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Mixed Soliton Solutions of MNLS/DNLS Equations Based on Hirota Method

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Abstract

Using Hirota's bilinear derivative method to derive single-breather solutions for the modified nonlinear Schrödinger (MNLS) equation and the derivative nonlinear Schrödinger (DNLS) equation under non-vanishing boundary conditions, along with explicit mixed solutions combining breather-type and pure solitons. The collision dynamics between pure and breather-type solitons in a mixed solution has been graphically demonstrated and analyzed. Furthermore, by setting specific parameter to zero, we naturally obtain corresponding single-breather solution and its explicit mixed solutions with pure solitons for the DNLS equation. The mixed soliton solution can asymptotically degenerate into a simple algebraic summation of a simple pure soliton and a breather in the infinite past or the infinite future, which was graphically validated.

Keywords

MNLS Equation, DNLS Equation, Nonlinear Equation, Hirota's Bilinear Derivative Transform, Soliton, Breather

1. Introduction

The modified nonlinear Schrödinger (MNLS) equation

$$iu_t + u_{xx} + i\left(|u|^2 u\right)_x + 2\gamma |u|^2 u = 0 \tag{1}$$

and the derivative nonlinear Schrödinger (DNLS) equation

$$iu_t + u_{xx} + i\left(|u|^2 u\right)_x = 0 \tag{2}$$

are two very important and fully integrable nonlinear models that are closely related and gauge equivalent to each other. Under zero or non-zero boundary conditions, the two equations have broad applications in Alfven solitary wave in plasma physics [1]-[3], optical soliton theory in single-mode fiber transmission [4]-[6], and weak nonlinear electromagnetic wave in (anti)ferromagnetic media or dielectrics under external magnetic fields [7] [8]. Obviously, DNLS Equation (2) can be regarded as the limit case of MNLS Equation (1) as the parameter γ tends to zero. Therefore, combining these two equations for discussion can solve them once for all, and has the effect of yielding twice result with half effort, and killing two birds with one stone.

Strictly integrable nonlinear equations mean that they have some exact analytical solutions. In the zero boundary case, D. C. Kaup and A. C. Newell (1978) first proposed an inverse scattering transform (IST) method to obtain the single soliton solution of Equation (2) [9] [10]; Ref. [11] (1990) also obtained the soliton solution of Equation (2) using the Darboux transform. Subsequently, various methods for solving the DNLS equation emerged, such as the Bäcklund transform [12], improved IST method [13], Hirota bilinear derivative transform [14], Marchenko method [15], and so on. For the DNLS equation under non-zero boundary conditions, in order to overcome the difficulty of integrating multi-value functions on the Riemann surface, Ref. [16]-[19] improved the IST method by introducing an affine parameter to avoid multi-value problems. Ref. [20] [21] applied this improved IST method and several linear algebraic techniques to attain different kinds of soliton solutions such as breathers and pure solitons, and their mixed solutions for the DNLS equation under non-zero boundary condition.

Compared to the complex and intricate calculation process of the IST method, Hirota's bilinear derivative transformation (HBDT for brevity, or Hirota method) is a direct method which combines perturbation theory with truncation technique of bilinear exponential functions. In recent years, this method has been widely used in the study of nonlinear equations and soliton theory, and has shone brilliantly [22]-[26]. This paper chose Hirota method to search for the special mixed soliton solutions of MNLS/DNLS equation. Unlike the usual solution forms taken by other scholars, in this paper, a typical form of soliton solutions was deliberately chosen for the MNLS/DNLS equation (Equation (4) below), based upon which the HBDT was applied, greatly simplifying the expression of two functions, f, g with two variables. Using this typical form of solution to implement HBDT, reference [14] obtained single soliton, double soliton, and even multi-soliton solution of the DNLS equation under zero boundary condition; References [22] [23] used Hirota method to obtain spatial periodic solutions of the DNLS/MNLS equations under the background of a simple-harmonic wave, and further got a rogue-wave solution in the long-wave limit; Reference [24] used Hirota method to obtain a simple soliton solution of the DNLS/MNLS equation under non-zero boundary condition. Ref. [24] [25] specifically pointed out that the breather solution of MNLS cannot satisfy a complex constant boundary condition, but a linear exponential function of x at infinity. On the contrary, the breather solution of the DNLS equation only satisfies a complex constant boundary condition at infinity. The key reason is that the gauge transformation between the solutions and boundary conditions of the DNLS and MNLS equations contains linear exponential functions [13] [17]. Ref. [25] further obtained the bright/dark pure 2-soliton solutions and soliton anti-oliton pair solutions of the DNLS/MNLS equation under non-zero boundary condition.

On the basis of the previous work, this paper still chooses the typical form of soliton solutions for the MNLS/DNLS equations, applies Hirota bilinear derivative transformation, and directly obtains the explicit mixed solutions of single breathers and pure solitons for these two equations. We can especially observe a fact that this mixed solution can gradually degenerate into a linear superposition of breather and pure soliton solutions, thus intuitively demonstrating the elastic collision process among solitons and the independent propagation of each soliton in the mixed solution.

The paper is organized as follows: Chapter 2 briefly introduces Hirota's theory of bilinear derivative transformation; Chapter 3 discusses the process of searching for the breather solutions of MNLS/DNLS based on Hirota method, and in Chapter 4, we derived further the mixed solution of breathers and pure solitons for MNLS/DNLS; Finally, Chapter 5 discusses and summarizes the mixed solution.

2. Hirota's Bilinear Derivative Transformation

Japanese scholar Hirota was the first to introduce bilinear derivative transformation, with the initial motivation of dealing with some bivariate partial differential equations, and then applied to solve soliton solutions of bivariate nonlinear integrable equations [27]-[29]. Operator D represents a derivative transformation, which is defined as

$$\mathrm{D}_t^m \mathrm{D}_x^n f(x,t) \bullet g(x,t) = (\partial/\partial t - \partial/\partial t')^m (\partial/\partial x - \partial/\partial x')^n f(x,t) g(x',t')\Big|_{t'=t,x'=x} \quad (3)$$

where the operation symbol $\bullet$ represents the ordered product between two functions $f(x,t)$ and $g(x,t)$, which can be omitted when $m=n=1$. The detailed calculation rules and examples can be found in references [14] [23]-[25] [29]. In addition, in order to search for soliton solutions by use of Hirota's method, we need to transform the nonlinear partial differential equation to be a bilinear form of equation containing only operator D, and then decompose it into several independent bilinear equations, which may require introduction of non-zero parameters.

Unlike the fractional or logarithmic forms commonly used in other integrable models, for the MNLS/DNLS equation, the typical solution form is chosen in this paper just as that used in Res. [18]-[25]:

$$u(x,t) = g\,\overline{f}/f^2 \quad (4)$$

here $\overline{f}$ is the complex conjugate of function $f(x,t)$; and $g(x,t)$. Apply Hirota's bilinear derivative transformation to MNLS Equation (1) and use the above solution form, and decompose it into the following bilinear-form equation group [23]-[25]

$$\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda\right) g \bullet f = 0 \tag{5}$$

$$\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda\right) f \bullet \overline{f} = 2\gamma g\overline{g} \tag{6}$$

$$\mathrm{D}_x f \bullet \overline{f} = (i/2) g\overline{g} \tag{7}$$

To deal with the non-zero boundary condition, we need to introduce a non-zero parameter λ which depend on the specific form of the functions f, g as well as the complex constant boundary condition at infinity.

3. Breather Solution of MNLS/DNLS Equation

Now expand the functions f and g to be infinite series with a perturbation parameter ϵ as follows

$$f = \sum_{i=0}^{\infty} \epsilon^i f^{(i)}, \quad g = \sum_{i=0}^{\infty} \epsilon^i g^{(i)} \tag{8}$$

Substitute expressions (8) into the system of Equations (5)-(7), compare the coefficients of the same power terms ϵ^n on both sides of each equation, and obtain

$$\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda\right)\left(\sum_{j=0}^{n} g^{(j)} \bullet f^{(n-j)}\right) = 0 \tag{9}$$

$$\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda + 4i\gamma\mathrm{D}_x\right)\left(\sum_{j=0}^{n} f^{(j)} \bullet \overline{f^{(n-j)}}\right) = 0 \tag{10}$$

$$\mathrm{D}_x\left(\sum_{j=0}^{n} f^{(j)} \bullet \overline{f^{(n-j)}}\right) = (i/2)\left(\sum_{j=0}^{n} g^{(j)} \bullet \overline{g^{(n-j)}}\right) \tag{11}$$

Thus, the MNLS equation is transformed into a bilinear form, namely the system of Equations (9)-(11) mentioned above. For any given positive integer n, there are a total of $6n + 3$ equations in above equation group.

4. Process of Searching for Breather Solutions

Hirota method has been succeeded in searching for one or two pure-soliton solution of MNLS/DNLS equation under non-zero boundary condition in ref. [24] [25]. This paper only focuses on the 1 + 1 type of mixed breather and pure soliton by means of Hirota method. The discussion on the solution of the DNLS equation in ref. [16] [20] [21] [24] provide us with much inspiration. Its 1-pure soliton solution is as follows:

$$u = \rho\left(1 + \mathrm{e}^{i3\xi_1^{(0)}} \mathrm{e}^{\eta}\right)\left(1 + \mathrm{e}^{-i\xi_1^{(0)}} \mathrm{e}^{\eta}\right) \Big/ \left(1 + \mathrm{e}^{i\xi_1^{(0)}} \mathrm{e}^{\eta}\right)^2 \tag{12}$$

Reference [24] specifically discusses and points out that $\mathrm{e}^{i\xi_1^{(0)}}$ in the above equation corresponds to the pole in the integration circuit in the IST method, while according to references [16] [20] [21], the 1-pure soliton solution has only two symmetrical poles, the double pure soliton solution has four poles, and similarly, the single breather solution also has four poles. Although the arrangement of the poles is not the same on the contour integral for the two cases—double soliton case and the breather case, this gives us great inspiration that the single breath solution should have a mathematical structure similar to that of the double soliton solution. In fact, a careful observation of the process of obtaining the n-breather solution and m-pure soliton solution using the IST method in references [21] [30] also confirms our above discussion and speculation. However, due to the fact that the two pairs of poles of the IST coefficient corresponding to the breather are two pairs of geometric inversion points about a circle, describing its solution requires three additional constraints compared to the conventional double soliton solution, which will be discussed later. Below are brief steps for exploring breather solutions. Firstly, set truncation conditions

$$f^{(i)}=0,\ g^{(i)}=0,(i>2) \tag{13}$$

Therefore, from the obtained equation group (9)-(11), we can obtain a total of 15 bilinear derivative sub equations of MNLS, which are listed as follows:

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)f^{(0)}\cdot\overline{f^{(0)}}=0 \tag{14.a}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)\left(f^{(0)}\cdot\overline{f^{(1)}}+f^{(1)}\cdot\overline{f^{(0)}}\right)=0 \tag{14.b}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)\left(f^{(0)}\cdot\overline{f^{(2)}}+f^{(1)}\cdot\overline{f^{(1)}}+f^{(2)}\cdot\overline{f^{(0)}}\right)=0 \tag{14.c}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)\left(f^{(1)}\cdot\overline{f^{(2)}}+f^{(2)}\cdot\overline{f^{(1)}}\right)=0 \tag{14.d}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)f^{(2)}\cdot\overline{f^{(2)}}=0 \tag{14.e}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda\right)g^{(0)}\cdot f^{(0)}=0 \tag{15.a}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda\right)\left(g^{(0)}\cdot f^{(1)}+g^{(1)}\cdot f^{(0)}\right)=0 \tag{15.b}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda\right)\left(g^{(0)}\cdot f^{(2)}+g^{(1)}\cdot f^{(1)}+g^{(2)}\cdot f^{(0)}\right)=0 \tag{15.c}$$

$$\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda\right)\left(g^{(1)}\cdot f^{(2)}+g^{(2)}\cdot f^{(1)}\right)=0 \tag{15.d}$$

$$\mathrm{D}_x f^{(2)}\cdot\overline{f^{(2)}}=(i/2)g^{(2)}\overline{g^{(2)}} \tag{15.e}$$

$$\mathrm{D}_x f^{(0)}\cdot\overline{f^{(0)}}=(i/2)g^{(0)}\overline{g^{(0)}} \tag{16.a}$$

$$\mathrm{D}_x\left(f^{(0)}\cdot\overline{f^{(1)}}+f^{(1)}\cdot\overline{f^{(0)}}\right)=(i/2)\left(g^{(0)}\overline{g^{(1)}}+g^{(1)}\overline{g^{(0)}}\right) \tag{16.b}$$

$$\mathrm{D}_x\left(f^{(0)}\cdot\overline{f^{(1)}}+f^{(1)}\cdot\overline{f^{(1)}}+f^{(2)}\cdot\overline{f^{(0)}}\right)=(i/2)\left(g^{(0)}\overline{g^{(2)}}+g^{(1)}\overline{g^{(1)}}+g^{(2)}\overline{g^{(0)}}\right) \tag{16.c}$$

$$\mathrm{D}_x\left(f^{(1)}\cdot\overline{f^{(2)}}+f^{(2)}\cdot\overline{f^{(1)}}\right)=(i/2)\left(g^{(1)}\overline{g^{(2)}}+g^{(2)}\overline{g^{(1)}}\right) \tag{16.d}$$

$$\mathrm{D}_x f^{(2)}\cdot\overline{f^{(2)}}=(i/2)g^{(2)}\overline{g^{(2)}} \tag{16.e}$$

For the zero-order expansion terms of two functions f,g, *i.e.* the ground state, considering Hirota method, the key method for seeking soliton solutions of non-linear integrable equations is to expand the unknown functions into a perturbation series with coefficients of linear exponential functions. Without loss of generality, we choose $f^{(0)}, g^{(0)}$ to be of following form

$$f^{(0)} = \mathrm{e}^{i\xi_0},\ \xi_0 = \omega_0 t + k_0 x,\quad g^{(0)} = \rho \mathrm{e}^{i\alpha_0},\ \alpha_0 = a_0 t + b_0 x \tag{17}$$

Noting that they satisfy the same differential equation as $f^{(1)}, g^{(1)}$, it can be assumed that the argumentation angles of $f^{(1)}/g^{(1)}$ differ from that of $f^{(0)}/g^{(0)}$ only by a constant [24] [25], *i.e.*

$$f^{(1)} = a_1 f^{(0)} \mathrm{e}^{i\xi_1^{(0)}} \mathrm{e}^{\eta_1} + a_2 f^{(0)} \mathrm{e}^{i\xi_2^{(0)}} \mathrm{e}^{\eta_2} \equiv a_1 f_1^{(1)} + a_2 f_2^{(1)} \tag{18.a}$$

$$g^{(1)} = b_1 g^{(0)} \mathrm{e}^{i\alpha_1^{(0)}} \mathrm{e}^{\eta_1} + b_2 g^{(0)} \mathrm{e}^{i\alpha_2^{(0)}} \mathrm{e}^{\eta_2} \equiv b_1 g_1^{(1)} + b_2 g_2^{(1)} \tag{18.b}$$

In the above equations

$$\eta_j = \tau_j t + \kappa_j x + \eta_j^{(0)};\quad j = 1, 2 \tag{19}$$

Note that all the parameters to be solved in the above expressions are real numbers. The first-order perturbation terms $f^{(1)}, g^{(1)}$ correspond to the terms involving with pure soliton and breather, therefore it should have two terms in the summation, and terms $f^{(2)}, g^{(2)}$ correspond to the interaction between two solitons in each soliton pair, either the pure soliton pair or the pair in a single breather term. By substituting the first Equation of (18.a), into the Equation (14.b), we can obtain

$$\begin{aligned}&\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda + 4i\gamma \mathrm{D}_x\right)\left(\overline{a_1} f^{(0)} \cdot \overline{f_1^{(1)}} + a_1 f_1^{(1)} \cdot \overline{f^{(0)}}\right)\\ &+\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda + 4i\gamma D_x\right)\left(\overline{a_2} f^{(0)} \cdot \overline{f_2^{(1)}} + a_2 f_2^{(1)} \cdot \overline{f^{(0)}}\right) = 0\end{aligned} \tag{20}$$

using the above Equation (20), the relationship among parameters can be found

$$\overline{a}_1 = a_1,\ \overline{a}_2 = a_2 \tag{21}$$

Similarly, using Equations (15.b) and (18.b), we can obtain

$$a_1 = b_1, a_2 = b_2 \tag{22}$$

Now considering the interaction between $f^{(2)}, g^{(2)}$, the interaction strength parameters A, and B should be introduced. Therefore, the specific form of $f^{(2)}, g^{(2)}$ can be set as

$$f^{(2)} = f^{(0)} a_1 a_2 A \mathrm{e}^{i\xi_1^{(0)} + i\xi_2^{(0)}} \mathrm{e}^{\eta_1 + \eta_2};\ g^{(2)} = g^{(0)} b_1 b_2 B \mathrm{e}^{i\alpha_1^{(0)} + i\alpha_2^{(0)}} \mathrm{e}^{\eta_1 + \eta_2} \tag{23}$$

It is easy to verify that the Equation (14.e) is automatically satisfied, that is

$$\begin{aligned}&\left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda + 4i\gamma \mathrm{D}_x\right) f^{(2)} \cdot \overline{f^{(2)}}\\ &= \left|a_1 a_2 A\right|^2 \left(i\mathrm{D}_t + \mathrm{D}_x^2 - \lambda + 4i\gamma \mathrm{D}_x\right) f^{(0)} \cdot \overline{f^{(0)}} = 0\end{aligned} \tag{24}$$

Similarly, the Equation (15.e) is automatically satisfied by the Equation (24). In addition, in order to satisfy the Equation (16.e), parameters A, B must meet

$$A^2 = \left|B\right|^2 \tag{25}$$

Meanwhile, Equation (14.d) becomes

$$\begin{aligned}&\left(i\mathrm{D}_t+\mathrm{D}_x^2-\lambda+4i\gamma\mathrm{D}_x\right)\left(f^{(1)}\cdot\overline{f^{(2)}}+f^{(2)}\cdot\overline{f^{(1)}}\right)\\&=\left(i\tau_2+4ik_0\kappa_2+\kappa_2^2+4i\gamma\kappa_2\right)\mathrm{e}^{i\xi_2^{(0)}}\left(A-\overline{A}\right)\mathrm{e}^{\eta_1}\\&\quad+\left(i\tau_1+4ik_0\kappa_1+\kappa_1^2+4i\gamma\kappa_1\right)\mathrm{e}^{i\xi_1^{(0)}}\left(A-\overline{A}\right)\mathrm{e}^{\eta_2}\\&=0\end{aligned} \tag{26}$$

This leads to

$$A=\overline{A} \tag{27}$$

which means A is a real number. Similarly, Equation (15.d) will lead to

$$A=B \tag{28}$$

By substituting this obtained relationship into Equation (15.e), it can be found that Equation (15.e) is automatically satisfied by (27) and (28). By reusing the Equation (14.d), and the expression for interaction strength parameter A inside the soliton solution of the MNLS equation can be obtained, which is

$$\begin{aligned}A=1+4\kappa_1\kappa_2\sin\xi_1^{(0)}\sin\xi_2^{(0)}\Big/&\\\left[2\kappa_1\kappa_2\cos\left(\xi_1^{(0)}+\xi_2^{(0)}\right)-\kappa_2^2\sin\xi_1^{(0)}\Big/\sin\xi_2^{(0)}-\kappa_1^2\sin\xi_2^{(0)}\Big/\sin\xi_1^{(0)}\right]&\end{aligned} \tag{29}$$

In fact, the Equation (15.c) (16.c) contain the same information as the Equation (14.c), and they can also be used to obtain the expression of the interaction strength parameter A. Thus, we obtain all the parameters that govern the MNLS equation, and the function f_2, g_2 can be expressed as

$$f_2=\mathrm{e}^{ik_0x}\left(1+a_1\mathrm{e}^{i\xi_1^{(0)}}\mathrm{e}^{\eta_1}+a_2\mathrm{e}^{i\xi_2^{(0)}}\mathrm{e}^{\eta_2}+a_1a_2A\mathrm{e}^{i\xi_1^{(0)}+i\xi_2^{(0)}}\mathrm{e}^{\eta_1+\eta_2}\right) \tag{30}$$

$$g_2=\rho\mathrm{e}^{ib_0x}\left(1+a_1\mathrm{e}^{i\alpha_1^{(0)}}\mathrm{e}^{\eta_1}+a_2\mathrm{e}^{i\alpha_2^{(0)}}\mathrm{e}^{\eta_2}+a_1a_2A\mathrm{e}^{i\alpha_1^{(0)}+i\alpha_2^{(0)}}\mathrm{e}^{\eta_1+\eta_2}\right) \tag{31}$$

Substitute above results into following formula

$$u_2=g_2\overline{f_2}\big/f_2^2 \tag{32}$$

we thus obtain the explicit single breather solution for MNLS equation. We need to further give the constraint relationships among other parameters. Firstly, substitute the obtained results of $f^{(0)}$ and $g^{(0)}$ into (14.a), (15.a), (16.a) to obtain the expression of the parameters k_0, b_0. then secondly, express each parameter as a function of ρ and γ in order to fit the parameters in the theoretical solution abiding by the given specific boundary condition. Further we substitute the obtained expression $f^{(0)}$, $g^{(0)}$, $f^{(1)}$, $g^{(1)}$ into (14.b), (15.b), (16.b), the expressions for parameters $\alpha_j^{(0)}$, Δ, κ_j, τ_j, $\xi_j^{(0)}$ can be attained. Here, we view ρ, γ, $\xi_j^{(0)}$, a_j as independent adjustable parameters, while other parameters can be represented by the adjustable parameters mentioned above. In the end, we obtained the following set of expressions

$$k_0=(1/4)|\rho|^2;\ b_0=(1/4)|\rho|^2\pm\sqrt{(1/4)|\rho|^4+2\gamma|\rho|^2} \tag{33.a}$$

$$\alpha_j^{(0)}=\xi_j^{(0)}+\arccos\left(\cos^2\xi_j^{(0)}+(\Delta-1)\sin^2\xi_j^{(0)}\right) \tag{33.b}$$

$$\Delta = 2\left(|\rho|^2 \mp \sqrt{|\rho|^4 + 8\gamma|\rho|^2} + 4\gamma\right)\Big/|\rho|^2 \quad (33.c)$$

$$\kappa_j = (1/2)|\rho|^2\left(\cos\alpha_j^{(0)} - \cos\xi_j^{(0)}\right)\Big/\sin\xi_j^{(0)} \quad (33.d)$$

$$\tau = -4(\gamma + k_0)\kappa_j + \cot\xi_j^{(0)}\kappa_j^2;\ \ j = 1,2 \quad (33.e)$$

For given adjustable parameter parameters, the spatiotemporal evolution characteristics of the breather solution of the MNLS equation can be plotted. In addition, unlike the double soliton solution, the breather solution must be subject to the following constraints

$$\xi_1^{(0)} = \xi_2^{(0)} \quad (34)$$

$$a_1 = -a_2 \quad (35)$$

$$\eta_1 = -\eta_2 \quad (36)$$

Pay attention to the constraint relationship (35), which is why the last two terms on the right side of Equations (30) and (31) have the same negative signs, which leads to the characteristic of the breather.

On the other hand, it can be verified that the solution function u does tend towards linear exponential boundary conditions at infinity. $x \to \pm\infty$; $u \to \rho e^{i(k_0 - 3b_0)x} e^{i\phi_0}$. After setting the parameters, the spatiotemporal evolution characteristics of the breather solution of the MNLS equation are shown in **Figure 1**.

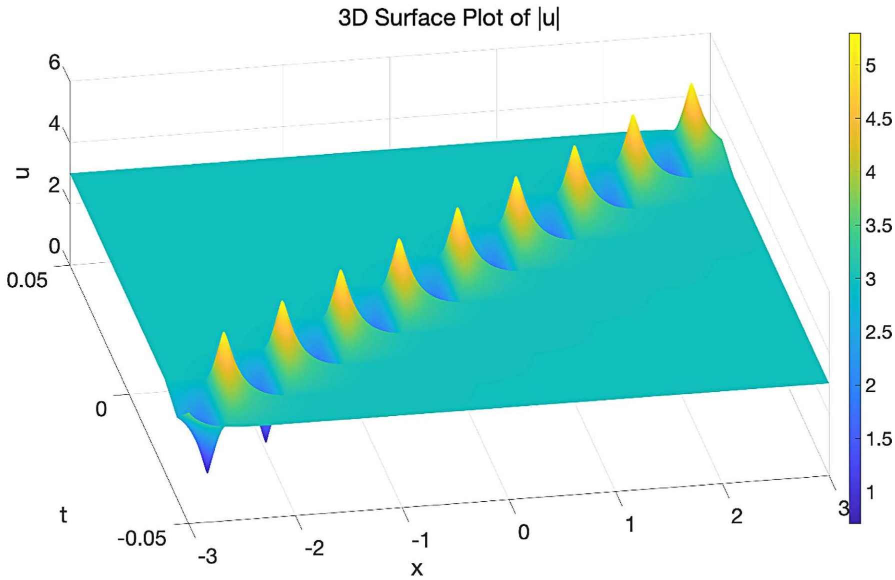

Figure 1. The single breather solution of MNLS equation $\rho = 3$, $\xi_1^{(0)} = \xi_2^{(0)} = \frac{3\pi}{8}$, $a_1 = -a_2 = 1$, $\gamma = -1$.

When the parameter are set $\gamma \to 0$, the MNLS equation will degenerate into the DNLS equation, and the expression of the functions f and g, and the breather solution to DNLS equation will be immediately obtained.

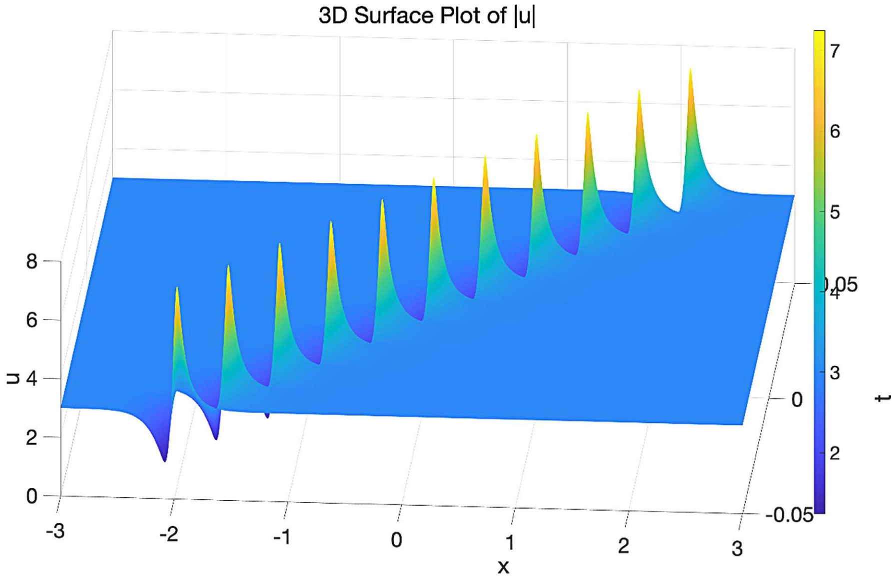

Figure 2. The single breather solution of DNLS equation $\rho = 3$, $\xi_1^{(0)} = \xi_2^{(0)} = \frac{3\pi}{8}$, $a_1 = -a_2 = 1$, $\gamma = 0$.

Keeping the constraints unchanged, the parameters become as follows

$$A = \sin^2\left(\xi_1^{(0)} - \xi_2^{(0)}\right) \Big/ \sin^2\left(\xi_1^{(0)} + \xi_2^{(0)}\right);\ k_0 = (1/4)|\rho|^2 \tag{37}$$

$$b_0 = (3/4)|\rho|^2;\ \alpha_j^{(0)} = 3\xi_j^{(0)} \tag{38}$$

$$\kappa_j = -|\rho|^2 \sin 2\xi_j^{(0)};\ \tau_j = |\rho|^4 \sin 2\xi_j^{(0)}\left(1 + 2\cos^2 \xi_j^{(0)}\right);\ j = 1,2 \tag{39}$$

And as $x \to \pm\infty$, constant boundary condition $u \to \rho e^{i\phi_0}$ were met. The spatiotemporal evolution of the single breather solution to the DNLS equation is shown in Figure 2.

5. 1 + 1 Type of Mixed Soliton Solution of DNLS/MNLS Equation

For the mixed soliton solution of the DNLS/MNLS equation, we need to add a first-order perturbation term to express the interaction between the two solitons of the 1-pure soliton and the single breathing soliton. Now the truncation condition has changed to

$$f^{(i)} = 0,\ g^{(i)} = 0,\ i > 3 \tag{40}$$

Therefore, from the equation system (5)-(7), the following 21 sub-equations of the three equation systems of MNLS can be obtained:

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right) f^{(0)} \cdot \bar{f}^{(0)} = 0 \tag{41.a}$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(0)} \cdot \bar{f}^{(1)} + f^{(1)} \cdot \bar{f}^{(0)}\right) = 0$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(1)} \cdot \overline{f}^{(3)} + f^{(2)} \cdot \overline{f}^{(2)} + f^{(3)} \cdot \overline{f}^{(1)}\right) = 0 \quad (41.b)$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(0)} \cdot \overline{f}^{(2)} + f^{(1)} \cdot \overline{f}^{(1)} + f^{(2)} \cdot \overline{f}^{(0)}\right) = 0 \quad (41.c)$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(0)} \cdot \overline{f}^{(3)} + f^{(1)} \cdot \overline{f}^{(2)} + f^{(2)} \cdot \overline{f}^{(1)} + f^{(3)} \cdot \overline{f}^{(0)}\right) = 0 \quad (41.d)$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(1)} \cdot \overline{f}^{(3)} + f^{(2)} \cdot \overline{f}^{(2)} + f^{(3)} \cdot \overline{f}^{(1)}\right) = 0 \quad (41.e)$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right)\left(f^{(2)} \cdot \overline{f}^{(3)} + f^{(3)} \cdot \overline{f}^{(2)}\right) = 0 \quad (41.f)$$

$$\left(iD_t + D_x^2 - \lambda + 4i\gamma D_x\right) f^{(3)} \cdot \overline{f}^{(3)} = 0 \quad (41.g)$$

$$\left(iD_t + D_x^2 - \lambda\right) f^{(0)} \cdot g^{(0)} = 0 \quad (42.a)$$

$$\left(iD_t + D_x^2 - \lambda\right)\left(f^{(0)} \cdot g^{(1)} + f^{(1)} \cdot g^{(0)}\right) = 0 \quad (42.b)$$

$$\left(iD_t + D_x^2 - \lambda\right)\left(f^{(0)} \cdot g^{(2)} + f^{(1)} \cdot g^{(1)} + f^{(2)} \cdot g^{(0)}\right) = 0 \quad (42.c)$$

$$\left(iD_t + D_x^2 - \lambda\right)\left(f^{(0)} \cdot g^{(3)} + f^{(1)} \cdot g^{(2)} + f^{(2)} \cdot g^{(1)} + f^{(3)} \cdot g^{(0)}\right) = 0 \quad (42.d)$$

$$\left(iD_t + D_x^2 - \lambda\right)\left(f^{(1)} \cdot g^{(3)} + f^{(2)} \cdot g^{(2)} + f^{(3)} \cdot g^{(1)}\right) = 0 \quad (42.e)$$

$$\left(iD_t + D_x^2 - \lambda\right)\left(f^{(2)} \cdot g^{(3)} + f^{(3)} \cdot g^{(2)}\right) = 0 \quad (42.f)$$

$$\left(iD_t + D_x^2 - \lambda\right) f^{(3)} \cdot g^{(3)} = 0 \quad (42.g)$$

$$D_x f^{(0)} \cdot \overline{f}^{(0)} = (i/2) g^{(0)} \overline{g}^{(0)} \quad (43.a)$$

$$D_x\left(f^{(0)} \cdot \overline{f}^{(1)} + f^{(1)} \cdot f^{(0)^*}\right) = (i/2)\left(g^{(0)} \overline{g}^{(1)} + g^{(1)} \overline{g}^{(0)}\right) \quad (43.b)$$

$$D_x\left(f^{(0)} \cdot \overline{f}^{(2)} + f^{(1)} \cdot \overline{f}^{(1)} + f^{(2)} \cdot \overline{f}^{(0)}\right) = (i/2)\left(g^{(0)} \overline{g}^{(2)} + g^{(1)} \overline{f}^{(1)} + g^{(2)} \overline{g}^{(0)}\right) \quad (43.c)$$

$$\begin{aligned} &D_x\left(f^{(0)} \cdot \overline{f}^{(3)} + f^{(1)} \cdot \overline{f}^{(2)} + f^{(2)} \cdot \overline{f}^{(1)} + f^{(3)} \cdot \overline{f}^{(0)}\right) \\ &= (i/2)\left(g^{(0)} \overline{g}^{(3)} + g^{(1)} \overline{g}^{(2)} + g^{(2)} \overline{g}^{(1)} + g^{(3)} \overline{g}^{(0)}\right) \end{aligned} \quad (44.d)$$

$$D_x\left(f^{(1)} \cdot \overline{f}^{(3)} + f^{(2)} \cdot \overline{f}^{(2)} + f^{(3)} \cdot \overline{f}^{(1)}\right) = (i/2)\left(g^{(1)} \overline{g}^{(3)} + g^{(2)} \overline{f}^{(2)} + g^{(3)} \overline{g}^{(1)}\right) \quad (44.e)$$

$$D_x\left(f^{(2)} \cdot \overline{f}^{(3)} + f^{(3)} \cdot \overline{f}^{(2)}\right) = (i/2)\left(g^{(2)} \overline{g}^{(3)} + g^{(3)} \overline{g}^{2}\right) \quad (44.f)$$

$$D_x f^{(3)} \cdot \overline{f}^{(3)} = (i/2) g^{(3)} \overline{g}^{(3)} \quad (44.g)$$

Now provide the perturbation expressions for the various orders of the bilinear function to be solved, and the ground state of the bilinear function remains as follows

$$f^{(0)} = \mathrm{e}^{i\xi_0}, \xi_0 = \omega_0 t + k_0 x, \quad g^{(0)} = \rho \mathrm{e}^{i\alpha_0}, \alpha_0 = a_0 t + b_0 x \quad (45)$$

The first-order disturbance becomes

$$f^{(1)} = a_1 f^{(0)} \mathrm{e}^{i\xi_1^{(0)}} \mathrm{e}^{\eta_1} + a_2 f^{(0)} \mathrm{e}^{i\xi_2^{(0)}} \mathrm{e}^{\eta_2} + a_3 f^{(0)} \mathrm{e}^{i\xi_3^{(0)}} \mathrm{e}^{\eta_3} \quad (46)$$

$$g^{(1)} = b_1 g^{(0)} \mathrm{e}^{i\alpha_1^{(0)}} \mathrm{e}^{\eta_1} + b_2 g^{(0)} \mathrm{e}^{i\alpha_2^{(0)}} \mathrm{e}^{\eta_2} + b_3 g^{(0)} \mathrm{e}^{i\alpha_3^{(0)}} \mathrm{e}^{\eta_3} \quad (47)$$

The second-order disturbance becomes

$$f^{(2)} = a_1 a_2 A_{12} f^{(0)} e^{i\xi_1^{(0)}+i\xi_2^{(0)}} e^{\eta_1+\eta_2} + a_1 a_3 A_{13} f^{(0)} e^{i\xi_1^{(0)}+i\xi_3^{(0)}} e^{\eta_1+\eta_3} + a_2 a_3 A_{23} f^{(0)} e^{i\xi_2^{(0)}+i\xi_3^{(0)}} e^{\eta_2+\eta_3} \tag{48}$$

$$g^{(2)} = b_1 b_2 B_{12} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_2^{(0)}} e^{\eta_1+\eta_2} + b_1 b_3 B_{13} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_3^{(0)}} e^{\eta_1+\eta_3} + b_2 b_3 B_{23} g^{(0)} e^{i\alpha_2^{(0)}+i\alpha_3^{(0)}} e^{\eta_2+\eta_3} \tag{49}$$

The newly added third-order perturbation expression is

$$f^{(3)} = a_1 a_2 a_3 A_{12} A_{13} A_{23} f^{(0)} e^{i\xi_1^{(0)}+i\xi_2^{(0)}+i\xi_3^{(0)}} e^{\eta_1+\eta_2+\eta_3} \tag{50}$$

$$g^{(3)} = b_1 b_2 b_3 B_{12} B_{13} B_{23} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_2^{(0)}+i\alpha_3^{(0)}} e^{\eta_1+\eta_2+\eta_3} \tag{51}$$

By using similar methods for calculation and discussion, we can still obtain the following relationship between parameters

$$a_1 = \overline{a}_1,\ a_2 = \overline{a}_2,\ a_3 = \overline{a}_3 \tag{52}$$

$$a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3 \tag{53}$$

as well as

$$A_{12} = \overline{A}_{12},\ A_{13} = \overline{A}_{13},\ A_{23} = \overline{A}_{23} \tag{54}$$

$$A_{12} = B_{12},\ A_{13} = B_{13},\ A_{23} = B_{23} \tag{55}$$

By substituting (53) (55) into expressions (47) (49) (51), we obtain

$$g^{(1)} = a_1 g^{(0)} e^{i\alpha_1^{(0)}} e^{\eta_1} + a_2 g^{(0)} e^{i\alpha_2^{(0)}} e^{\eta_2} + a_3 g^{(0)} e^{i\alpha_3^{(0)}} e^{\eta_3} \tag{56}$$

$$g^{(2)} = a_1 a_2 A_{12} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_2^{(0)}} e^{\eta_1+\eta_2} + a_1 a_3 A_{13} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_3^{(0)}} e^{\eta_1+\eta_3} + a_2 a_3 A_{23} g^{(0)} e^{i\alpha_2^{(0)}+i\alpha_3^{(0)}} e^{\eta_2+\eta_3} \tag{57}$$

$$g^{(3)} = a_1 a_2 a_3 A_{12} A_{13} A_{23} g^{(0)} e^{i\alpha_1^{(0)}+i\alpha_2^{(0)}+i\alpha_3^{(0)}} e^{\eta_1+\eta_2+\eta_3} \tag{58}$$

Based on above result, we can obtain the expression of functions f_{1+1}, g_{1+1} for the 1 + 1 type of mixed soliton solution of the DNLS/MNLS equation as follows:

$$f_{1+1} = f^{(0)} + f^{(1)} + f^{(2)} + f^{(3)} \tag{59}$$

$$g_{1+1} = g^{(0)} + g^{(1)} + g^{(2)} + g^{(3)} \tag{60}$$

By substituting (45), (46), (48), (50) into expression (59), and substituting (45), (47), (49), (51) into expression (60), the explicit form of bilinear functions can be obtained. Substitute the two equations (59), (60) into the typical form (4) of soliton solutions

$$u_{1+1} = g_{1+1}\, \overline{f}_{1+1} \big/ f_{1+1}^2 \tag{61}$$

Thus the mixed soliton solution of one pure soliton and one breather for MNLS equation is obtained, but due to the lengthy expression, it is not specifically listed.

For the 1 + 1 mixed soliton solution of MNLS equation, the interaction strength parameter becomes

$$A_{ij}=1+\frac{4\kappa_1\kappa_2\sin\xi_i^{(0)}\sin\xi_j^{(0)}}{2\kappa_1\kappa_2\cos\left(\xi_i^{(0)}+\xi_j^{(0)}\right)-\kappa_2^2\sin\xi_i^{(0)}/\sin\xi_j^{(0)}-\kappa_1^2\sin\xi_j^{(0)}/\sin\xi_i^{(0)}} \tag{62a}$$

Similarly as $\gamma\to 0$, the interaction strength parameter for the DNLS equation becomes

$$A_{ij}=\frac{\sin^2\left(\xi_i^{(0)}-\xi_j^{(0)}\right)}{\sin^2\left(\xi_i^{(0)}+\xi_j^{(0)}\right)} \tag{62b}$$

In addition, the mixed solutions of the MNLS/DNLS equation still abide by three constraint conditions (34)-(36), and can be proved to satisfy especially the same linear exponential function /constant boundary conditions as the single breathers for the MNLS/DNLS equations, which are omitted here. The spatiotemporal evolution characteristics of mixed soliton solutions in the MNLS equation are shown in **Figure 3** and **Figure 4**.

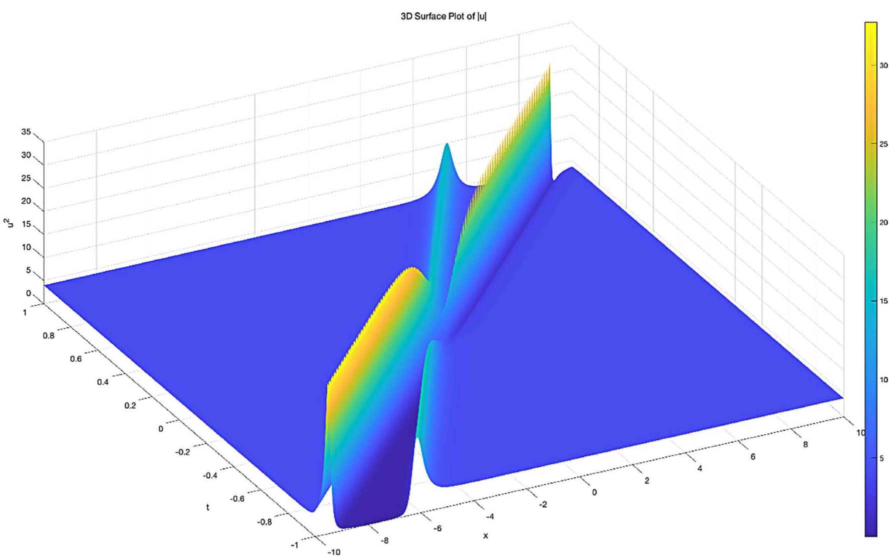

Figure 3. The mixed 1-breather and 1-pure bright soliton of MNLS equation $\rho=3$, $\xi_1^{(0)}=\xi_2^{(0)}=\frac{\pi}{5}$, $a_1=-a_2=1$, $\xi_3^{(0)}=\frac{\pi}{4}$, $a_3=-1$, $\gamma=-1$.

The evolution of the mixed solution of breather and pure soliton in DNLS equation in time and space is shown in **Figure 5** and **Figure 6**.

The justification of Hirota method is realized by the successful truncation of the perturbation series. This paper meets the demand by making our results satisfy every subset equation group listed above, the unwritten equations are higher perturbation terms which are naturally satisfied by letting the coefficients of those higher terms to be zero. Sorry for our unclear description of the steps of truncating the perturbation series.

In order, we should briefly analyze the evolution characteristics of the solution in space and time, especially the internal collision process. **Figure 3** and **Figure 4** respectively show the evolution process of approaching-collision-separation

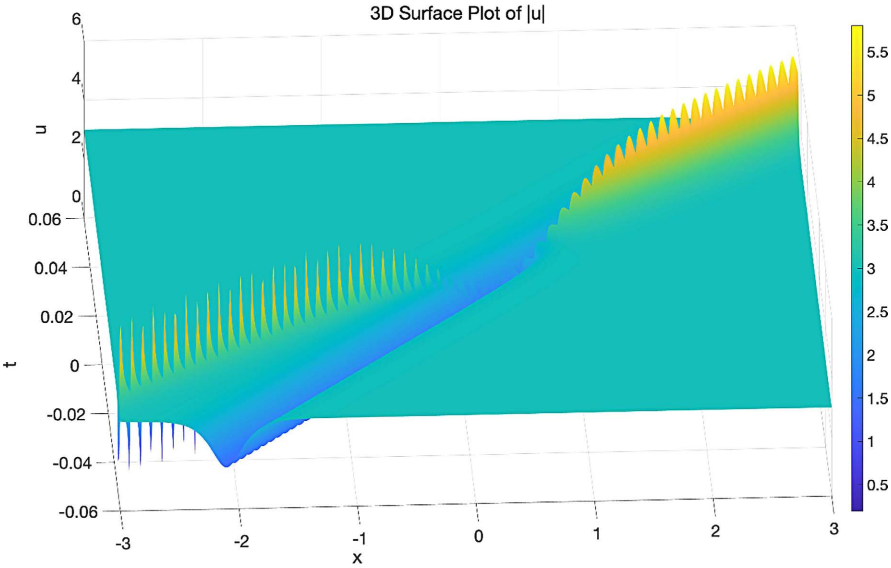

Figure 4. The mixed 1-breather and 1-pure dark soliton of MNLS equation $\rho = 3$, $\xi_1^{(0)} = \xi_2^{(0)} = \dfrac{\pi}{2.9}$, $a_1 = -a_2 = 1$, $\xi_3^{(0)} = \dfrac{\pi}{4.2}$, $a_3 = 1$.

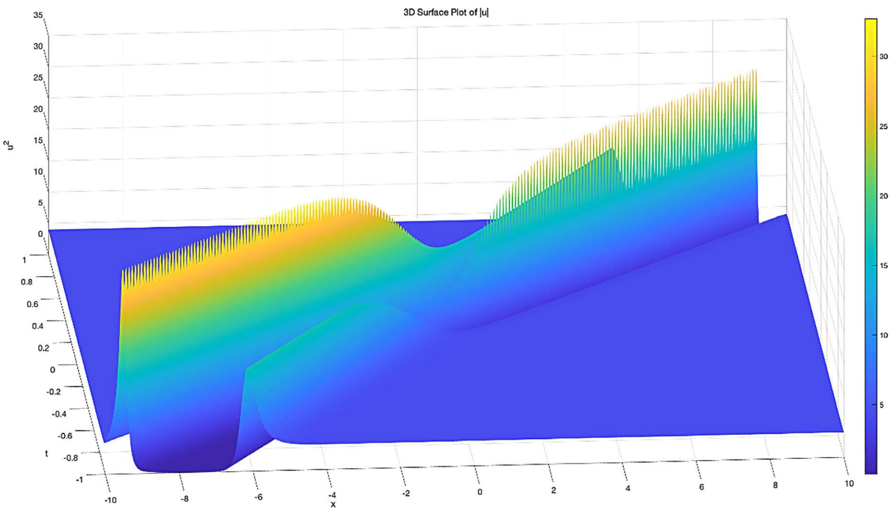

Figure 5. The mixed 1-breather and 1-pure bright soliton of DNLS equation $\rho = 3$, $\xi_1^{(0)} = \xi_2^{(0)} = \dfrac{\pi}{5}$, $a_1 = -a_2 = 1$, $\xi_3^{(0)} = \dfrac{\pi}{4}$, $a_3 = -1$.

within a mixed breather and pure bright/dark solitons for the MNLS equation in space and time, namely the elastic collision process inside the mixed solution. **Figure 5** and **Figure 6** show the evolution process of approaching-collision-separation within a mixed breather and pure bright/dark solitons and for the DNLS equation in space and time, namely the elastic collision process inside the mixed solution. We have graphically demonstrated the characteristic of the propagation independence and shape invariance of breather and pure solitons within a mixed solution, before and after collision, but it can be seen that there is a remarkable phase shift between breather and pure solitons after collision, that is,

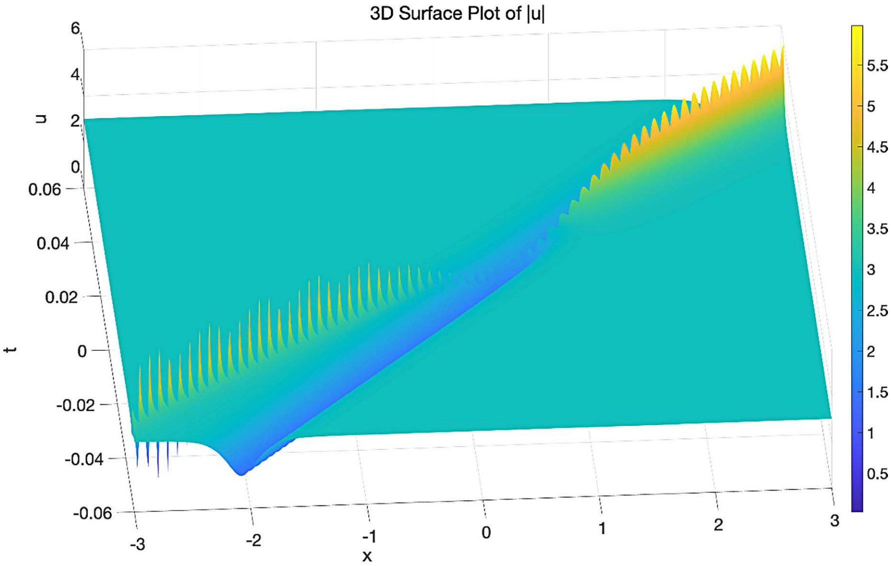

Figure 6. The mixed 1-breather and 1-pure dark soliton of DNLS equation $\rho = 3$, $\xi_1^{(0)} = \xi_2^{(0)} = \frac{\pi}{3}$, $a_1 = -a_2 = 1$, $\xi_3^{(0)} = \frac{\pi}{4.2}$, $a_3 = 1$.

a constant phase shift, and the centers of breather and pure soliton also undergo a constant shift. It can also be seen that whether in the infinite past before the collision or in the infinite future after the collision, the breather and pure solitons in the mixed solution will asymptotically depart from each other, that is, become an algebraic superposition of independent breather and pure solitons. Limited to the paper size, we hadn't given a direct proof of our results. The asymptotic analysis for the behavior of the mixed solution can be found in references [13] [18] [21].

6. Summary and Outlook

This paper is based on Hirota's bilinear derivative transformation to search for the breathers and their mixed solutions with pure solitons of the MNLS/DNLS equations under non-zero boundary conditions. We first chose a typical and unique solution form of soliton to perform bilinear derivative transformation on the MNLS/DNLS equation. Secondly, through careful observation and reasonable analogue, we guessed that the breather solution and bi-pole solution form have the similar mathematical structures, which makes our solving process clear and natural. The spatiotemporal evolution graph of the mixed solution of the MNLS/DNLS equation clearly shows the elastic collision process between breather and pure soliton, as well as the constant phase shift and soliton center shift that occur after the collision. It particularly indicates that the mixed soliton solution will gradually approach a simple superposition of independent breather and pure soliton solutions, whether in the infinite past before the collision or in the infinite future after the collision, which is consistent with the results and descriptions given in references [21] [30], indicating the rationality and correctness of our obtained solution.

The Hirota method can be used to further obtain the mixed solution of $m+n$-type breather and pure soliton for the MNLS/DNLS equation under non-zero boundary conditions. We already have its construction method and clues to the solution, but due to space limitation, we will elaborate on it in another paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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