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# On Eigenvalue and Maximum Principle Type Problems Involving the p-Laplacian with Nonlinear Boundary Conditions

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## Abstract

In this present work, we consider an eigenvalue-type problem involving the p-Laplacian operator under nonlinear boundary conditions, in case $g \equiv 0$, and we prove the existence of an isolated eigenvalue plus a continuous family of eigenvalues. We also study the maximum principle-type for $(\mathcal{P}_{\lambda,g})$ with $g \geq 0$, $g \not\equiv 0$.

## Keywords

Weak Solution, Eigenvalue Problem, Maximum Principle, p-Laplacian Operator, Nonlinear Boundary Conditions, Existence of Solutions

## 1. Introduction

We consider, for a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with smooth boundary, the following problem

$$(\mathcal{P}_{\lambda,g}) \quad \begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda f(x,u) + g(x) & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := div\left(|\nabla u|^{p-2} \nabla u\right)$ is the p-laplacian operator, $1 < p < +\infty$, $\dfrac{\partial u}{\partial \nu}$ represents an exterior normal derivative of $u$ and $f$ and $g$ are given functions satisfying some conditions which will be specified later on. In this paper, we focus on the existence of a set of eigenvalues of the problem $(\mathcal{P}_\lambda)$, *i.e.* in the case $g \equiv 0$ given by

$$(\mathcal{P}_\lambda)\quad \begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

and existence of positive solution (maximum principle) of $(\mathcal{P}_{\lambda,g})$.

Obviously, the problem $(\mathcal{P}_\lambda)$ is not a typical eigenvalue problem since it does not have a homogenous structure. The usually eigenvalue problem is obtained if

$$f(x,t) = |t|^{p-2} t.$$

It can also be considered with a weighted function and in that case the problem is given by

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda m(x) |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{1}$$

It is well known that the eigenvalue problem (1) has a sequence of positive eigenvalues $\lambda_n$, such that $\lambda_n \to +\infty$ when $n \to +\infty$ and the first eigenvalue $\lambda_1(m)$ is defined by

$$\lambda_1(m) = \inf\left\{ \int_\Omega \left( |\nabla u|^p + |u|^p \right), u \in W^{1,p}(\Omega), \int_{\partial\Omega} m |u|^p = 1 \right\}. \tag{2}$$

For those results, see [1] [2] for $m \equiv 1$, [3] for $m$ indefinite weight with $m^+ \not\equiv 0$, $m \in L^\alpha(\partial\Omega)$, for $\alpha > \dfrac{N-1}{p-1}$ if $1 < p \le N$ and $\alpha \ge 1$ if $p > N$ and [4]-[6] for more general problem. In [5], the authors proved that the first eigenvalue $\lambda_1(m)$ is simple, isolated and principal, that is, every eigenfunction associated to the first eigenvalue $\lambda_1(m)$ has a constant sign. They also proved that $\lambda_1(m)$ is the only principal eigenvalue of (1) and given a definition of the second positive eigenvalue of (1) as the following

$$\lambda_2(m) = \min\left\{ \lambda \in \mathbb{R}; \lambda \text{ is an eigenvalue and } \lambda > \lambda_1(m) \right\}. \tag{3}$$

Recently, some authors have been attracted by the similar eigenvalue problems to $(\mathcal{P}_\lambda)$: in [7], the authors consider the Laplacian, *i.e.* $p = 2$, with Dirichlet boundary condition in a bounded domain, in [8] the author consider the p-Laplace operator in exterior domain and the work [9] consider the fractional (s,p)-Laplacian case. Under different formes given above, the authors proved that the problem $(\mathcal{P}_\lambda)$ possesses a continuous family of eigenvalues plus an isolated eigenvalue.

In case of $f(x,t) = m(x)|u|^{p-2} u$, the maximum principle for problem $(\mathcal{P}_{\lambda,g})$ has been considered in [10] with $m \equiv 1$ and [11] with an indefinite weight $m$ and reference therein.

Motived by results proved in [7]-[9] in one hand, we will analyse the eigenvalue problem $(\mathcal{P}_\lambda)$ with nonlinear boundary condition in a bounded domain $\Omega$ of $\mathbb{R}^N$. On another hand, we consider a new problem involving the p-Laplacian by adding a function source given by $(\mathcal{P}_{\lambda,g})$ in order to study the existence of positive solution (maximum principle) for some values of parameter $\lambda$. Until now, no work has considered this last problem.

This paper is organized as follows: In Section 2, we recall some basic definitions and we review some properties of the principal eigenvalues of the $p$-Laplacian under Steklov boundary conditions and give the hypothesis on the functions $m, f$ and $g$. We prove in Section 3 the existence of a continuous family of eigenvalues plus an isolated eigenvalue for $(\mathcal{P}_\lambda)$ in Theorem 3. At the end, we prove in Section 4 some results concerning the maximum principle and existence of solutions for problem $(\mathcal{P}_{\lambda,g})$ in Theorem 9 and Theorem 10 respectively.

## 2. Notations and Preliminaries

Throughout this paper, $\Omega$ will be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ with smooth boundary $\partial\Omega$, $\nu$ its outer normal vector defined every where. The real-valued functions $m$ and $g$ will always be assumed to belong in $C^r(\partial\Omega)$, for some $0 < r < 1$ and the real-valued function $f$ is defined as follows $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,s) := \begin{cases} h(x,s) & \text{if } s \geq 0 \\ m(x)|s|^{p-2}s & \text{if } s < 0, \end{cases} \tag{4}$$

with $m : \partial\Omega \to (0, +\infty)$ and $h : \partial\Omega \times [0, +\infty] \to \mathbb{R}$ is a Caratheodory function, that is, $h$ is measurable in $x \in \Omega$ for all $s \in \mathbb{R}$ and continuous in $s \in \mathbb{R}$; and $h$ satisfying the following hypothesis:

$(H_1)$ • there exists a positive constant $\kappa \in (0,1)$ such that $|h(x,s)| \leq \kappa m(x) s^{p-1}$ for any $s \geq 0$ and a.e. $x \in \Omega$;

$(H_2)$• there exists $s_0 > 0$ such that $H(x, s_0) := \int_0^{s_0} h(x,t)\,\mathrm{d}t > 0$, for a.e. $x \in \Omega$;

$(H_3)$ • $\lim_{s\to+\infty} \dfrac{h(x,s)}{s^{p-1}} = 0$, uniformly in $x$.

Throughout this work, we denote by $W^{1,p}(\Omega), 1 < p < \infty$, the classical Sobolev space endowed with its natural norm

$$\|u\|_{1,p} := \left( \int_\Omega \left( |\nabla u|^p + |u|^p \right) \right)^{\frac{1}{p}}.$$

The Lebesgue norm of $L^p(\Omega)$ will be denoted by $\|.\|_p$, and the one of $L^p(\partial\Omega)$ by $\|\cdot\|_{p,\partial\Omega}$, for any $p \in [1, +\infty]$. If $S \subset \mathbb{R}^N$ is measurable set, $|S|$ denotes the Lebesgue measure of S. Let $u \in W^{1,p}(\Omega)$, we denote by

$$u^+(x) := \max\{u(x), 0\},\ u^-(x) := \max\{-u(x), 0\},\ \forall x \in \Omega;$$

$$\partial\Omega^+ = \partial\Omega \cap \{u > 0\},\ \partial\Omega^- = \partial\Omega \cap \{u < 0\}.$$

Here we will denote by $p^* := \dfrac{Np}{(N-p)^+}$ the classical critical Sobolev's exponent and by $p_* := \dfrac{(N-1)p}{(N-p)^+}$ the critical Sobolev's exponent for the trace inclusion.

We are interested in the weak solutions of $(\mathcal{P}_{\lambda,g})$, *i.e.*, functions $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |u|^{p-2} uv = \lambda \int_{\partial\Omega} f(x,u)v + \int_{\partial\Omega} gv,$$

holds for all $v \in W^{1,p}(\Omega)$.

The following proposition give results on regularity of weak solutions of $(\mathcal{P}_{\lambda,g})$.

**Proposition 1.** Let $u \in W^{1,p}(\Omega)$ be a solution of problem $(\mathcal{P}_{\lambda,g})$. Then

i) $u \in L^\infty(\Omega)$;

ii) $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$ and there exists a constant $C > 0$, depending on $\|u\|_{1,p}, \|m\|_\infty, p, \Omega$, such that $\|u\|_{C^{1,\alpha}(\bar{\Omega})} \le C$.

*Proof.* Using hypothesis ($H_1$) on $h$ and assumptions on $m$ and $f$, the results follow [12] [13]. One can also see [14] [15]. □

The following are results for the strong maximum principle.

**Proposition 2.** Let $u \in W^{1,p}(\Omega)$ be a weak solution of $(\mathcal{P}_{\lambda,g})$ such that $u \ge 0$ with $g \ge 0$, $g \not\equiv 0$. Then $u > 0$ in $\bar{\Omega}$.

*Proof.* Let u be a solution of $(\mathcal{P}_{\lambda,g})$ with $u \ge 0$. By the Harnack's inequality, Theorems 5, 6 in [16] and by Hopf maximum principle, see [17], it follows that $u > 0$ a.e. in $\bar{\Omega}$. □

## 3. On Existence of Eigenvalues

In this section, we consider the eigenvalue-type problem involving the p-Laplacian $(\mathcal{P}_{\lambda,g})$ with $g \equiv 0$ given by

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

By définition, a real value $\lambda$ is said to be an eigenvalue of problem $(\mathcal{P}_\lambda)$ if there exists $u \in W^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |u|^{p-2} uv = \lambda \int_{\partial\Omega} f(x,u)v, \ \forall v \in W^{1,p}(\Omega). \tag{5}$$

Then u from above equation will be called an eigenfunction corresponding to the eigenvalue $\lambda$.

Using the definition (4) of function $f$, problem $(\mathcal{P}_\lambda)$ is equivalent to

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda \left[ h(x,u^+) - m(x)(u^-)^{p-1} \right] & \text{on } \partial\Omega. \end{cases} \tag{6}$$

We say that a real value $\lambda$ is an eigenvalue of problem $(\mathcal{P}_\lambda)$ or equivalent version (6) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v+\int_\Omega |u|^{p-2}uv=\lambda\int_{\partial\Omega}\left[h\left(x,u^+\right)-m\left(u^-\right)^{p-1}\right]v,\ \forall v\in W^{1,p}(\Omega). \quad (7)$$

Moreover, $u$ from equation (7) will be called an eigenfunction corresponding to the eigenvalue $\lambda$.

Let $u\in W^{1,p}(\Omega)$, we know by Lemma B.2 in [2] that $u^+,u^-,|u|$ are in $W^{1,p}(\Omega)$ and

$$\nabla u^+=\begin{cases}\nabla u & \text{if } u>0,\\ 0 & \text{if } u\le 0,\end{cases}$$

$$\nabla u^-=\begin{cases}0 & \text{if } u\ge 0,\\ \nabla u & \text{if } u<0,\end{cases}$$

$$\nabla|u|=\begin{cases}\nabla u & \text{if } u>0,\\ 0 & \text{if } u=0,\\ -\nabla u & \text{if } u<0.\end{cases}$$

Furthermore, $\left(u|_{\partial\Omega}\right)^+=u^+|_{\partial\Omega}$ and $\left(u|_{\partial\Omega}\right)^-=u^-|_{\partial\Omega}$.

Consequently, the weak formulation (7) of problem (6) can be reformulated as follows: a real number $\lambda$ is an eigenvalue of (6) if and only if there exists $u\in W^{1,p}(\Omega)\setminus\{0\}$ such that

$$\begin{aligned}&\int_\Omega\left(\left|\nabla u^+\right|^{p-2}\left|\nabla u^-\right|^{p-2}\right)\nabla u\nabla v+\int_\Omega\left[\left(u^+\right)^{p-2}+\left(u^-\right)^{p-2}\right]uv\\ &=\lambda\int_\Omega\left[h\left(x,u^+\right)-m\left(u^-\right)^{p-1}\right]v,\ \forall v\in W^{1,p}(\Omega).\end{aligned} \quad (8)$$

The following theorem is the main result of this section.

**Theorem 3.** Let $m\in C^r(\partial\Omega)$, for some $0<r<1$, $f$ be the function given by (4) and $h$ satisfying the assumptions ($H_1$)-($H_3$). Then $\lambda_1(m)$ defined by (2) is an isolated eigenvalue of $(\mathcal{P}_\lambda)$. Moreover any $\lambda\in(0,\lambda_1(m))$ is not an eigenvalue of $(\mathcal{P}_\lambda)$ but there exists $\Lambda_1(m)>\lambda_1(m)$ such that any $\lambda\in(\Lambda_1(m),+\infty)$ is an eigenvalue of $(\mathcal{P}_\lambda)$.

Theoreom 3 will be proved using Lemmas 4 - 8.

**Lemma 4.** No real number $\lambda\in(0,\lambda_1(m))$ is an eigenvalue of problem $(\mathcal{P}_\lambda)$ (which is equivalent to problem (6)).

*Proof.* Assume, by contradiction, that $\lambda$ with $\lambda\in(0,\lambda_1(m))$ is an eigenvalue of problem (6). Thus, by choosing $v=u^+$ and $v=u^-$ in (8), we get the following equalities

$$\int_\Omega\left(\left|\nabla u^+\right|^p+\left(u^+\right)^p\right)=\lambda\int_{\partial\Omega}h\left(x,u^+\right)u^+; \quad (9)$$

and

$$\int_\Omega\left(\left|\nabla u^-\right|^p+\left(u^-\right)^p\right)=\lambda\int_{\partial\Omega}m\left(u^-\right)^p. \quad (10)$$

Using (H$_1$), the definition (2) of $\lambda_1(m)$ and relation (9), we get

$$\lambda_1(m)\int_{\partial\Omega} m(u^+)^p \le \int_\Omega \left(\left|\nabla u^+\right|^p + (u^+)^p\right) = \lambda\int_{\partial\Omega} h(x,u^+)u^+ \le \lambda\int_{\partial\Omega} m(u^+)^p. \quad (11)$$

Similary, using definition (2) of $\lambda_1(m)$ and relation (10), we get

$$\lambda_1(m)\int_{\partial\Omega} m(u^-)^p \le \int_\Omega \left(\left|\nabla u^-\right|^p + (u^-)^p\right) = \lambda\int_{\partial\Omega} m(u^-)^p. \quad (12)$$

Since $\lambda$ is an eigenvalue for problem (6), then $u$ must not vanish everywhere in $\Omega$. We have, either, $u^+ \not\equiv 0$, or $u^- \not\equiv 0$. With (11) and (12), if $\lambda$ is an eigenvalue of (6), then we must have $\lambda > \lambda_1(m)$. And the result follows. □

**Lemma 5.** The real value $\lambda_1(m)$ is an eigenvalue of problem $(\mathcal{P}_\lambda)$ (or equivalent problem (6)).

*Proof.* Since $\lambda_1(m)$ is the first positive eigenvalue of (1) with $m(x) > 0$ for any $x \in \partial\Omega$, then it is simple and principal, and the eigenfunction associated to $\lambda_1(m)$ does not change his sign over $\Omega$, see [3] [5] [6]. Then there exists a function $\varphi \in W^{1,p}(\Omega)$, with $\varphi(x) < 0$ for any $x \in \Omega$ such that

$$\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla v + \int_\Omega |\varphi|^{p-2}\varphi v = \lambda_1(m)\int_{\partial\Omega} m|\varphi|^{p-2}\varphi v, \forall v \in W^{1,p}(\Omega);$$

which is equivalent to

$$\int_\Omega \left|\nabla\varphi^-\right|^{p-2}\nabla\varphi^-\nabla v + \int_\Omega \left|\varphi^-\right|^{p-2}\varphi^- v = \lambda_1(m)\int_{\partial\Omega} m\left|\varphi^-\right|^{p-2}\varphi^- v, \forall v \in W^{1,p}(\Omega).$$

We deduce from the last equation that the relation (8) holds thrue with $u = \varphi$ and $\lambda = \lambda_1(m)$. Consequently, $\lambda_1(m)$ is an eigenvalue of problem (6) and the result follows. □

**Lemma 6.** $\lambda_1(m)$ is an isolated eigenvalue of problem $(\mathcal{P}_\lambda)$ or equivalent problem (6).

*Proof.* By Lemma 4, $\lambda_1(m)$ is isolated to the left. It remains to show that $\lambda_1(m)$ is isolated to the right. Let take $\lambda > \lambda_1(m)$ be an eigenvalue of (6) and $u \in W^{1,p}(\Omega)$ its corresponding eigenfunction. We assume that $u^+$ does not vanish everywhere in $\Omega$. Then using (H$_1$), (2) and relation (9), we get

$$\lambda_1(m)\int_{\partial\Omega} m(u^+)^p \le \int_\Omega \left(\left|\nabla u^+\right|^p + (u^+)^p\right) = \lambda\int_{\partial\Omega} h(x,u^+)u^+ \le \kappa\lambda\int_{\partial\Omega} m(u^+)^p.$$

Using the fact that $\kappa \in (0,1)$, we realize that $\lambda_1(m) < \lambda$ in case $u^+ \not\equiv 0$. It follows that for $\lambda \in \left(\lambda_1(m), \frac{\lambda_1(m)}{\kappa}\right)$ is an eigenvalue of (6) if $u^+ \equiv 0$. And then if $\lambda \in \left(\lambda_1(m), \frac{\lambda_1(m)}{\kappa}\right)$ is an eigenvalue of (6), is also an eigenvalue of (1) with corresponding eigenfunction negative in $\Omega$. But it well know that $\lambda_1(m)$ is the only principal eigenvalue of problem (1), and there exists $\delta > 0$ such that the interval $(\lambda_1(m), \lambda_1(m) + \delta)$ can not contain an eigenvalue of (1). And so, any $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$ can not be an eigenvalue of problem (6) with

$\delta=\min\left\{\frac{\lambda_1(m)}{\kappa}-\lambda_1(m),\lambda_2(m)-\lambda_1(m)\right\}$, where $\lambda_2(m)$ is the second eigenvalue of (1) given by (3). □

In the following, we consider the eigenvalue problem given by

$$\begin{cases}-\Delta_p u+|u|^{p-2}u=0 & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}=\lambda h(x,u^+) & \text{on } \partial\Omega.\end{cases} \tag{13}$$

By définition, $\lambda$ is said to be an eigenvalue of (13) if and only if there exists $u\in W^{1,p}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega|\nabla u|^{p-2}\nabla u\nabla v+\int_\Omega|u|^{p-2}uv=\lambda\int_{\partial\Omega}h(x,u^+)v,\ \forall v\in W^{1,p}(\Omega). \tag{14}$$

Moreover, $u$ from the above relation will be called an eigenfunction associated to the eigenvalue $\lambda$.

Observely, $\lambda$ is greather than $\lambda_1(m)$. We notice that if $\lambda$ is an eigenvalue of (13), the associated eigenfunction u is positive. Indeed, take $v=u^-$ in (14), we have:

$$\int_\Omega\left(|\nabla u^-|^p+(u^-)^p\right)=-\lambda\int_{\partial\Omega}h(x,u^+)u^-.$$

And then, one has

$$\|u^-\|_{1,p}^p=-\lambda\int_{\partial\Omega}h(x,u^+)u^-=0.$$

We deduce $u^-\equiv 0$ and thus $u\geq 0$. Consequently, we conclude that the eigenvalues of problem (13) admits nonnegative corresponding eigenfunctions. According to the above discussion one get that an eigenvalue of problem (13) is also an eigenvalue of problem (6).

For each $\lambda>0$, the energy functional associated to the problem (13) is given by

$$\Phi_\lambda(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p+|u|^p\right)-\lambda\int_{\partial\Omega}H(x,u^+),$$

with $H(x,s)=\int_0^s h(x,t)\mathrm{d}t$. It well-known that $\Phi_\lambda\in C^1\left(W^{1,p}(\Omega),\mathbb{R}\right)$ and its derivative is given by

$$\langle\Phi_\lambda'(u),v\rangle=\int_\Omega|\nabla u|^{p-2}\nabla u\nabla v+\int_\Omega|u|^{p-2}uv-\lambda\int_{\partial\Omega}h(x,u^+)v,\ \forall v\in W^{1,p}(\Omega).$$

Then $\lambda>0$ is an eigenvalue of problem (13) if and only if the corresponding eigenfunction $u\in W^{1,p}(\Omega)$ is a critical nontrivial point of functional $\Phi_\lambda$.

In order to show the existence of solution, we prove the following lemma

**Lemma 7.** $\Phi_\lambda$ is bounded below and coercive.

*Proof.* From assumption ($H_3$), we deduce that

$$\lim_{s\to+\infty}\frac{H(x,s)}{m(x)s^p}=0,\ \text{uniformly in } \Omega.$$

Then, for fixed $\lambda>0$, there exists a positive constant $C_\lambda$ such that

$$\lambda H(x,s) \le \frac{\lambda_1(m)}{2p} m(x) s^p + C_\lambda m(x), \ \forall s \ge 0 \text{ a.e. } x \in \Omega.$$

Here $\lambda_1(m)$ is given by (2). Consequently, for each $u \in W^{1,p}(\Omega)$ we get from (2) and the above relation that:

$$\begin{aligned}\Phi_\lambda(u) &\ge \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) - \frac{\lambda_1(m)}{2p}\int_{\partial\Omega} m u^p - C_\lambda \int_{\partial\Omega} m \\ &\ge \frac{1}{2p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) - C_\lambda \int_{\partial\Omega} m \\ &\ge \beta \|u\|_{1,p}^p - C_\lambda \|m\|_\infty |\partial\Omega|;\end{aligned}$$

where $\beta = \frac{1}{2p}$. The last relation show that $\Phi_\lambda$ is bounded below and coercive. □

The proof of the following lemma follows the proof of Lemma 5. in [7].

**Lemma 8.** There exists a real value $\Lambda_1 > 0$ such that, assumming $\lambda > \Lambda_1$, we have

$$\inf_{W^{1,p}(\Omega)} \Phi_\lambda < 0.$$

*Proof.* The hypothesis ($H_2$) affirms that there exists $s_0 > 0$ such that $H(x, s_0) > 0$. Let $\partial\Omega_1$ be a compact subset of $\partial\Omega$ large enough and $u_0 \in C^1(\Omega) \subset W^{1,p}(\Omega)$ such that $u_0(x) = s_0$ for any $x \in \partial\Omega_1$ and $0 \le u_0(x) \le s_0$ for any $x \in \partial\Omega_1^C$ where $\partial\Omega_1^C = \partial\Omega \setminus \partial\Omega_1$. We have

$$\begin{aligned}\int_{\partial\Omega} H(x,u_0) &\ge \int_{\partial\Omega_1} H(x,s_0) - \int_{\partial\Omega_1^C} \kappa m u_0^p \\ &\ge \int_{\partial\Omega_1} H(x,s_0) - C s_0^p \left|\partial\Omega_1^C\right| > 0,\end{aligned}$$

with $C = \kappa \|m\|_\infty$. Then, we infer that

$$\Phi_\lambda(u_0) \le \frac{1}{p}\|u_0\|_{1,p}^p - \lambda\left(\int_{\partial\Omega_1} H(x,s_0) - C s_0^p \left|\partial\Omega_1^C\right|\right) < 0$$

as soon as

$$\lambda > \frac{\frac{1}{p}\|u_0\|_{1,p}^p}{\left(\int_{\Omega_1} H(x,s_0) - C s_0^p \left|\partial\Omega_1^C\right|\right)}.$$

We deduce the existence of a positive constant $\Lambda_1 > 0$ such that for any $\lambda > \Lambda_1$ we have $\inf_{W^{1,p}(\Omega)} \Phi_\lambda < 0$. □

Lemma 7 and Lemma 8 show that for $\lambda > 0$ sufficiently large, the functional $\Phi_\lambda$ possesses a negative global minimum and thus for any $\lambda > \Lambda_1$ it attains its infimum in $W^{1,p}(\Omega)$, (see [18], Theorem 1.2). Consequently, we deduce that any $\lambda > 0$ large enough is an eigenvalue of problem (13) and consequently of problem (6). Joining that result and Lemmas 4, 5, 6, we conclude the proof of Theorem 3.

## 4. On Maximum Principle-Type

This section treats the existence of positive solution of the problem

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda f(x,u) + g(x) & \text{on } \partial\Omega, \end{cases}$$

which is equivalent to

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta_p u + |u|^{p-2} u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda \left[ h(x,u^+) - m(x)(u^-)^{p-1} \right] + g(x) & \text{on } \partial\Omega. \end{cases}$$

Solutions of problem $(\mathcal{P}_{\lambda,g})$ are understood in weak sens, *i.e.*, function $u \in W^{1,p}(\Omega)$ is a solution of $(\mathcal{P}_{\lambda,g})$ if it satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |u|^{p-2} uv = \lambda \int_{\partial\Omega} f(x,u) v + \int_{\partial\Omega} gv, \ \forall v \in W^{1,p}(\Omega), \tag{15}$$

or equivalent

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |u|^{p-2} uv \\ &= \lambda \int_{\partial\Omega} \left[ h(x,u^+) - m(u^-)^{p-1} \right] v + \int_{\partial\Omega} gv, \ \forall v \in W^{1,p}(\Omega). \end{aligned} \tag{16}$$

In the following, we prove the main results of this section.

**Theorem 9.** Assume that $g \geq 0$, $g \not\equiv 0$ and $\lambda \in (0, \lambda_1(m)]$. Then every solution of problem $(\mathcal{P}_{\lambda,g})$ is positive (maximum principle).

*Proof.* Assume by contradiction that $u^- \not\equiv 0$ and take $v = u^-$ as test function in $(\mathcal{P}_\lambda, g)$. We have

$$\begin{aligned} \int_\Omega \left( |\nabla u^-|^p + (u^-)^p \right) &= \lambda \left( -\int_{\partial\Omega} h(x,u^+) u^- + \int_{\partial\Omega} m(u^-)^p \right) - \int_{\partial\Omega} gu^- \\ &= \lambda \int_{\partial\Omega} m(u^-)^p - \int_{\partial\Omega} gu^-. \end{aligned} \tag{17}$$

We distinguish two cases:

In first time, we consider $\lambda \in (0, \lambda_1(m))$.

If $\int_{\partial\Omega} m(u^-)^p > 0$, then we deduce from definition 2 of $\lambda_1(m)$ and relation 17 that

$$(\lambda_1(m) - \lambda) \int_{\partial\Omega} m(u^-)^p \leq -\int_{\partial\Omega} gu^- \leq 0$$

which implies that $\lambda_1(m) \leq \lambda$, a contradiction.

If $\int_{\partial\Omega} m(u^-)^p \leq 0$, it follows from equality 17 that for any $\lambda \in (0, \lambda_1(m))$

$$0 \leq \int_\Omega \left( |\nabla u^-|^p + (u^-)^p \right) = \lambda \int_{\partial\Omega} m(u^-)^p - \int_{\partial\Omega} gu^- \leq 0.$$

And then, we deduce that $u^- \equiv 0$, a contradiction.

In the second time, we consider the case of $\lambda = \lambda_1(m)$. It follows from definition 2 of $\lambda_1(m)$ and relation 17 that

$$0 \leq \int_\Omega \left( |\nabla u^-|^p + (u^-)^p \right) - \lambda \int_{\partial\Omega} m(u^-)^p = -\int_{\partial\Omega} gu^- \leq 0,$$

which implies that $\int_{\partial\Omega} gu^- = 0$ and consequently $u^-$ is an eigenfunction associated to $\lambda_1(m)$ and so $u^- > 0$, a contradiction. Hence, in all cases, we get that $u \ge 0$ and the conclusion follows from the Proposition 2. □

**Theorem 10.** Assume that $g \ge 0$, $g \not\equiv 0$. Then the problem $(\mathcal{P}_{\lambda,g})$ admits a positive solution if $0 < \lambda \le \lambda_1(m)$.

*Proof.* In order to prove the existence of solution of $(\mathcal{P}_{\lambda,g})$ for $\lambda \in (0, \lambda_1(m)]$, we consider the energy functional $\phi_\lambda$ associated to $(\mathcal{P}_{\lambda,g})$ given by

$$\phi_\lambda(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) - \lambda\left[\int_{\partial\Omega} H(x,u^+) - \int_{\partial\Omega} m(u^-)^p\right] - \int_{\partial\Omega} gu;$$

with $H(x,s) = \int_0^s h(x,t)\,\mathrm{d}t$. From Theorem 9 we know that every solution of $(\mathcal{P}_{\lambda,g})$ with $\lambda \in (0, \lambda_1(m)]$ is positive. Consequently, the above energy functional $\phi_\lambda$ is equivalent to $\Psi_\lambda$ given by

$$\Psi_\lambda(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) - \lambda\int_{\partial\Omega} H(x,u^+) - \int_\Omega gu.$$

It well know that $\Psi_\lambda \in C^1\left(W^{1,p}(\Omega), \mathbb{R}\right)$ and its derivative is given by

$$\begin{aligned}\langle \Psi'_\lambda(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2} uv\right) \\ &\quad - \lambda\int_{\partial\Omega} h(x,u^+)v - \int_\Omega gv, \ \forall u,v \in W^{1,p}(\Omega).\end{aligned}$$

Then, for each $\lambda \in (0, \lambda_1(m)]$, $u \in W^{1,p}(\Omega)$ is solution of problem $(\mathcal{P}_{\lambda,g})$ if and only if it is a nontrivial critical point of functional $\Psi_\lambda$.

Let us prove that $\Psi_\lambda$ is coercive and weakly lower semicontinuous. From (2) and ($H_3$) one has

$$\begin{aligned}\Psi_\lambda(u) &= \frac{1}{p}\int_\Omega \left(|\nabla u|^p + u^p\right) - \lambda\int_{\partial\Omega} H(x,u^+) - \int_{\partial\Omega} gu \\ &\ge \frac{1}{p}\int_\Omega \left(|\nabla u|^p + u^p\right) - \frac{\lambda_1(m)}{2p}\int_{\partial\Omega} mu^p - C_\lambda \int_{\partial\Omega} m - \|g\|_\infty \|u\|_{1,\partial\Omega} \qquad (18) \\ &\ge c_1\|u\|_{1,p}^p - C_\lambda \|m\|_\infty |\partial\Omega| - \|g\|_\infty \|u\|_{1,p}\end{aligned}$$

where $c_1 = \dfrac{1}{2p}$. The last inequaly show that $\Psi_\lambda$ is coercive.

Since $u \mapsto h(.,u)$, $u \mapsto \int_{\partial\Omega} mu^p$ and $u \mapsto \int_{\partial\Omega} gu$ are continuous, it follows from the weakly lower semicontinuity of $u \mapsto \int_\Omega \left(|\nabla u|^p + u^p\right)$ that the functional $\Psi_\lambda$ is weakly lower semicontinuous. And then, we get the existence of, at least, one solution. □

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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