrm{d}x = \sum_{i=0}^{m} \frac{c_i}{n^i} + \frac{(-1)^{m+1}}{n^{m+1}} \int_{kT}^{(k+1)T} f^{(m+1)}(x) h_{m+1}(nx) \mathrm{d}x, \tag{2.3}$$

where $c_0 = M_1 \int_{kT}^{(k+1)T} f(x) \mathrm{d}x$, and for $i = 1,2,\cdots,m$

$$c_i = (-1)^i kTM_i \left[f^{(i-1)}((k+1)T) - f^{(i-1)}(kT) \right] + (-1)^i M_{i+1} \int_{kT}^{(k+1)T} f^{(i)}(x) \mathrm{d}x.$$

Proof. For $i = 0,1,\cdots,m$, we aim to show

$$\int_{kT}^{(k+1)T} f(x) g(nx) \mathrm{d}x = \sum_{i=0}^{m} \frac{a_i}{n^{i+1}} + \sum_{j=0}^{m} \frac{b_j}{n^j} + \frac{(-1)^{m+1}}{n^{m+1}} \int_{kT}^{(k+1)T} f^{(m+1)}(x) h_{m+1}(nx) \mathrm{d}x,$$

where

$$b_i = (-1)^i M_{i+1} \int_{kT}^{(k+1)T} f^{(i)}(x) \mathrm{d}x,\ \ a_i = (-1)^{i+1} kTM_{i+1} \left[f^{(i)}((k+1)T) - f^{(i)}(kT) \right].$$

We use induction on m . According to (2.2b) and (2.2c), using integration by parts, we have

$$\begin{aligned}
& n \int_{kT}^{(k+1)T} f(x) g(nx) \mathrm{d}x \\
&= f(x) G_1(nx) \Big|_{kT}^{(k+1)T} - \int_{kT}^{(k+1)T} f'(x) G_1(nx) \mathrm{d}x \\
&= f(x) \left(M_1 \cdot nx + h_1(nx) \right) \Big|_{kT}^{(k+1)T} - nM_1 \int_{kT}^{(k+1)T} x f'(x) \mathrm{d}x - \int_{kT}^{(k+1)T} f'(x) h_1(nx) \mathrm{d}x \\
&= -kTM_1 \left[f((k+1)T) - f(kT) \right] + nM_1 \int_{kT}^{(k+1)T} f(x) \mathrm{d}x - \int_{kT}^{(k+1)T} f'(x) h_1(nx) \mathrm{d}x.
\end{aligned}$$

Thus

$$\int_{kT}^{(k+1)T} f(x) g(nx) \mathrm{d}x = \frac{a_0}{n} + b_0 - \frac{1}{n} \int_{kT}^{(k+1)T} f'(x) h_1(nx) \mathrm{d}x.$$

Therefore the theorem is true for $m = 0$.

Assume that Equation (2.3) holds for $m = s$, $0 \le s < r$. We shall prove it is also true for $m = s + 1$. Then

$$\begin{aligned}
& \int_{kT}^{(k+1)T} f(x) g(nx) \mathrm{d}x \\
&= \sum_{i=0}^{s} \frac{a_i}{n^{i+1}} + \sum_{j=0}^{s} \frac{b_j}{n^j} + \frac{(-1)^{s+1}}{n^{s+1}} \left(n \int_{kT}^{(k+1)T} f^{(s+1)}(x) h_{s+1}(nx) \mathrm{d}x \right).
\end{aligned} \tag{2.4}$$

Using (2.2c) and (2.2d), we have

$$\begin{aligned}
& n \int_{kT}^{(k+1)T} f^{(s+1)}(x) h_{s+1}(nx) \mathrm{d}x \\
&= \int_{kT}^{(k+1)T} f^{(s+1)}(x) \mathrm{d}\left(G_{s+2}(nx) \right) \\
&= f^{(s+1)}(x) \left(M_{s+2} \cdot nx + h_{s+2}(nx) \right) \Big|_{kT}^{(k+1)T} \\
&\quad - \int_{kT}^{(k+1)T} \left(M_{s+2} \cdot nx + h_{s+2}(nx) \right) f^{(s+2)}(x) \mathrm{d}x \\
&= -kTM_{s+2} \left(f^{(s+1)}((k+1)T) - f^{(s+1)}(kT) \right) + nM_{s+2} \int_{kT}^{(k+1)T} f^{(s+1)}(x) \mathrm{d}x \\
&\quad - \int_{kT}^{(k+1)T} f^{(s+2)}(x) h_{s+2}(nx) \mathrm{d}x.
\end{aligned}$$

Substituting the above result into the third term on the right-hand side of Equation (2.4), we obtain

$$\begin{aligned}&\int_{kT}^{(k+1)T} f(x)g(nx)\mathrm{d}x\\&=\sum_{i=0}^{s}\frac{a_i}{n^{i+1}}+\sum_{j=0}^{s}\frac{b_j}{n^j}+\frac{(-1)^{s+2}}{n^{s+2}}kTM_{s+2}\left(f^{(s+1)}((k+1)T)-f^{(s+1)}(kT)\right)\\&\quad+\frac{(-1)^{s+1}}{n^{s+1}}M_{s+2}\int_{kT}^{(k+1)T} f^{(s+1)}(x)\mathrm{d}x-\frac{(-1)^{s+1}}{n^{s+2}}\int_{kT}^{(k+1)T} f^{(s+2)}(x)h_{s+2}(nx)\mathrm{d}x\\&=\sum_{i=0}^{s+1}\frac{a_i}{n^{i+1}}+\sum_{j=0}^{s+1}\frac{b_j}{n^j}+\frac{(-1)^{s+2}}{n^{s+2}}\int_{kT}^{(k+1)T} f^{(s+2)}(x)h_{s+2}(nx)\mathrm{d}x.\end{aligned}$$

Therefore the theorem is true for $m=s+1$. This completes the proof. □

Letting $k=0$ and $m=r$ in Theorem 2.3, we have the following corollary.

Corollary 2.1 *Let* $r\in\mathbb{N}$ *be given. Let* $f\in C^{r+1}[0,T]$, *and* $g\in C[0,\infty)$ *be* T *-periodic. Then*

$$\int_0^T f(x)g(nx)\mathrm{d}x=\sum_{i=0}^{r}\frac{b_i}{n^i}+\frac{(-1)^{r+1}}{n^{r+1}}\int_0^T f^{(r+1)}(x)h_{r+1}(nx)\mathrm{d}x,$$

where $b_i=(-1)^i M_{i+1}\int_0^T f^{(i)}(x)\mathrm{d}x$ for $i=0,1,\cdots,r$.

3. Examples

The main results not only have important applications in integral calculations involved in mathematics, but also can be applied to the calculation of high-frequency Fourier coefficients of continuous functions involved in physics and engineering. This section gives the applications of our theorem in two examples.

Example 3.1 *Consider the asymptotic estimate of*

$$J_n=\int_{2\pi}^{4\pi}\frac{\sin nx}{x^2}\mathrm{d}x,\ \text{as } n\to\infty.$$

Let $f(x)=\frac{1}{x^2}$, $k=1$. Then, by Theorem 4.1, we obtain

$$\int_{2\pi}^{4\pi}\frac{\sin nx}{x^2}\mathrm{d}x=\frac{3}{16\pi^2}\cdot\frac{1}{n}-\frac{45}{128\pi^4}\cdot\frac{1}{n^3}+\frac{315}{512\pi^6}\cdot\frac{1}{n^5}-\frac{80325}{4096\pi^8}\cdot\frac{1}{n^7}+o\left(\frac{1}{n^7}\right).$$

Table 1. Asymptotic analysis of $J_{n,r+1}=\sum_{i=0}^{r}\frac{a_i}{n^{2i+1}}$.

n	$J_n(10^{-4})$	$J_{n,1}(10^{-4})$	$J_{n,2}(10^{-4})$	$J_{n,3}(10^{-4})$	$J_{n,4}(10^{-4})$
	1.8997361038	1.89977	1.899736101	1.899736102589	1.899736102589
	0.189977183239	0.1899772	0.18997718323804	0.18997718323805	0.18997718323805

Table 1 confirms that as n increases and more terms are included, the approximation $J_{n,r+1}$ converges to the exact value J_n, validating the asymptotic

expansion's accuracy.

Example 3.2 ([7], Problem 9.2) Consider the asymptotic estimate of

$$I_n = \int_0^{\pi} \frac{\sin x}{1+\cos^2 nx}\mathrm{d}x, \text{ as } n \to \infty.$$

Let $f(x)=\sin x$ and $g(x)=\frac{1}{1+\cos^2 x}$. By Corollary 2.1, we have

$$M_1 = \frac{1}{\pi}\int_0^{\pi} \frac{1}{1+\cos^2 x}\mathrm{d}x = \frac{\sqrt{2}}{2}, \ a_0 = M_1\int_0^{\pi}\sin x\mathrm{d}x = \sqrt{2},$$

and

$$a_{2i+1} = -M_{2i+2}\int_0^{\pi} f^{(2i+1)}(x)\mathrm{d}x = -M_{2i+2}\int_0^{\pi}\cos x\mathrm{d}x = 0, \ i = 0,1,2,\cdots,$$

$$\begin{aligned} a_2 &= \frac{1}{\pi}\int_0^{\pi}\left(G_2(x) - M_2\cdot x\right)\mathrm{d}x\int_0^{\pi}(-\sin x)\mathrm{d}x \\ &= -\frac{2}{\pi}\int_0^{\pi}\int_0^{x}\int_0^{y}\frac{1}{1+\cos^2 t}\mathrm{d}t\mathrm{d}y\mathrm{d}x + \frac{\sqrt{2}}{2\pi}\int_0^{\pi}x^2\mathrm{d}x + \pi M_2 \approx 0.1165, \end{aligned}$$

since

$$M_2 = \frac{1}{\pi}\int_0^{\pi}\left(G_1(t) - M_1\cdot t\right)\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi}\int_0^{x}\frac{1}{1+\cos^2 t}\mathrm{d}t\mathrm{d}x - \frac{\sqrt{2}\pi}{4} = 0.$$

Therefore,

$$\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}\mathrm{d}x = \sqrt{2} + \frac{a_2}{n^2} + o\left(\frac{1}{n^3}\right),$$

where $a_2 \approx 0.1165$.

We can continue this process and obtain the other terms of the asymptotic expansion.

Example 3.3 *Vibration analysis considering high-order polynomial external forces. For the mechanical system subjected to the external force* $f(x) = x^5$, *the high-frequency attenuation characteristics of its Fourier coefficients are analyzed on the interval* $[0, 2\pi]$.

Apply the Theorem 2.1, we have

$$a_0 = -32\pi^5, \ a_1 = 160\pi^3, \ a_2 = -240\pi.$$

Then

$$\int_0^{2\pi} x^5\sin(nx)\mathrm{d}x = -\frac{32\pi^5}{n} + \frac{160\pi^3}{n^3} - \frac{240\pi}{n^5} + \frac{1}{n^5}\int_0^{2\pi} f^5(x)\sin(nx)\mathrm{d}x.$$

The following is the processing of the remaining items

$$\frac{1}{n^5}\int_0^{2\pi} f^5(x)\sin(nx)\mathrm{d}x = \frac{120}{n^5}\frac{\sin(nx)}{n}\Bigg|_0^{2\pi} = 0.$$

Therefore

$$\int_0^{2\pi} x^5\sin(nx)\mathrm{d}x \sim -\frac{32\pi^5}{n} + \frac{160\pi^3}{n^3} - \frac{240\pi}{n^5}.$$

This asymptotic expansion can well analyze the suppression effect of the high-order changes of external forces on the high-frequency vibration mode. The originally complex high-frequency oscillation integral is transformed into an asymptotic expansion form that is easy to analyze, simplifying the solution process of the physical model.

4. Conclusion

We extend Andrica and Piticari's first two terms of asymptotic expansions of $\int_a^b f(x)\sin(nx)\,\mathrm{d}x$ to complete asymptotic expansions. Furthermore, a new extension of the Riemann-Lebesgue lemma is obtained by generalising their first two terms of asymptotic expansions to complete asymptotic expansions of $\int_a^b f(x)g(nx)\,\mathrm{d}x$. Through the study of this type of integral sequence, it can be directly applied to the integral examples in practical applications, avoiding repetitive integration by parts during calculation, and the attenuation rate dependent on n can be explicitly displayed.

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Authors' Contributions

Na Li, Yong-Guo Shi and Kelin Li wrote the main manuscript, and reviewed the manuscript.

Conflicts of Interest

The author declares no competing interests.

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