

Embedding the Einstein Tensor in the Klein-Gordon Equation Using Geometric Algebra $Cl_{3,0}$

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Abstract

In this paper, we will use Geometric Algebra to be able to embed the Klein-Gordon equation for a particle in a non-Euclidean field (gravitational field). This way, we will obtain an expression similar to the Dirac equation, but with a slight change in one of the terms. This variation is produced and depends on the curvature of the space where the particle lies in (the Ricci scalar). In a similar manner, we will find variations in the equation for the energy of a particle and in the Einstein gravitational equation that will depend again on the value of the Ricci scalar (the curvature of the space where the particle lies in). An important outcome will be an equation that limits the value of the Ricci scalar depending on the value of the mass that provokes it (the value of the mass, not the mass density), highly reducing the possibilities of arriving at singularities. In fact, the value of this R has been found to be equal to the cosmological constant (both in the order of $1E-52$), making it a perfect candidate for the dark energy. Also, the magnetic-like effects of gravitation coming from the equations are sufficient to explain the rotation of the galaxies (NGC 1560, NGC 3198 and NGC 3115) without the need for dark matter.

Keywords

Geometric Algebra, Einstein Tensor, Klein-Gordon Equation, Bra-Ket Product, Non-Euclidean Metric

1. Introduction

In this paper, we will embed the Klein-Gordon equation for a particle in a non-Euclidean field (gravitational field) using Geometric Algebra and the Einstein equations. This will lead to new equations that we will show in the paper.

2. Geometric Algebra $Cl_{3,0}$: Basis Vectors

There is a discipline in mathematics that is called Geometric Algebra [1] [2], also known as Clifford Algebras.

In the specific Geometric Algebra $Cl_{3,0}$, it is considered a three-dimensional space, so we need three independent vectors to define a basis. The classical definition of a basis is shown in **Figure 1**.

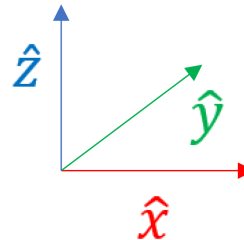


Figure 1. Basis vectors in three-dimensional space.

In this paper, we will use the nomenclature e_i (without any hat or vector sign) to name these three vectors instead of the classical $\hat{x} \hat{y} \hat{z}$. Above, I have considered an orthonormal basis as an example.

But in the general case, this is not even necessary. The only necessary constraint to form a basis is that the three vectors are linearly independent (that is, they do not lie on the same plane). An example is shown in **Figure 2**.

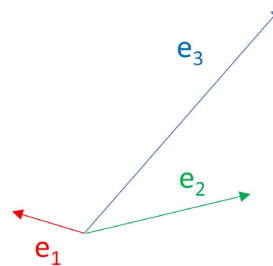


Figure 2. General basis in three-dimensional space.

In geometric algebra, it is defined as an operation called the geometric product. The geometric product is not represented by any symbol. It is the implicit operation when two vectors are represented one after the other.

Its definition is:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

Being:

$$e_i \cdot e_j = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

The classical definition of the scalar product. The product of the two norms (the length) of the vectors by the cosine of the angle formed by them (we have called it α_{ij} in this case).

The result of the scalar product is a number, a scalar. An important property of the scalar product is that it is commutative:

$$e_i \cdot e_j = e_j \cdot e_i = \|e_i\| \|e_j\| \cos(\alpha_{ij})$$

As the cosine of the angle is included in the product, you can check that when e_i and e_j are perpendicular (right angle), the scalar product is zero. And the vectors are colinear (the angle is zero), the scalar product is just the product of the modules of the vectors.

The other element of the geometric product above is:

$$e_i \wedge e_j$$

What it is called the outer, exterior or wedge product of the two vectors.

The result of this operation is not a number. It is another entity that is not a number and not a vector. It is called a bivector. The bivector is an entity that represents an oriented surface area (in a same way that a vector “represents” an oriented line segment). See **Figure 3**.

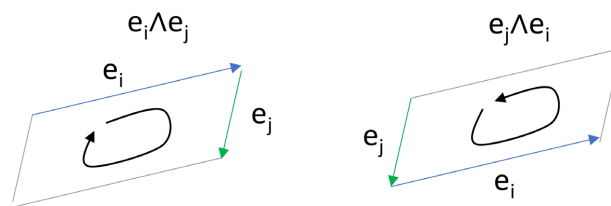


Figure 3. Bivectors.

It can be checked above that the module (area of the surface) when reversing the order of the exterior product is the same. But the orientation (its sign) changes. So, the exterior product is anticommutative:

$$e_i \wedge e_j = -e_j \wedge e_i$$

The module (area of the surface) of the exterior product is:

$$e_i \wedge e_j = e_j \wedge e_i = e_i e_j \sin(\alpha_{ij})$$

You can see that when the vectors are colinear (the angle is zero), the exterior product result is zero. And when the vectors are perpendicular, the module of the exterior product is the product of the modules of the vectors.

Coming back to the definition of the geometric product:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j$$

We can see that when we perform the square of a vector, this is, the product of a vector by itself (the vector is colinear with itself, its angle is zero) the result is:

$$(e_i)^2 = e_i e_i = e_i \cdot e_i + e_i \wedge e_i = \|e_i\| \|e_i\| \cdot 1 + 0 = \|e_i\| \|e_i\| = \|e_i\|^2$$

So, the square of a vector is its norm squared. The important thing here, is that the result is just a number. It is not a vector, it is not a bivector, it is just a number. We have converted a vector to a number just multiplying it by itself.

If now, we multiply (geometric product) two perpendicular vectors (the angle between them is a right angle):

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = 0 + e_i \wedge e_j = e_i \wedge e_j$$

So, you can see that the result is a pure bivector. It does not include vectors or scalars, just a bivector.

If we reverse the angle, we have:

$$e_j e_i = e_j \cdot e_i + e_j \wedge e_i = 0 + e_j \wedge e_i = e_j \wedge e_i = -e_i \wedge e_j = -e_i e_j$$

So, when two vectors are perpendicular, not only the exterior product, but also the geometric product is anticommutative.

From the equations above, we can obtain the following equations.

$$e_i \cdot e_j = \frac{1}{2}(e_i e_j + e_j e_i)$$

$$e_i \wedge e_j = \frac{1}{2}(e_i e_j - e_j e_i)$$

The demonstration comes directly from the definition of the geometric product. If we sum a geometric product by its reverse, we put the definition of geometric product, we take into account that the scalar product is commutative and the exterior product anticommutative:

$$\begin{aligned} e_i e_j + e_j e_i &= e_i \cdot e_j + e_i \wedge e_j + e_j \cdot e_i + e_j \wedge e_i \\ &= e_i \cdot e_j + e_i \wedge e_j + e_i \cdot e_j - e_i \wedge e_j = 2(e_i \cdot e_j) \end{aligned}$$

$$e_i \cdot e_j = \frac{1}{2}(e_i e_j + e_j e_i)$$

If instead of summing, we subtract:

$$\begin{aligned} e_i e_j - e_j e_i &= e_i \cdot e_j + e_i \wedge e_j - e_j \cdot e_i - e_j \wedge e_i \\ &= e_i \cdot e_j + e_i \wedge e_j - e_i \cdot e_j + e_i \wedge e_j = 2(e_i \wedge e_j) \end{aligned}$$

$$e_i \wedge e_j = \frac{1}{2}(e_i e_j - e_j e_i)$$

We will see in next chapters that when we apply the exterior product instead of the geometric product of two vectors, this means that we want only the result that appears in the plane they form (in the bivector they form). And we “remove” from the result the scalars (that will appear with the scalar product of the vectors) and also, we remove the possible result in vectors (in more complicated products that we will see in next chapters).

Another point to comment is that the exterior product of bivectors (instead of vectors) is defined in the opposite way (summing instead of subtracting). I am not going to enter into details, you can check it in [2].

$$(e_i e_j) \wedge (e_r e_s) = \frac{1}{2}(e_i e_j e_r e_s + e_r e_s e_j e_i)$$

The same way, the scalar product of bivectors is also defined as the opposite of vectors. See [2].

$$(e_i e_j)(e_r e_s) = \frac{1}{2}(e_i e_j e_r e_s - e_r e_s e_j e_i)$$

Also, to remark that the geometric product is always associative and distributive as you can see in [2]. But in general, it is not commutative or anticommutative as commented (it depends on the specific product). We will see more examples in the following chapters.

To conclude this chapter about geometric algebra, we will define the trivector. When two vectors are exterior multiplied, they form a bivector as seen above. The same way, when three vectors are exterior multiplied, they create an oriented volume, called the trivector (Figure 4).

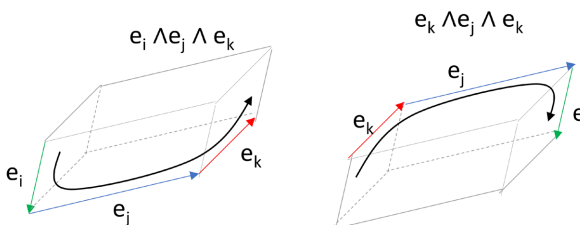


Figure 4. Trivectors.

You can see again that when we reverse the vectors, we get the same volume (module of the trivector) but with different orientation (sign):

$$e_i \wedge e_j \wedge e_k = -e_k \wedge e_j \wedge e_i$$

We will check more things regarding reversion and change of signs in the next chapter.

3. Geometric Algebra Cl_{3,0}: Different Types of Bases

3.1. Orthonormal Basis

In an orthonormal basis, the norm of the basis vectors is equal to one. And the basis vectors are perpendicular to each other.

So, from the properties commented in Chapter 2, we can obtain the following equations (for orthonormal basis):

$$(e_i)^2 = e_i e_i = e_i \cdot e_i = 1$$

$$e_i e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_i \quad (\text{when } i \neq j)$$

$$e_i \cdot e_j = e_j \cdot e_i = 0 \quad (\text{when } i \neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1 e_1 = 1$$

$$(e_2)^2 = e_2 e_2 = 1$$

$$(e_3)^2 = e_3 e_3 = 1$$

$$e_1 e_2 = -e_2 e_1$$

$$e_2 e_3 = -e_3 e_2$$

$$e_3 e_1 = -e_1 e_3$$

We can define the inverse of a vector and name it e^i , as the vector that fulfills (Einstein summation is not implied here):

$$(e_i)^{-1} e_i \equiv e^i e_i = 1 = e_i (e_i)^{-1} \equiv e_i e^i$$

To calculate e^i , we can post-multiply by e_i :

$$(e_i)^{-1} e_i e_i \equiv e^i e_i e_i = 1 \cdot e_i$$

$$e^i (e_i)^2 = e_i$$

$$e^i \cdot 1 = e_i$$

$$e^i = e_i = (e_i)^{-1}$$

So, in orthonormal metric, the inverse of a basis vector is itself. It is important to remark here that in Geometric Algebra, there are no covectors (or 1-forms). There are only scalars, bivectors, trivectors... We will see that the concept of covector in Geometric Algebra is just a vector that is the inverse of another vector.

In traditional algebra, you cannot define the inverse of a vector, so it is used a different type of element. In Geometric Algebra, the covectors are also vectors. And in fact, the product of inverse vectors by vectors outputs scalars as it would be expected by the product of a covector by a vector.

3.2. Geometric Algebra $Cl_{3,0}$: Orthogonal but Not Orthonormal Basis

In an orthogonal basis, the vectors are perpendicular to each other. But in general, the norm of the vectors is not one. In Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from zero.

The 3 in the name $Cl_{3,0}$, makes reference to that there are 3 basis vectors with positive norm. The 0 in the name $Cl_{3,0}$, makes reference to that there are no basis vectors with negative norm. And the absence of a third number makes reference to that there are no basis vectors with zero norm.

From the properties commented in Chapter 2, we can obtain the following equations (for orthogonal, not orthonormal basis):

$$(e_i)^2 = e_i e_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$

$$e_i e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_i \quad (\text{when } i \neq j)$$

$$e_i \cdot e_j = e_j \cdot e_i = 0 \quad (\text{when } i \neq j)$$

Making the equations explicit for three dimensions:

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$

$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$

$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$

$$\begin{aligned} e_1 e_2 &= -e_2 e_1 \\ e_2 e_3 &= -e_3 e_2 \\ e_3 e_1 &= -e_1 e_3 \end{aligned}$$

where the g_{ii} makes reference to the metric tensor components. See papers [3] [4]. Take into account that when you multiply two colinear vectors (and a vector is colinear with itself), its geometric product is equal to the scalar product. And this is exactly the definition of g_{ii} (the scalar product of e_i with itself).

The definition of the inverse of a vector, and naming it e^i , is the vector that fulfills (not Einstein summation is implied here):

$$(e_i)^{-1} e_i \equiv e^i e_i = 1 = e_i (e_i)^{-1} \equiv e_i e^i$$

To calculate e^i , we can post multiply by e_i :

$$\begin{aligned} (e_i)^{-1} e_i e_i &\equiv e^i e_i e_i = 1 \cdot e_i \\ e^i (e_i)^2 &= e_i \\ e^i \|e_i\|^2 &= e_i \\ e^i g_{ii} &= e_i \\ e^i &= \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2} = (e_i)^{-1} \end{aligned}$$

So, in orthogonal metric the inverse of a basis vector is itself divided by its norm squared (by g_{ii}). Everything commented regarding covectors in 3.1 applies also here.

One important consequence of this, is that if the basis vectors are orthogonal (as in this chapter), all the basis vectors and all the inverse of the basis vectors are also orthogonal among them (when $i \neq j$). This is:

$$\begin{aligned} e^i \cdot e_j &= \frac{e_i}{g_{ii}} \cdot e_j = \frac{1}{g_{ii}} (e_i \cdot e_j) = \frac{1}{2g_{ii}} (e_i e_j + e_j e_i) = 0 \\ e^i \cdot e^j &= \frac{e_i}{g_{ii}} \cdot \frac{e_j}{g_{jj}} = \frac{1}{2g_{ii}g_{jj}} (e_i \cdot e_j) = \frac{1}{2g_{ii}g_{jj}} (e_i e_j + e_j e_i) = 0 \end{aligned}$$

In the last equation (but when $i = j$), we get:

$$e^i \cdot e^i = (e^i)^2 = \frac{e_i}{g_{ii}} \cdot \frac{e_i}{g_{ii}} = \frac{1}{g_{ii}g_{ii}} (e_i \cdot e_i) = \frac{1}{g_{ii}g_{ii}} (e_i e_i) = \frac{1}{(g_{ii})^2} \cdot 1 = \frac{1}{(g_{ii})^2}$$

These last properties apply also to Chapter 3.1 (orthonormal basis) but in that case, the elements g_{ii} or g_{jj} are always 1.

3.3. Geometric Algebra $Cl_{3,0}$: Non-Orthogonal (and Therefore Not Orthonormal) Basis

In a non-orthogonal basis, the vectors are not perpendicular from each other. And in general, the norm of the vectors is not one. As commented in 3.2, in Geometric Algebra $Cl_{3,0}$, the norm of the basis vectors is always positive and different from

zero.

From the properties commented in Chapter 2 and also in [3], we can obtain the following equations (for orthogonal, not orthonormal basis):

$$\begin{aligned} (e_i)^2 &= e_i e_i = \|e_i\|^2 = g_{ii} \\ e_i e_j &= 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i \\ e_i \cdot e_j &= e_j \cdot e_i = g_{ij} = g_{ji} \\ e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = g_{ij} + e_i \wedge e_j \end{aligned}$$

Making the equations explicit for three dimensions:

$$\begin{aligned} (e_1)^2 &= e_1 e_1 = \|e_1\|^2 = g_{11} \\ (e_2)^2 &= e_2 e_2 = \|e_2\|^2 = g_{22} \\ (e_3)^2 &= e_3 e_3 = \|e_3\|^2 = g_{33} \\ e_1 e_2 &= 2g_{12} - e_2 e_1 = 2g_{21} - e_2 e_1 \\ e_2 e_3 &= 2g_{23} - e_3 e_2 = 2g_{32} - e_3 e_2 \\ e_3 e_1 &= 2g_{31} - e_1 e_3 = 2g_{13} - e_1 e_3 \end{aligned}$$

where the g_{ij} makes reference again to the metric tensor components (the scalar products of the basis vectors). See paper [3] for more information. You can obtain the above equations from the definition of scalar product in geometric algebra as commented in Chapter 2.

$$\begin{aligned} e_i \cdot e_j &= g_{ij} = \frac{1}{2}(e_i e_j + e_j e_i) \\ 2g_{ij} &= e_i e_j + e_j e_i \\ e_i e_j &= 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i \end{aligned}$$

Now, we will define again the inverse of the basis vectors and name them e^i . To obtain the inverse of the basis vectors in this case, you have to get the inverse of the metric tensor, so you are able to define a vector e^i that fulfills for every i and every j the following (Einstein summation does not apply):

$$\begin{aligned} (e_i)^{-1} e_i &\equiv e^i e_i = 1 = e_i (e_i)^{-1} \equiv e_i e^i \\ e^i \cdot e_j &= e_i \cdot e^j = \frac{1}{2}(e_i e^j + e^j e_i) = 0 \text{ for } i \neq j \\ e^i \cdot e_j &= \delta_j^i \end{aligned}$$

where δ_j^i is the Kronecker Delta, that is equal to 1 when $i = j$ and 0 when $i \neq j$.

If we multiply two inverse vectors between them, in non-orthogonal metric, we do not obtain zero as a general case. See below:

$$e^i \cdot e^j = \frac{1}{2}(e^i e^j + e^j e^i) = g^{ij} = g^{ji}$$

So:

$$e^i e^j = 2g^{ij} - e^j e^i$$

And:

$$e^i e^i = (e^i)^2 = e^i \cdot e^i = g^{ii}$$

In this paper, we will work mainly with orthogonal (or orthonormal basis), so do not worry about these above points. For more info regarding how to invert the metric, you have a lot of literature [5]-[10].

What we will do in general is to make all the calculations with orthogonal metrics and then try to generalize to the case of non-orthogonal metric applying the above relations.

3.4. Geometric Algebra Cl_{3,0}: Sum of Geometric Products of Basis Vectors

We will calculate the following sum. Take into account that the product inside the sum is geometric (not scalar) and that we have not imposed anything regarding the basis (it can be not orthogonal).

$$S = \sum_{i=1}^3 \sum_{j=1}^3 e_i e_j$$

If we operate, we get:

$$\begin{aligned} S &= e_1 e_1 + e_1 e_2 + e_1 e_3 + e_2 e_1 + e_2 e_2 + e_2 e_3 + e_3 e_1 + e_3 e_2 + e_3 e_3 \\ &= e_1 e_1 + e_2 e_2 + e_3 e_3 + (e_1 e_2 + e_2 e_1) + (e_2 e_3 + e_3 e_2) + (e_3 e_1 + e_1 e_3) \\ &= e_1 \cdot e_1 + e_2 \cdot e_2 + e_3 \cdot e_3 + 2(e_1 \cdot e_2) + 2(e_2 \cdot e_3) + 2(e_3 \cdot e_1) \end{aligned}$$

As the scalar product is always symmetric (independently if the basis is orthogonal or not) we can convert the elements that are multiplied by 2, in the sum of two scalar products reversed (with the same result).

$$\begin{aligned} S &= e_1 \cdot e_1 + e_2 \cdot e_2 + e_3 \cdot e_3 + e_1 \cdot e_2 + e_2 \cdot e_1 + e_2 \cdot e_3 + e_3 \cdot e_2 + e_3 \cdot e_1 + e_1 \cdot e_3 \\ &= \sum_{i=1}^3 \sum_{j=1}^3 e_i \cdot e_j = \sum_{i=1}^3 \sum_{j=1}^3 g_{ij} \end{aligned}$$

So:

$$\sum_{i=1}^3 \sum_{j=1}^3 e_i e_j = \sum_{i=1}^3 \sum_{j=1}^3 e_i \cdot e_j = \sum_{i=1}^3 \sum_{j=1}^3 g_{ij}$$

As commented, this holds, independently of the type of metric. And in fact, it holds even for more than three dimensions, but I have preferred to do it explicitly for three dimensions to avoid any doubt and avoid getting lost in the subindices.

Now, consider a symmetric tensor (or a symmetric matrix if you want) that has the components a^{ij} :

$$a^{ij} = a^{ji}$$

And now want to perform the sum (don't worry, I will explain the reason of all this later):

$$\sum_{i=1}^3 \sum_{j=1}^3 a^{ij} e_i e_j$$

Making the same calculation as above (and only if a^{ij} is symmetric), we will obtain a similar result:

$$\sum_{i=1}^3 \sum_{j=1}^3 a^{ij} e_i e_j = \sum_{i=1}^3 \sum_{j=1}^3 a^{ij} (e_i \cdot e_j) = \sum_{i=1}^3 \sum_{j=1}^3 a^{ij} g_{ij}$$

Or using the Einstein notation to simplify:

$$a^{ij} e_i e_j = a^{ij} (e_i \cdot e_j) = a^{ij} g_{ij} \quad \text{only if } a^{ij} = a^{ji}$$

Similarly, we can obtain:

$$a_{ij} e^i e^j = a_{ij} (e^i \cdot e^j) = a_{ij} g^{ij} \quad \text{only if } a_{ij} = a_{ji}$$

But if:

$$a_i^j e^j e_j = a_i^j (e^j \cdot e_j) = a_i^j \delta_j^j = a_i^i \quad \text{only if } a_i^j = a_j^i$$

$$a_i^j e_j e^i = a_i^j (e_j \cdot e^i) = a_i^j \delta_j^i = a_i^i \quad \text{only if } a_i^j = a_j^i$$

Where the last move of above equations is a property of the Kronecker Delta that you can check in [5]-[10].

3.5. Geometric Algebra Cl_{3,0}: Expanding the Basis

One of the properties of the Geometric Algebra is that the number of elements that conform the algebra of a certain realm is more than the number of dimensions of that realm. In three dimensions, we have three basis vectors as commented, but we have 8 different elements that conform that algebra, that are:

- The scalars;
- The three vectors;
- The three bivectors;
- One trivector.

We will call these elements with these names:

$$e_0 \rightarrow \text{scalars}$$

$$e_1 \ e_2 \ e_3$$

$$e_4 = e_2 e_3$$

$$e_5 = e_3 e_1$$

$$e_6 = e_1 e_2$$

$$e_7 = e_1 e_2 e_3$$

Regarding e_0 , I will comment later. In Geometric Algebra, probably you would expect $e_0 = 1$. And this is the natural move, but I will come back to this later, as commented.

The elements e_4, e_5, e_6 are bivectors whose square is negative, as we will see now. And e_7 is the trivector whose square is also negative, as we will see.

In general, we will work with orthogonal (not necessarily orthonormal) basis. About the non-orthogonal case, we will talk explicitly in certain points of the paper. If nothing is said, along the paper, we will work with orthogonal metric that fulfills the following, already commented, relations:

$$(e_i)^2 = e_i e_i = e_i \cdot e_i = \|e_i\|^2 = g_{ii}$$

$$e_i e_j = e_i \wedge e_j = -e_j \wedge e_i = -e_j e_i$$

$$e_i \cdot e_j = e_j \cdot e_i = 0 \quad (\text{when } i \neq j)$$

This is, in 3 dimensions:

$$(e_1)^2 = e_1 e_1 = \|e_1\|^2 = g_{11}$$

$$(e_2)^2 = e_2 e_2 = \|e_2\|^2 = g_{22}$$

$$(e_3)^2 = e_3 e_3 = \|e_3\|^2 = g_{33}$$

$$e_1 e_2 = -e_2 e_1$$

$$e_2 e_3 = -e_3 e_2$$

$$e_3 e_1 = -e_1 e_3$$

The last three equations are key in orthogonal metric and are the ones that will make working with bivectors or the trivector much easier. Because they permit us to swap the order of the vectors in any geometric product, just adding a minus sign for each swap. This means that the result will be the same if we make an even number of swaps. And the result will be the negative of the original if we make an odd number of swaps.

An example. We have the following trivectors and we want to sum them:

$$7e_1 e_2 e_3 + 2e_2 e_1 e_3$$

We swap e_2 and e_1 in the second element and we add a minus sign. This is the same as using one of the equations above.

$$7e_1 e_2 e_3 - 2e_1 e_2 e_3 = 5e_1 e_2 e_3$$

But, take into account that when a basis vector is squared, it is converted to a number, so it does not count as a vector anymore. It is just a number that can be moved in the product not changing signs.

For more info regarding this type of operation, you can check [1]-[4] [11]-[13].

As commented, all these swapping's with changing of sign can only be applied in orthogonal bases. In non-orthogonal bases, you should apply the equations in the beginning of Chapter 3.3.

Knowing this rule, I would just show the squares of the bivectors and the trivector to check that they are in fact negative:

$$(e_4)^2 = (e_2 e_3)^2 = e_2 e_3 e_2 e_3 = -e_2 e_3 e_3 e_2 = -e_2 g_{33} e_2 = -g_{33} e_2 e_2 = -g_{33} g_{22}$$

$$(e_5)^2 = (e_3 e_1)^2 = e_3 e_1 e_3 e_1 = -e_3 e_1 e_1 e_3 = -e_3 g_{11} e_3 = -g_{11} e_3 e_3 = -g_{11} g_{33}$$

$$(e_6)^2 = (e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_2 e_2 e_1 = -e_1 g_{22} e_1 = -g_{22} e_1 e_1 = -g_{22} g_{11}$$

$$\begin{aligned} (e_7)^2 &= (e_1 e_2 e_3)^2 = e_1 e_2 e_3 e_1 e_2 e_3 = +e_1 e_2 e_3 e_3 e_1 e_2 \\ &= g_{33} e_1 e_2 e_1 e_2 = -g_{33} e_1 e_1 e_2 e_2 = -g_{33} g_{11} g_{22} \end{aligned}$$

Remind that the g_{ij} are just numbers, so you can move them as you want along the product. I keep the order obtained in the operations to facilitate the understanding, but you can swap them as you want not changing the sign or the result.

Just to close the chapter, I will comment that an entity that is composed by the sum of scalars, vectors, bivectors etc... is called a multivector. As an example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3e_1$$

This entity A is called a multivector. We will see that in Geometric Algebra any object can be defined by a multivector expression.

The most important comment of this section is the following. In Geometric Algebra, once you have defined the number of dimensions (in this case 3) and the consequent degrees of freedom (or different basis vectors and their combinations, in this case 8, from e_0 to e_7), it does not matter how many operations (sums, geometric products, even exponentials etc...) you do, the number of basis vectors and their combinations are always the same (8 in this case). You can multiply the times you want any multivector by another one, you will only finish with 8 coefficients that multiply 8 basis vectors from e_0 to e_7 (considering also basis vectors their product combinations). Nothing else. This is key in Geometric Algebra and its power.

If you are familiarized with matrices, tensors or tensors products, you know that in those cases the number of elements could grow to infinite (the number of dimensions also). In Geometric Algebra, there is a limit. And this KEY, as we will see.

3.6. Geometric Algebra $Cl_{3,0}$: Comments about e_0 and e_7

Before, I have commented that the natural move is that:

$$e_0 = 1$$

And in general, this is what I would have written in any of my previous papers. But in this case, as we will see later, it is possible that we need a “degree of freedom more” or the possibility that e_0 is a scalar function that depends on certain parameters that we will see later.

So, instead of defining e_0 equal to 1, we will define it as a scalar (this is important, it is a scalar or a function whose output is a scalar, not vectors, not bivectors etc...):

$$e_0 = \sqrt{g_{00}}$$

$$(e_0)^2 = \|e_0\|^2 = g_{00}$$

As commented g_{00} , is a scalar or a function that outputs a scalar (positive-definite). The problem is the conceptual meaning of e_0 and g_{00} . Normally, g_{00} would mean the scalar product of vectors. In this case, it is not that. It is a function that appears only at certain operations that we will see later.

Regarding the possible values of g_{00} are (we will comment later):

$$g_{00} = 1$$

$$g_{00} = \|e_1\|^2 \|e_2\|^2 \|e_3\|^2$$

$$g_{00} = \frac{1}{\|e_1\|^2 \|e_2\|^2 \|e_3\|^2}$$

g_{00} = independent scalar function (positive definite)

As commented, we will keep this nomenclature of g_{00} as in the end it is discovered that it is equal to 1 or to whatever other result we will substitute in the equations. If we put directly that it is equal to 1, it will be more difficult to modify the equations.

Anyhow, for the shake of simplicity, for orthonormal or orthogonal metrics, we will consider $e_0 = 1$ as it most probably is, except in exceptional situations. For non-orthogonal metric, we will keep it indicated as e_0 .

Regarding e_7 the important property as commented is this:

$$(e_7)^2 = (e_1e_2e_3)^2 = e_1e_2e_3e_1e_2e_3 = -g_{33}g_{11}g_{22}$$

This means, its square is negative, and it is a “neutral” vector. Meaning “neutral” that it does not have any “preferred” direction or orientation. The bivectors e_4, e_5, e_6 have also negative square but with “preferred” directions.

$$(e_4)^2 = (e_2e_3)^2 = e_2e_3e_2e_3 = -g_{33}g_{22}$$

$$(e_5)^2 = (e_3e_1)^2 = e_3e_1e_3e_1 = -g_{11}g_{33}$$

$$(e_6)^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -g_{22}g_{11}$$

But e_7 has a negative square and does not point anywhere specific. It applies to the volume in general (not a surface or a line). If you have read the papers [11]-[13] probably you have already seen the possibility that the time vector can be associated with e_7 (the trivector). The reason is that the square of e_7 is negative and that taking this consideration is completely coherent with Dirac Equation, Maxwell equations and Gell-Mann matrices [11]-[14].

When we come to general relativity, the thing gets more complicated. We will see that depending on the context, the scalars e_0 (as considered in APS [15]) or the trivector e_7 can represent time depending on the context. We will see later, but first we need to understand the spinor in Geometric Algebra to understand the different possible contexts.

What we will keep from previous papers [4] [11]-[14] is that as the square of e_7 is negative and does not have any preferred direction. So, when the imaginary unit i is used in traditional algebra, we will substitute it in Geometric Algebra by the trivector e_7 . The reason is that in Geometric Algebra there are already elements as e_7 (appearing in a natural way) whose square is negative.

And the imaginary unit i is used in traditional algebra as an “unknown or generic” element whose square is negative. In Geometric Algebra, what you have to do is, depending on the context, to use the corresponding already existing element in the Algebra (of all the ones whose square is negative) instead of using i . As commented, we will use e_7 for the reasons commented above.

4. The Reverse of a Multivector and the Reverse Product

If we have multivector, the reverse of it can be defined as a multivector with the

same coefficients but where all the products of basis vectors are reversed. An example:

$$A = 3 + 2e_1 - 3e_1 + 7e_3e_1 + 2e_2e_3 - 5e_1e_2e_3$$

Its reverse will be:

$$A^\dagger = 3 + 2e_1 - 3e_1 + 7e_1e_3 + 2e_2e_3 - 5e_3e_2e_1$$

This, in orthogonal metric (not in general) can be converted using Chapter 3.2 equations into:

$$A^\dagger = 3 + 2e_1 - 3e_1 - 7e_3e_1 - 2e_2e_3 + 5e_1e_2e_3 = A^*$$

Being A^* the conjugate multivector. This means, in orthogonal metric the reverse of a multivector is the same as a conjugate of the multivector. The conjugate means changing the sign of the elements whose square is negative (this means: bivectors and trivector) and keeping the same sign for scalars and vectors (whose square is positive).

In a non-orthogonal metric, you should use equations in Chapter 3.3 instead of those in Chapter 3.2, so in a general case, reverse and conjugate will not be the same.

Anyhow, as commented, in this paper we will focus on orthogonal basis, so here reverse and conjugate will be the same in most cases (but this is not true for a general case).

Calculating the reverse for the different basis vectors, we have:

$$\begin{aligned} e_0^\dagger &= e_0 & e_1^\dagger &= e_1 & e_2^\dagger &= e_2 & e_3^\dagger &= e_3 \\ e_4^\dagger &= (e_2e_3)^\dagger & & & & & &= e_3e_2 \\ e_5^\dagger &= (e_3e_1)^\dagger & & & & & &= e_1e_3 \\ e_6^\dagger &= (e_1e_2)^\dagger & & & & & &= e_2e_1 \\ e_7^\dagger &= (e_1e_2e_3)^\dagger & & & & & &= e_3e_2e_1 \end{aligned}$$

One important property is that a product of basis vectors multiplied by its reverse is always positive definite (also in non-orthogonal metrics):

$$\begin{aligned} e_0e_0^\dagger &= e_0e_0 = \|e_0\|^2 = g_{00} \\ e_1e_1^\dagger &= e_1e_1 = \|e_1\|^2 = g_{11} \\ e_2e_2^\dagger &= e_2e_2 = \|e_2\|^2 = g_{22} \\ e_3e_3^\dagger &= e_3e_3 = \|e_3\|^2 = g_{33} \\ e_4e_4^\dagger &= e_2e_3(e_2e_3)^\dagger = e_2e_3e_3e_2 = e_2g_{33}e_2 = g_{33}e_2e_2 = g_{33}g_{22} \equiv g_{44} \\ e_5e_5^\dagger &= e_3e_1(e_3e_1)^\dagger = e_3e_1e_1e_3 = e_3g_{11}e_3 = g_{11}e_3e_3 = g_{11}g_{33} \equiv g_{55} \\ e_6e_6^\dagger &= e_1e_2(e_1e_2)^\dagger = e_1e_2e_2e_1 = e_1g_{22}e_1 = g_{22}e_1e_1 = g_{22}g_{11} \equiv g_{66} \\ e_7e_7^\dagger &= e_1e_2e_3(e_1e_2e_3)^\dagger = e_1e_2e_3e_3e_2e_1 = g_{33}e_1e_2e_2e_1 = g_{33}g_{22}e_1e_1 = g_{33}g_{22}g_{11} \equiv g_{77} \end{aligned}$$

Where I have defined the g_{ii} as the result of these products also for basis vectors with $i > 3$. And also, as commented it is defined a g_{00} as the square for e_0 to have one degree of freedom more (even that very probably defining it as 1, should be ok, meaning just a that pre-normalization has been de-facto done).

As you can guess, the reverse product is just defined as multivector by the reverse of other (or the same) multivector following the rules commented above.

An important thing to comment, is that the reverse should not be mixed up with the inverse.

The inverse of a product of basis vectors is defined as the inverse of each basis vector in reverse order. This is, for example:

$$(e_7)^{-1} = (e_1 e_2 e_3)^{-1} = (e_3)^{-1} (e_2)^{-1} (e_1)^{-1} = e^3 e^2 e^1 = e^7$$

where in the last steps above, I have used the definition of the superscripts as defined in Chapters 3.1, 3.2 and 3.3, as the inverse of the basis vectors. We can check that this hold:

$$e_7 e^7 = e_1 e_2 e_3 e^3 e^2 e^1 = e_1 e_2 \cdot 1 \cdot e^2 e^1 = e_1 \cdot 1 \cdot e^1 = 1$$

So, in fact, it corresponds to the inverse of e_7 . The same applies, to the rest of vectors:

$$\begin{aligned} (e_1)^{-1} &= e^1 & (e_2)^{-1} &= e^2 & (e_3)^{-1} &= e^3 \\ (e_4)^{-1} &= (e_2 e_3)^{-1} = (e_3)^{-1} (e_2)^{-1} = e^3 e^2 = e^4 \\ (e_5)^{-1} &= (e_3 e_1)^{-1} = (e_1)^{-1} (e_3)^{-1} = e^1 e^3 = e^5 \\ (e_6)^{-1} &= (e_1 e_2)^{-1} = (e_2)^{-1} (e_1)^{-1} = e^2 e^1 = e^6 \\ (e_7)^{-1} &= (e_1 e_2 e_3)^{-1} = (e_3)^{-1} (e_2)^{-1} (e_1)^{-1} = e^3 e^2 e^1 = e^7 \end{aligned}$$

So, you can see that the inverse also reverses the order, but besides that, it inverts the basis vectors (converts the subscripts in superscripts and vice-versa).

5. Spinor in Geometric Algebra Cl_{3,0}

A spinor in matrix notation has this form:

$$\psi = \begin{pmatrix} \psi_{1r} + \psi_{1i}i \\ \psi_{2r} + \psi_{2i}i \\ \psi_{3r} + \psi_{3i}i \\ \psi_{4r} + \psi_{4i}i \end{pmatrix}$$

As you can see, it has eight parameters:

$$\psi_{1r} \ \psi_{1i} \ \psi_{2r} \ \psi_{2i} \ \psi_{3r} \ \psi_{3i} \ \psi_{4r} \ \text{and} \ \psi_{4i}$$

In Geometric Algebra, the spinor has this form:

$$\psi = \psi^\mu e_\mu = \psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7$$

where e_i are the elements (scalars, vectors, bivectors and trivector) as defined in Chapter 3.5.

The ψ^i are the coefficients of the spinor or wavefunction. You can see that they are also eight as in the matrix notation. You can find a relation between both in [11] [14] and [16]. There you can find that that relation is coherent with Dirac Equation and Strong Force Interaction (Gell-Mann matrices).

For this paper, we will just stick to that these 8 coefficients are sufficient to define a spinor or wavefunction. And calculating them is what we need to define the state of a particle or a related field.

6. Probability Density and Probability Current

As we saw in [14], we can calculate probability density and probability current multiplying the reverse of the wavefunction by itself, this way:

$$\psi^\dagger \psi = (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger) (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7)$$

where all the vectors, bivectors and the trivector and their reverses, are as defined in Chapter 4 and previous ones.

Only in the case of orthogonal metric (not in the general case), this can be simplified as (the reverse is the same as the conjugate):

$$\begin{aligned} \psi^\dagger \psi &= \psi^* \psi \\ &= (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7) (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \end{aligned}$$

As you can see in **Annex A2**, the result of this multiplication is for the orthogonal case is:

$$\psi^\dagger \psi = \rho + \vec{j}$$

Being:

$$\begin{aligned} \rho &= (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} \\ &\quad + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33} \\ \vec{j} &= 2(\psi^0 \psi^1 - \psi^2 \psi^6 g_{22} + \psi^3 \psi^5 g_{33} + \psi^4 \psi^7 g_{22} g_{33}) e_1 \\ &\quad + 2(+\psi^0 \psi^2 + \psi^1 \psi^6 g_{11} - \psi^4 \psi^3 g_{33} + \psi^5 \psi^7 g_{33} g_{11}) e_2 \\ &\quad + 2(+\psi^0 \psi^3 - \psi^1 \psi^5 g_{11} + \psi^2 \psi^4 g_{22} + \psi^6 \psi^7 g_{11} g_{22}) e_3 \end{aligned}$$

Being ρ the probability and \vec{j} the fermionic current.

But we can say that even in the general case where the basis is not orthogonal or even if the product above is defined another way, the result will have for sure have this form:

$$\psi^\dagger \psi = j^\mu e_\mu$$

In **Annexes A1 - A4**, you can find that in whatever metric you are or however this product is defined (in **Annex A4**, it is shown an example using the inverse product instead of the reverse product), the result will always have this form:

$$\psi^\dagger \psi = j^\mu e_\mu$$

where μ and ν go from 0 to 7 in the most general case. This means, independently of the metric, independently if the product is correctly defined or are some elements pending (see **Annexes A1 - A4** for details), what it is true is that the result, will have the form above.

Even if we calculate wrongly the coefficients of j^μ , we can continue with our study as these coefficients will represent a general case. In case they change the value, we will change the operations done, but the study following will be perfectly correct as the meaning of the coefficients j^μ is general. This is the power of geometric algebra. We know the form of the results even if we have calculated them wrong. We know that the result will have 8 components j^μ (very important, scalar coefficients or functions that output a scalar) multiplying 8 basis vectors (considering their product combinations also, this means, considering them from e_0 to e_7).

Last comment to make are the measuring units of this $j^\mu e_\nu$. For the j^0 component the units are density of probability in 3D space, this means probability/cubic length. Probability does not have units, so it is L^{-3} .

The components j^1 to j^3 are called the probability current and its units are density of probability multiplied by velocity. As probability does not have units, the density has L^{-3} and the speed has LT^{-1} , the total units are $L^{-2}T^{-1}$. To make these units coherent with j^0 , we have to multiply j^0 by c (the speed of light) or the opposite, to divide the components of j^1 to j^3 by it.

As commented, for orthonormal or orthogonal bases, j^μ only has components from 0 to 3. For the general case, it would have components from 0 to 7 and the measuring units should be harmonized with the units that have the components from 0 to 3. But we will not care about that now, we will just consider that we can find a coherent following expression with coherent units:

$$\psi^\dagger \psi = j^\mu e_\mu$$

Just to finalize, I will comment that to be consequent with certain papers in the literature [17], sometimes I will use the following nomenclature, but you can check that the concept is the same, just changing the name of j to V , and the dummy index from μ to ρ :

$$\psi^\dagger \psi = j^\mu e_\mu = V^\rho e_\rho$$

7. Definition of Covariant Operator in Geometric Algebra

We will define the following operator:

$$e^\mu \nabla_\mu$$

where ∇_μ is the covariant derivative. This means, if it is applied to a scalar function, it will be just the partial derivative with respect to μ of it. If f is a scalar function:

$$e^\mu \nabla_\mu f = e^\mu \frac{\partial f}{\partial e^\mu}$$

where the partial derivative is taken with respect to the coordinate variable that corresponds to the vector e_μ . This means, that ∂e^1 would mean derivative with respect to the coordinate variable associated to e_1 (typically x in cartesian coordinates, or r in polar coordinates or called e^1 in the general case). It is important to recall that in this paper, the coefficients that multiply the vectors are scalars (not “covectors”), so the rule above, apply to them (to the coefficients). It does not apply to the vectors as you can see below.

If the function includes vectors, apart from the partial derivative of the coefficients that multiply these vectors, we will have to apply the covariant derivative to the vectors.

The covariant derivative of the basis vectors (you can check this in different literature of General Relativity or Riemann geometries [5]-[10] [17]) are the Christoffel symbols.

So, applying the product rule of derivation we get:

$$e^\mu \nabla_\mu (f^\nu e_\nu) = e^\mu (\nabla_\mu f^\nu) e_\nu + e^\mu f^\nu (\nabla_\mu e_\nu)$$

And it is important that we are keeping the same order of the vectors. Remember they are not commutative in the general case.

Now, for the scalar coefficients f^ν , we can use the same equation shown before (partial derivative equation). For the other term (the covariant derivative of a basis vector), we will use the Christoffel symbols as they are defined [5]-[10] [17].

$$e^\mu \nabla_\mu (f^\nu e_\nu) = e^\mu (\nabla_\mu f^\nu) e_\nu + e^\mu f^\nu (\nabla_\mu e_\nu) = e^\mu \frac{\partial f^\nu}{\partial e^\mu} e_\nu + e^\mu f^\nu \Gamma_{\mu\nu}^\lambda e_\lambda$$

As the partial derivative of the coefficients of f and the Christoffel symbols are just scalars (yes, in this context, Christoffel symbols are just scalars that multiply vectors), we can move the vectors as follows:

$$e^\mu \nabla_\mu (f^\nu e_\nu) = e^\mu \frac{\partial f^\nu}{\partial e^\mu} e_\nu + e^\mu f^\nu \Gamma_{\mu\nu}^\lambda e_\lambda = e^\mu e_\nu \frac{\partial f^\nu}{\partial e^\mu} + e^\mu e_\lambda f^\nu \Gamma_{\mu\nu}^\lambda$$

Another thing to comment is that we can calculate also the covariant derivative of the inverse of a vector this way [5]-[10] [17].

$$\begin{aligned} \nabla_\beta (e_\mu (e^\alpha)^{-1}) &= \nabla_\beta (e_\mu e^\alpha) = \nabla_\beta (\delta_\mu^\alpha) = 0 \\ \nabla_\beta (e_\mu) e^\alpha + e_\mu \nabla_\beta (e^\alpha) &= \Gamma_{\beta\mu}^\lambda e_\lambda e^\alpha + e_\mu \nabla_\beta (e^\alpha) = 0 \\ e_\mu \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\lambda e_\lambda e^\alpha \\ e_\mu \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\lambda \delta_\lambda^\alpha \\ e_\mu \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\alpha \\ e^\mu e_\mu \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\alpha e^\mu \\ \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\alpha e^\mu \end{aligned}$$

So, this above, and the already commented classical definition covariant derivative of basis vector:

$$\nabla_{\beta}(e_{\alpha}) = \Gamma^{\mu}_{\beta\alpha} e_{\mu}$$

They are the equations we will need in following chapters. Also, to comment something that we will need in some steps. The geometric product is not commutative in general. But sometimes we will have to commute the vectors. To do so, we have to consider one of these three scenarios:

- The metric is orthogonal. So, the geometric product is the same as scalar product, and therefore commutative.
- We are in a situation as in Chapter 3.4. This is, the symmetry of the sums in certain situations, “convert” the geometric products in scalar products. So, the same as commented above applies.
- The other option is directly that we are forced to change the definition of the operators, using scalar products instead of geometric products. As an example, in certain situations, we can say, instead of using the operator:

$$e^{\mu} \nabla_{\mu}$$

We could decide to use:

$$e^{\mu} \cdot \nabla_{\mu}$$

Loosing generality (all the non-commutative elements will be lost), rigor and probably some solutions, but as a way to move forward.

Just to finish, we will define the reverse (the reverse not the inverse) of the covariant operator to a function f as:

$$(e^{\mu} \nabla_{\mu} f)^{\dagger} = f \nabla_{\mu}^{\dagger} e^{\mu} = (f \nabla_{\mu}^{\dagger}) e^{\mu} = (\nabla_{\mu} f) e^{\mu}$$

This means, when we see the reverse operator, we have to take into account these things:

- The operator applies to the function on the **left** of it (not on the right as it is usual).
- The vector that accompanies it, it is located on the right of the operator, not on the left as defined from the non-reverse operator.

Probably you are asking why the vector that accompanies the function is not reversed as well. In general, I would say that the logic thing would be to reverse it, creating sometimes changes on signs (or even real changes in result in non-orthogonal metric). In this paper, I will keep it as not reversed to facilitate the things and the message, but it could be that in the future, the definition, changes to be reversed.

Also, you can ask why the f is not reversed as well. The answer is that to keep the symmetry, it should be reversed. But to simplify the nomenclature, we will keep f not reversed, and just indicate it directly in the expression if this is the case.

Another thing we could think about is that if the operator is reversed, we should add a minus sign to the derivative as we are deriving in the opposite direction to

the one represented by the variable. This is true in fact. But as we will always make double derivatives (in the left and in the right, see later), in the end, this will only lead to a change of sign in the final results, not affecting the implicit meaning. Anyhow, this is something that probably has to be taken into account in the future (and also if it is needed or not to reverse the vectors that accompany the derivative/del operator).

The last comment is that in Geometric Algebra everything is done keeping symmetries. When a double operator has to be applied (like a Laplacian) it is not generally done as a double operator on the left. Instead, it is done like a simple operator in the left and another simple operator on the right (that is applying to the elements on the left).

The reason for this is that in geometric algebra the order of the vectors matters. As it is not the same pre-multiplying than post-multiplying. Because the products are not in general commutative or anticommutative, it depends on the product itself (the number of vectors and its grade). So, the only way to keep the symmetries is to keep the balance of operators on the left and in the right as much as possible.

When this happens, we will have the convention that we will start applying the reverse del operator (the one in the right, and afterwards the non-reverse del operator, the one in the left). This is just by convention. Taking into account that normally we work with commutators in our calculations, a change of this will only lead to a change of signs in the final results.

Apart from this, this will let us also facilitate the factorization of the equations that will be key to simplify them in following chapters.

8. Ricci Tensor in Geometric Algebra

As we can see in different papers [5]-[10] [17], the Ricci Tensor can be considered as the Laplacian of the basis vectors. Taking into account what we have commented about the covariant derivative in the previous chapter, we can calculate the Laplacian as a covariant derivative on the left and another covariant derivative on the right to keep the symmetry. And to be in the most general case as possible, instead of applying to the basis vectors, I will apply to a complete field that includes coefficients and vectors:

$$V^\rho e_\rho$$

If you want to apply only to basis vectors just consider:

$$V^\rho = 1 \text{ for every } \rho$$

And:

$$V_{,\mu}^\rho = 0$$

where the comma represents partial derivative with respect to e^μ .

Ok, so let's apply the operator defined in Chapter 7 to $V^\rho e_\rho$ to the left and the reverse of it, to the right. We will start operating the one of the right (the reverse

operator). This is just by convention as commented in Chapter 7. If we do the opposite, we will obtain a different result. But we will see that it does not even really matters, as we will perform the reverse operation later.

$$\begin{aligned}
 & e^\mu \nabla_\mu V^\rho e_\rho \nabla_v^\dagger e^v \\
 & e^\mu \nabla_\mu \left((V^\rho e_\rho \nabla_v^\dagger) e^v \right) \\
 & e^\mu \nabla_\mu \left(\nabla_v V^\rho e_\rho \right) e^v \\
 & e^\mu \nabla_\mu \left((\nabla_v V^\rho e_\rho) e^v \right)
 \end{aligned}$$

It is very important to remark that the coefficients V^ρ are just scalars. Their covariant derivative is just the partial derivative.

And for the vectors, we will apply the equations shown in Chapter 7:

$$\begin{aligned}
 \nabla_\beta (e_\alpha) &= \Gamma_{\beta\alpha}^\mu e_\mu \\
 \nabla_\beta (e^\alpha) &= -\Gamma_{\beta\mu}^\alpha e^\mu
 \end{aligned}$$

And to remark that in this context, the Christoffel symbols are just scalar coefficients, that multiply vectors. So, the covariant derivative of the Christoffel symbol itself is the partial derivative. The covariant of the vectors that accompany them will be done naturally following the derivative product rule.

We start calculating, the expression inside the brackets:

$$\nabla_v V^\rho e_\rho = V_{,v}^\rho e_\rho + V^\rho \Gamma_{v\rho}^\sigma e_\sigma$$

I change the name of the dummy coefficients for convenience and to follow [17]:

$$\nabla_v V^\rho e_\rho = V_{,v}^\rho e_\rho + V^\sigma \Gamma_{v\sigma}^\rho e_\rho$$

Now, I just post-multiply by the vector that appeared in the original equation at the beginning of the paper:

$$(\nabla_v V^\rho e_\rho) e^v = V_{,v}^\rho e_\rho e^v + V^\sigma \Gamma_{v\sigma}^\rho e_\rho e^v$$

Now, I proceed with the covariant derivative that was in the left (that applies to all the expression above, including the two vectors):

$$\begin{aligned}
 \nabla_\mu \left((\nabla_v V^\rho e_\rho) e^v \right) &= \nabla_\mu \left(V_{,v}^\rho e_\rho e^v + V^\sigma \Gamma_{v\sigma}^\rho e_\rho e^v \right) \\
 &= V_{,v\mu}^\rho e_\rho e^v + V_{,v}^\rho \Gamma_{\rho\mu}^\sigma e_\sigma e^v - V_{,v}^\rho e_\rho \Gamma_{\mu\sigma}^\nu e^\sigma + V_{,\mu}^\sigma \Gamma_{v\sigma}^\rho e_\rho e^v \\
 &\quad + V^\sigma \Gamma_{v\sigma,\mu}^\rho e_\rho e^v + V^\sigma \Gamma_{v\sigma}^\rho \Gamma_{\rho\mu}^\lambda e_\lambda e^v - V^\sigma \Gamma_{v\sigma}^\rho e_\rho \Gamma_{\mu\lambda}^\nu e^\lambda
 \end{aligned}$$

I change again the name of dummy variables to follow [17] nomenclature:

$$\begin{aligned}
 & V_{,v\mu}^\rho e_\rho e^v + V_{,v}^\lambda \Gamma_{\lambda\mu}^\rho e_\rho e^v - V_{,\lambda}^\rho e_\rho \Gamma_{\mu\nu}^\lambda e^\nu + V_{,\mu}^\sigma \Gamma_{v\sigma}^\rho e_\rho e^v \\
 & + V^\sigma \Gamma_{v\sigma,\mu}^\rho e_\rho e^v + V^\sigma \Gamma_{v\sigma}^\lambda \Gamma_{\lambda\mu}^\rho e_\rho e^v - V^\sigma \Gamma_{\lambda\sigma}^\rho e_\rho \Gamma_{\mu\nu}^\lambda e^\nu \\
 & V_{,v\mu}^\rho e_\rho e^v + V_{,v}^\lambda \Gamma_{\lambda\mu}^\rho e_\rho e^v - V_{,\lambda}^\rho \Gamma_{\mu\nu}^\lambda e_\rho e^v + V_{,\mu}^\sigma \Gamma_{v\sigma}^\rho e_\rho e^v \\
 & + V^\sigma \Gamma_{v\sigma,\mu}^\rho e_\rho e^v + V^\sigma \Gamma_{v\sigma}^\lambda \Gamma_{\lambda\mu}^\rho e_\rho e^v - V^\sigma \Gamma_{\lambda\sigma}^\rho \Gamma_{\mu\nu}^\lambda e_\rho e^v
 \end{aligned}$$

Now, we pre-multiply by the vector as it was stated in original equation in the

beginning of the chapter:

$$\begin{aligned}
 e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu &= e^\mu \nabla_\mu \left((\nabla_\nu V^\rho e_\rho) e^\nu \right) \\
 &= V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu \\
 &\quad + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu
 \end{aligned}$$

Now, we calculate the result with the operations reversed. This is, the operator on the left with respect to ν and the reverse operator in the right with respect to μ :

$$\begin{aligned}
 e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu &= e^\nu \nabla_\nu \left((\nabla_\mu V^\rho e_\rho) e^\mu \right) \\
 &= V^\rho_{,\mu\nu} e^\nu e_\rho e^\mu + V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu - V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu + V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\nu e_\rho e^\mu \\
 &\quad + V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\nu e_\rho e^\mu + V^\sigma \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\lambda\nu} e^\nu e_\rho e^\mu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu
 \end{aligned}$$

Now, let's calculate the subtraction of one to another (let's say the commutator of this operation):

$$\begin{aligned}
 e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu &= V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu \\
 &\quad + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu \\
 &\quad - V^\rho_{,\mu\nu} e^\nu e_\rho e^\mu - V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\nu e_\rho e^\mu + V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu - V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\nu e_\rho e^\mu \\
 &\quad - V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\nu e_\rho e^\mu - V^\sigma \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\lambda\nu} e^\nu e_\rho e^\mu + V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\nu e_\rho e^\mu
 \end{aligned}$$

To be able to perform, this operation we have to be able to “move” vectors inside the products. This can only be done if we are in one of three cases commented in Chapter 7. This is: orthogonal metric, summation of symmetric elements (Chapter 3.4) or changing the geometric product by the scalar product in the definition of the covariant operator.

So, we will consider that we are in one of these three cases and let's move the position of the vectors inside the products at our convenience:

$$\begin{aligned}
 e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu &= V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu \\
 &\quad + V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu \\
 &\quad - V^\rho_{,\mu\nu} e^\mu e_\rho e^\nu - V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\mu e_\rho e^\nu + V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\mu e_\rho e^\nu - V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\mu e_\rho e^\nu \\
 &\quad - V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\lambda\nu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\mu e_\rho e^\nu
 \end{aligned}$$

We see that the only elements left (the ones that do not cancel) are the ones in bold. See [17] for more info.

$$\begin{aligned}
 &V^\rho_{,\nu\mu} e^\mu e_\rho e^\nu + V^\lambda_{,\nu} \Gamma^\rho_{\lambda\mu} e^\mu e_\rho e^\nu - V^\rho_{,\lambda} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu + V^\sigma_{,\mu} \Gamma^\rho_{\nu\sigma} e^\mu e_\rho e^\nu \\
 &+ V^\sigma \Gamma^\rho_{\nu\sigma,\mu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\lambda\mu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\mu\nu} e^\mu e_\rho e^\nu \\
 &- V^\rho_{,\mu\nu} e^\mu e_\rho e^\nu - V^\lambda_{,\mu} \Gamma^\rho_{\lambda\nu} e^\mu e_\rho e^\nu + V^\rho_{,\lambda} \Gamma^\lambda_{\nu\mu} e^\mu e_\rho e^\nu - V^\sigma_{,\nu} \Gamma^\rho_{\mu\sigma} e^\mu e_\rho e^\nu \\
 &- V^\sigma \Gamma^\rho_{\mu\sigma,\nu} e^\mu e_\rho e^\nu - V^\sigma \Gamma^\rho_{\mu\sigma} \Gamma^\lambda_{\lambda\nu} e^\mu e_\rho e^\nu + V^\sigma \Gamma^\rho_{\lambda\sigma} \Gamma^\lambda_{\nu\mu} e^\mu e_\rho e^\nu
 \end{aligned}$$

This is:

$$\begin{aligned}
 &= V^\sigma \left(\Gamma^\rho_{\nu\sigma,\mu} + \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} - \Gamma^\rho_{\mu\sigma,\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu} \right) e^\mu e_\rho e^\nu \\
 &= V^\sigma \left(\Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\lambda_{\nu\sigma} \Gamma^\rho_{\lambda\mu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu} \right) e^\mu e_\rho e^\nu
 \end{aligned}$$

As V^σ and the Christoffel symbols are just scalars in this context I can move it freely inside the product.

$$\begin{aligned} &= \left(\Gamma_{\nu\sigma,\mu}^\rho - \Gamma_{\mu\sigma,\nu}^\rho + \Gamma_{\lambda\mu}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\lambda\nu}^\rho \Gamma_{\mu\sigma}^\lambda \right) V^\sigma e^\mu e_\rho e^\nu \\ &= R_{\sigma\mu\nu}^\rho V^\sigma e^\mu e_\rho e^\nu \end{aligned}$$

where $R_{\sigma\mu\nu}^\rho$ is the Riemann tensor, as commented in [17].

Now, if we consider that we are within one of the three cases commented in Chapter 7, we can consider that this product is scalar and therefore:

$$e^\mu e_\rho = e^\mu \cdot e_\rho = \delta_\rho^\mu$$

So:

$$R_{\sigma\mu\nu}^\rho V^\sigma e^\mu e_\rho e^\nu = R_{\sigma\mu\nu}^\rho V^\sigma \delta_\rho^\mu e^\nu = R_{\sigma\mu\nu}^\mu V^\sigma e^\nu$$

Now checking [17], we can see that the last element is the Ricci tensor.

$$R_{\sigma\mu\nu}^\mu V^\sigma e^\nu = R_{\sigma\nu} V^\sigma e^\nu$$

So, summing up we can say that (in the last step, I have just used the property that dummy indices can be renamed as convenience):

$$e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu = R_{\sigma\nu} V^\sigma e^\nu = R_{\mu\nu} V^\mu e^\nu$$

If we want to isolate the Ricci tensor, we could do:

$$\begin{aligned} (R_{\sigma\nu} V^\sigma e^\nu) e_\nu V_\sigma &= R_{\sigma\nu} V^\sigma e^\nu e_\nu V_\sigma = R_{\sigma\nu} V^\sigma \cdot 1 \cdot V_\sigma = R_{\sigma\nu} V^\sigma V_\sigma = R_{\sigma\nu} \\ (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu V_\sigma &= (R_{\sigma\nu} V^\sigma e^\nu) e_\nu V_\sigma = R_{\sigma\nu} \\ R_{\sigma\nu} &= (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu V_\sigma \end{aligned}$$

If we want to calculate the Ricci scalar [5]-[10] [17], we can do:

$$\begin{aligned} R &= g^{\sigma\nu} R_{\sigma\nu} = g^{\sigma\nu} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu V_\sigma \\ &= (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) g^{\sigma\nu} e_\nu V_\sigma \end{aligned}$$

Another way to obtain it (but not isolating it):

$$\begin{aligned} &(e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) = R_{\sigma\nu} V^\sigma e^\nu \\ g^{\sigma\lambda} g^{\nu\theta} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) &= g^{\sigma\lambda} g^{\nu\theta} R_{\sigma\nu} V^\sigma e^\nu \\ g^{\sigma\lambda} g^{\nu\theta} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) &= R^{\lambda\theta} V^\sigma e^\nu \\ g_{\lambda\theta} g^{\sigma\lambda} g^{\nu\theta} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) &= g_{\lambda\theta} R^{\lambda\theta} V^\sigma e^\nu \\ g_{\lambda\theta} g^{\sigma\lambda} g^{\nu\theta} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) &= R V^\sigma e^\nu \\ g^{\sigma\nu} (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) &= R V^\sigma e^\nu \end{aligned}$$

9. Klein-Gordon Equation of a Field

We consider the definition of stress-energy tensor of a scalar field [18]-[20]. We will not use natural units. It is better to use real units with factors so we can control that the measuring units of the variables are coherent:

$$G_{\mu\nu} = T_{\mu\nu} = 2\hbar^2 \partial_\mu \phi \partial_\nu \phi - \hbar^2 g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - g_{\mu\nu} m^2 c^2 \phi^2$$

We divide by $2m$:

$$T_{\mu\nu} = \frac{\hbar^2}{m} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\mu\nu} mc^2 \phi^2$$

It is important to check that the measuring units are coherent. $\frac{\hbar^2}{m}$ units are Energy·L². But there are always two derivatives with respect two spatial coordinates that creates a L⁻². So, the units of the first two elements are energy. The last element mc^2 is energy also. So, in principle ok. But the stress energy tensor should have units that are Energy·L⁻³. Do not worry, we will solve this later, as the field that only appears in the right-hand side elements will have L⁻³ units, leaving everything ok.

The first thing we will do is to apply the operator we defined in Chapter 7. But as there are some vectors missing to be able to do that, we will just multiply and divide by them, leaving everything ok.

$$T_{\mu\nu} = \frac{\hbar^2}{m} e_\mu e^\mu \partial_\mu \phi \partial_\nu \phi e^\nu e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha e^\alpha \partial_\alpha \phi \partial_\beta \phi e^\beta e_\beta - \frac{1}{2} g_{\mu\nu} mc^2 \phi^2$$

$$T_{\mu\nu} = \frac{\hbar^2}{m} e_\mu (e^\mu \partial_\mu \phi \partial_\nu \phi e^\nu) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\alpha \partial_\alpha \phi \partial_\beta \phi e^\beta) e_\beta - \frac{1}{2} g_{\mu\nu} mc^2 \phi^2$$

And here's the drill. Instead of applying this to a scalar field as it was original conceived by the equation, we will apply it to a vector field. We have the tools commented in Chapters 7 and 8 to make all the operation so we can do it. We will apply to a general field that is:

$$V^\rho e_\rho$$

And the double derivatives will be left and reverse right derivatives (keeping the symmetries as always in geometric algebra), instead of two left derivatives.

$$T_{\mu\nu} = \frac{\hbar^2}{m} e_\mu (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\alpha \nabla_\alpha V^\rho e_\rho \nabla_\beta^\dagger e^\beta) e_\beta - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho$$

I add the following elements to the equation. I can do it, because its sum is zero:

$$-\frac{\hbar^2}{m} e_\mu (e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha) e_\beta + \frac{\hbar^2}{m} e_\mu (e^\nu \nabla_\nu V^\rho e_\rho \bar{\nabla}_\mu e^\mu) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha) e_\beta$$

Once added, we have:

$$T_{\mu\nu} = \frac{\hbar^2}{m} e_\mu (e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\alpha \nabla_\alpha V^\rho e_\rho \nabla_\beta^\dagger e^\beta) e_\beta - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho - \frac{\hbar^2}{m} e_\mu (e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha) e_\beta + \frac{\hbar^2}{m} e_\mu (e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha) e_\beta$$

Reordering:

$$\begin{aligned}
 T_{\mu\nu} &= \frac{\hbar^2}{m} e_\mu \left(e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu \right) e_\nu - \frac{\hbar^2}{m} e_\mu \left(e^\nu \nabla_\nu V^\rho e_\rho \bar{\nabla}_\mu e^\mu \right) e_\nu \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\alpha \nabla_\alpha V^\rho e_\rho \nabla_\beta^\dagger e^\beta \right) e_\beta \\
 &\quad + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right) e_\beta \\
 &\quad - \frac{1}{2} g_{\mu\nu} m c^2 V^\rho e_\rho + \frac{\hbar^2}{m} e_\mu \left(e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) e_\nu \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right) e_\beta
 \end{aligned}$$

Factorizing as possible:

$$\begin{aligned}
 T_{\mu\nu} &= \frac{\hbar^2}{m} e_\mu \left(e^\mu \nabla_\mu V^\rho e_\rho \nabla_\nu^\dagger e^\nu - e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) e_\nu \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\alpha \nabla_\alpha V^\rho e_\rho \nabla_\beta^\dagger e^\beta - e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right) e_\beta \\
 &\quad - \frac{1}{2} g_{\mu\nu} m c^2 V^\rho e_\rho + \frac{\hbar^2}{m} e_\mu \left(e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) e_\nu \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right) e_\beta
 \end{aligned}$$

Applying the relation to the Ricci tensor commented in 8:

$$\begin{aligned}
 T_{\mu\nu} &= \frac{\hbar^2}{m} e_\mu \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) e_\nu - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) e_\beta \\
 &\quad - \frac{1}{2} g_{\mu\nu} m c^2 V^\rho e_\rho + \frac{\hbar^2}{m} e_\mu \left(e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) e_\nu \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right) e_\beta
 \end{aligned}$$

Now, again we will suppose that the vectors can be moved inside the product, following one of the three possible cases commented in 7 (orthogonal metric, sum over symmetric elements or defining from the beginning that the products are scalar instead of geometric, losing solutions and rigor).

$$\begin{aligned}
 T_{\mu\nu} &= \frac{\hbar^2}{m} e_\mu e_\nu \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha e_\beta \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) \\
 &\quad - \frac{1}{2} g_{\mu\nu} m c^2 V^\rho e_\rho + \frac{\hbar^2}{m} e_\mu e_\nu \left(e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} e_\alpha e_\beta \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right)
 \end{aligned}$$

If the products are scalars (following the three cases in Chapter 7), the geometric product of two vectors is the metric (or delta if they are inverse).

$$\begin{aligned}
 T_{\mu\nu} &= \frac{\hbar^2}{m} g_{\mu\nu} \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} g_{\alpha\beta} \left(R_{\sigma\lambda} V^\sigma e^\lambda \right) \\
 &\quad - \frac{1}{2} g_{\mu\nu} m c^2 V^\rho e_\rho + \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu \right) \\
 &\quad - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} g^{\alpha\beta} g_{\alpha\beta} \left(e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha \right)
 \end{aligned}$$

Operating:

$$T_{\mu\nu} = \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{\hbar^2}{m} g_{\mu\nu} (e^\nu \nabla_\nu V^\rho e_\rho \nabla_\mu^\dagger e^\mu) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Changing the dummy variables names:

$$T_{\mu\nu} = \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha) - \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Operating:

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Now, I multiply by $e_\sigma e^\sigma$ to simplify the operations and get to the Ricci scalar.

I could obtain the same result, multiplying by $g^{\lambda\sigma} g_{\lambda\sigma}$:

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e^\lambda e_\sigma e^\sigma) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Here, I can move the vectors inside the product considering the 3 cases of Chapter 7 (this is not even necessary if I use $g^{\lambda\sigma} g_{\lambda\sigma}$ instead of $e_\sigma e^\sigma$):

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e_\sigma e^\lambda e^\sigma) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R_{\sigma\lambda} V^\sigma e_\sigma g^{\lambda\sigma}) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Now, I just change nomenclature of dummy indices:

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (g^{\lambda\rho} R_{\rho\lambda} V^\rho e_\rho) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

It is not clear of the following move can be done or not. If it cannot be done. Just substitute R by $g^{\lambda\rho} R_{\rho\lambda}$ in the following equations.

$$T_{\mu\nu} = \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (R V^\rho e_\rho) - \frac{1}{2} g_{\mu\nu} mc^2 V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) V^\rho e_\rho + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} (e^\beta \nabla_\beta V^\rho e_\rho \nabla_\alpha^\dagger e^\alpha)$$

Here, it comes another drill. We have seen that the solution to:

$$\psi^\dagger \psi = j^\mu e_\mu$$

And just changing nomenclature, we can consider that it has the form:

$$\psi^\dagger \psi = j^\mu e_\mu = V^\rho e_\rho$$

So why not applying the above equations to $\psi^\dagger\psi$ when appears $V^\rho e_\rho$? This is to apply the equation to collapsed waveform of a particle. This is, to its probability and fermionic current. As you know the units of $\psi^\dagger\psi$ is L^{-3} . This is because the probability does not have units, but $\psi^\dagger\psi$ represents the density of probability. This is probability divided by volume (L^{-3}). So here, we solve the issue of the measuring units. They are Energy· L^{-3} in all the elements.

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger\psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu$$

Another thing we could do to simplify even more, considering we can move the vectors freely inside the products and that they are scalar multiplied (3 cases of Chapter 7) is:

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu e_\nu \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} e_\mu e_\nu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) g_{\mu\nu} \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

Now, we can define a multivector (not even tensor):

$$T = g^{\mu\nu} T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) g^{\mu\nu} g_{\mu\nu} \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} g^{\mu\nu} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

$$T = g^{\mu\nu} T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

Which result is not a scalar. It is a multivector with elements in the eight vectors (scalars, 3 vectors, 3 bivectors and trivector).

Above, the stress-energy tensor is treated as independent of the particle, or the field we are considering. Below, we will see three examples of using this equation, taking into account possible relations between the particle and this tensor.

9.1. Considering That the Stress Energy Tensor Is Zero

If we consider that the stress energy tensor is zero (vacuum solution), we can calculate as follows:

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) g_{\mu\nu} \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

$$0 = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu$$

$$-\frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu = \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu$$

$$-\left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu = \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu$$

$$\begin{aligned} \left(-\frac{\hbar^2}{m}R + mc^2\right)e_\mu\psi^\dagger\psi e_\nu &= \frac{\hbar^2}{m}e_\mu\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)e_\nu \\ \frac{\hbar^2}{m}e_\mu\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)e_\nu &= \left(mc^2 - \frac{\hbar^2}{m}R\right)e_\mu\psi^\dagger\psi e_\nu \\ e_\mu\left(e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha\right)e_\nu &= \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)e_\mu\psi^\dagger\psi e_\nu \\ e^\beta\nabla_\beta\psi^\dagger\psi\nabla_\alpha^\dagger e^\alpha &= \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi \\ e^\beta\nabla_\beta\left(\nabla_\alpha\left(\psi^\dagger\psi\right)e^\alpha\right) &= \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi \end{aligned}$$

We can see that equation obtained, takes into account to calculate wavefunction not only the energy of the particle but also curvature conditions of the space-time in its position (scalar curvature R).

This is, it is like the energy to be taken into account is not mc^2 alone but also, we have to subtract an element depending on the Ricci scalar R . In fact, operating the factor:

$$\frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right) = \frac{m^2c^2}{\hbar^2} - R$$

Multiplying by \hbar^2c^2 (multiplying by constants do not change the meaning of the equation, it just escalates its values):

$$m^2c^4 - R\hbar^2c^2$$

Taking the square root to get Energy units:

$$\sqrt{m^2c^4 - R\hbar^2c^2} = mc^2\sqrt{1 - \frac{R\hbar^2c^2}{m^2c^4}} = mc^2\sqrt{1 - \frac{R\hbar^2}{m^2c^2}}$$

We can see that the classical energy of a mass at rest mc^2 is reduced by a factor depending on the Ricci scalar. We will get back to this later.

Coming back to the previous equation. If we perform the multiplication to the bracket, we can see that the equation is in fact a Klein-Gordon equation [18]-[20] with an extra element that depends on the Ricci scalar R . We can check easily that the units of $\frac{m^2c^2}{\hbar^2}$ and R are L^{-2} , so everything is coherent,

$$\begin{aligned} e^\beta\nabla_\beta\left(\nabla_\alpha\left(\psi^\dagger\psi\right)e^\alpha\right) &= \frac{m}{\hbar^2}\left(mc^2 - \frac{\hbar^2}{m}R\right)\psi^\dagger\psi \\ e^\beta\nabla_\beta\left(\nabla_\alpha\left(\psi^\dagger\psi\right)e^\alpha\right) &= \left(\frac{m^2c^2}{\hbar^2} - R\right)\psi^\dagger\psi \\ e^\beta\nabla_\beta\left(\nabla_\alpha\left(\psi^\dagger\psi\right)e^\alpha\right) &= \frac{m^2c^2}{\hbar^2}\psi^\dagger\psi - R\psi^\dagger\psi \end{aligned}$$

Coming back to the equation:

$$e^\beta \nabla_\beta (\nabla_\alpha (\psi^\dagger \psi) e^\alpha) = \frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi$$

We can see that the equation (as expected for a Klein-Gordon equation) can be factored (a la Dirac way) this way:

$$e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha e^\alpha = \frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi$$

$$e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha e^\alpha = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi^\dagger \psi \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)}$$

$$e^\beta \nabla_\beta \psi^\dagger = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi^\dagger$$

$$\psi \nabla_\alpha e^\alpha = \psi \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)}$$

$$(\nabla_\alpha \psi) e^\alpha = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi$$

In the end, the equations in alpha and beta are the same, just reversing sometimes or changing signs. We could simplify even more:

$$(\nabla_\alpha \psi) e^\alpha e_\alpha = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi e_\alpha$$

$$\nabla_\alpha \psi = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi e_\alpha$$

$$\nabla_\beta \psi^\dagger = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} e_\beta \psi^\dagger$$

Or performing the multiplication to the bracket:

$$\nabla_\alpha \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} \psi e_\alpha$$

$$\nabla_\beta \psi^\dagger = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} e_\beta \psi^\dagger$$

which we can see is just the Dirac equation [11] [16] [21] [22] with that extra-term that subtract the Ricci scalar to the $\frac{m^2 c^2}{\hbar^2}$ element.

One important thing is that in Geometric Algebra we do not work with imaginary numbers (only bivectors or trivector that make its function, you can check [1]-[4] [11]-[13] for more information). So, the element inside $\sqrt{\frac{m^2 c^2}{\hbar^2} - R}$ must be positive to keep the coherence.

So:

$$\frac{m^2 c^2}{\hbar^2} - R > 0$$

$$\frac{m^2 c^2}{\hbar^2} > R$$

$$R < \frac{m^2 c^2}{\hbar^2}$$

This means, there is a limit to the value of the Ricci scalar curvature depending on the mass. It is important to remark that the limit is in the absolute value of the mass, not to the mass density in volume, so the possibility of arriving to singularities is highly reduced.

If we represent the Dirac equation in standard matrix-tensor notation (not Geometric Algebra) as defined as [22] [23] (here the imaginary numbers are allowed):

$$-i\hbar\gamma^\mu \partial_\mu \psi + mc\psi = 0$$

$$i\hbar\gamma^\mu \partial_\mu \psi = mc\psi$$

$$i\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar} \psi$$

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2}} \psi$$

Using the equation obtained in this chapter, it should read:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} \psi$$

So, it will be the same but include this Ricci scalar that is subtracted from the element $\frac{m^2 c^2}{\hbar^2}$. Both of them have L⁻² units.

In **Annex A5**, I show how following a similar process we can get a modification of the Einstein equation, with this result:

$$\frac{8\pi G}{c^4} T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R \right) = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

One important conclusion of these equations is that the higher the energy of the mass (in Dirac equation) or the higher the stress-energy tensor (in Einstein equation), the Ricci scalar increases due to gravitational effects. As the Ricci scalar is being subtracted to the energy of the system (to the particle energy or the stress-energy tensor), the system will arrive to a balance avoiding singularities. This is summed up in the following equation that impose a limit to the Ricci scalar depending on the mass (not the mass density), reducing highly the possibilities of arriving to singularities:

$$R < \frac{m^2 c^2}{\hbar^2}$$

Another important conclusion is that in the Dirac equation, as we have now the

mass and the Ricci scalar (that depends on the mass), probably finding eigenvalues of equilibrium could lead to the discovery of the discrete values of the masses of the different particles.

And it would explain why there are families of three different masses per type of particle. They would correspond to the eigenvalues depending on the three possible values of the indices (1, 2, 3) corresponding to the three dimensions (their three corresponding eigen vectors in the 3 spatial dimensions of $Cl_{3,0}$).

9.2. Considering That the Stress Energy of the Particle Is the One of a Point Particle (This Option If Probably Wrong)

If we follow [24] [25], we can consider the stress energy tensor, just relates to the energy and momentum of the particle. Being coherent with the units, one option could be the energy density of the particle defined by its waveform collapse (squared by its reversed). The units are coherent $\text{Energy}\cdot\text{L}^{-3}$ and for the cross elements $\text{Force}\cdot\text{L}^{-2}$ (pressure) that has the same units as $\text{Energy}\cdot\text{L}^{-3}$. So, a definition would be:

$$T_{\mu\nu} = mc^2 e_\mu \psi^\dagger \psi e_\nu$$

But we have to take into account that in this context, the element mc^2 is reduced by the element containing the Ricci scalar (that appeared in Chapter 9), so we should use instead:

$$T_{\mu\nu} = \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu$$

I remind you that:

$$\begin{aligned} \psi^\dagger \psi &= \psi^\mu e_\mu^\dagger \psi^\nu e_\nu \\ &= (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger) \\ &\quad \times (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\ &= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1) \\ &\quad \times (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) \end{aligned}$$

So, this is in fact a complicate operation, not a trivial one, with one scalar as result. It has result in all 8 vectors (scalars, 3 vectors, 3 bivectors and the trivector).

You can see different examples of the calculation in **Annexes A1 - A4**. For example, the most simple on (orthonormal metric) **Annex A1**, gives:

$$\psi^\dagger \psi = \rho + \vec{j}$$

With:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

And:

$$\begin{aligned} \vec{j} &= 2(\psi^1 \psi^0 - \psi^2 \psi^6 + \psi^3 \psi^5 + \psi^4 \psi^7) e_1 \\ &\quad + 2(\psi^0 \psi^2 + \psi^1 \psi^6 - \psi^3 \psi^4 + \psi^5 \psi^7) e_2 \\ &\quad + 2(\psi^0 \psi^3 - \psi^1 \psi^5 + \psi^2 \psi^4 + \psi^6 \psi^7) e_3 \end{aligned}$$

So, considering the definition of the Stress Energy tensor, as commented above:

$$T_{\mu\nu} = mc^2 e_\mu \psi^\dagger \psi e_\nu$$

And introducing the equation found in the end of Chapter 9:

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) g_{\mu\nu} \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) \\ \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu &= \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu - \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu &= \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu &= \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ \frac{3}{2} \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu &= \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ 3 \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu &= \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ \frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu &= e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu &= \frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) e_\mu \psi^\dagger \psi e_\nu \\ e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha &= \frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi \\ e^\beta \nabla_\beta \left(\nabla_\alpha \left(\psi^\dagger \psi \right) e^\alpha \right) &= \frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi \\ e^\beta \nabla_\beta \left(\nabla_\alpha \left(\psi^\dagger \psi \right) e^\alpha \right) &= \frac{3m^2 c^2}{\hbar^2} \psi^\dagger \psi - 3R \psi^\dagger \psi \end{aligned}$$

We can see that we obtain an equation like the Klein-Gordon equation obtained in 9.1 but with a factor of 3. So, this result seems to be erroneous. Anyhow, we will continue operating.

If this was ok, the “Dirac” factorization would be:

$$\begin{aligned} e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha &= \frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi \\ e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha &= \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi^\dagger \psi \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \\ e^\beta \nabla_\beta \psi^\dagger &= \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi^\dagger \end{aligned}$$

$$\psi \nabla_{\alpha}^{\dagger} e^{\alpha} = \psi \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)}$$

$$(\nabla_{\alpha} \psi) e^{\alpha} = \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi$$

In the end, the equations in alpha and beta are the same, just reversing sometimes or changing signs. We could simplify even more:

$$(\nabla_{\alpha} \psi) e^{\alpha} e_{\alpha} = \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi e_{\alpha}$$

$$\nabla_{\alpha} \psi = \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi e_{\alpha}$$

$$\nabla_{\beta} \psi^{\dagger} = \sqrt{\frac{3m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} e_{\beta} \psi^{\dagger}$$

$$\nabla_{\alpha} \psi = \sqrt{\frac{3m^2 c^2}{\hbar^2} - 3R} \psi e_{\alpha}$$

$$\nabla_{\beta} \psi^{\dagger} = \sqrt{\frac{3m^2 c^2}{\hbar^2} - 3R} e_{\beta} \psi^{\dagger}$$

But as commented this assumption of considering the stress-energy tensor of a particle as considered in this chapter seems mistaken, and therefore its results. In fact, in the Klein-Gordon equation and in the Dirac equations a non-expected factor by 3 appears. So, this assumption and its consequent results seem wrong.

9.3. Introducing the Einstein Tensor in the Equation

Coming from the equation, we got in the end of Chapter 9:

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_{\mu} \psi^{\dagger} \psi e_{\nu} + \frac{1}{2} \frac{\hbar^2}{m} e_{\mu} \left(e^{\beta} \nabla_{\beta} \psi^{\dagger} \psi \nabla_{\alpha}^{\dagger} e^{\alpha} \right) e_{\nu}$$

And taking the Einstein General Relativity equation [5]-[10] [17]:

$$\frac{8\pi G}{c^4} T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

Operating this equation:

$$T_{\mu\nu} = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

$$T_{\mu\nu} = \frac{c^4}{8\pi G} R_{\mu\nu} - \frac{1}{2} \frac{c^4}{8\pi G} g_{\mu\nu} R + \Lambda \frac{c^4}{8\pi G} g_{\mu\nu}$$

$$T_{\mu\nu} = \frac{c^4}{8\pi G} R_{\mu\nu} - \frac{c^4}{16\pi G} g_{\mu\nu} R + \Lambda \frac{c^4}{8\pi G} g_{\mu\nu}$$

And now, we introduce in the equation in the end of Chapter 9:

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^{\dagger} \psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^{\beta} \nabla_{\beta} \psi^{\dagger} \psi \nabla_{\alpha}^{\dagger} e^{\alpha} \right)$$

$$\begin{aligned} & \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) \\ &= \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) \\ & \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) + \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi \\ & - \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0 \\ & \frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \left(\nabla_\alpha \left(\psi^\dagger \psi \right) e^\alpha \right) \right) + \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi \\ & - \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0 \end{aligned}$$

This equation above seems (and it is) very complicated but it can be solvable.

The unknown variables are:

- $\psi^0 \psi^1 \psi^2 \psi^3 \psi^4 \psi^5 \psi^6 \psi^7$;
- $g_{11} g_{22} g_{33} g_{23} g_{31} g_{12}$ and it could be also g_{00} if it is not 1 directly.

So, in total 14 (or 15) unknown variables. The equation above, only because it is a multivector equation, is converted into 8 equations (one per type of vector (3), bivectors (3), scalar and trivector). So not even counting that it is also a tensor equation also (probably the equations obtained as a tensor equation are linearly dependent to the ones of the multivector), we will have 8 equations.

The rest of the equations we will get from the continuity equation [24]:

$$e^\lambda \nabla_\lambda T = 0$$

With T defined as (end of Chapter 9):

$$T = g^{\mu\nu} T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi + \frac{1}{2} \frac{\hbar^2}{m} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right)$$

Or in tensor form:

$$e^\lambda \nabla_\lambda T_{\mu\nu} = 0$$

Being $T_{\mu\nu}$:

$$T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu$$

Or in the classical form of the divergence:

$$\nabla_\lambda T^{\lambda\rho} = 0$$

Being:

$$\begin{aligned} T^{\lambda\rho} &= g^{\lambda\mu} g^{\rho\nu} T_{\mu\nu} \\ &= \frac{1}{2} g^{\lambda\mu} g^{\rho\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu + \frac{1}{2} g^{\lambda\mu} g^{\rho\nu} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu \\ &= \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e^\lambda \psi^\dagger \psi e^\rho + \frac{1}{2} \frac{\hbar^2}{m} e^\lambda \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e^\rho \end{aligned}$$

These are the other 8 equations. So, in total, we have 16 equations to solve 14 or 15 variables, so it should be ok. The system is over dimensioned. This means, we can take some of the unknowns as parameters, or even normalize the system as convenience (making those free parameters whatever value we want to make a normalization).

Coming back to this equation:

$$\frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) + \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi - \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0$$

Putting it more symmetrically (considering we are in one of the three cases of Chapter 7):

$$\frac{1}{2} \frac{\hbar^2}{m} e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu + \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e_\mu \psi^\dagger \psi e_\nu - \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} e_\mu R e_\nu + e_\mu \Lambda g_{\mu\nu} \right) = 0$$

This equation, for sure can be factorized a la Dirac way somehow. But the quadratic equation solution has to be used, complicating the things. I will come back with this in next revisions of the paper.

The main difference between above equations and original Einstein Field equations is the number of dimensions considered. In Einstein equations, it is considered space-time (four dimensions) and the state of the bodies are defined by 4-vectors (position, momentum...).

In above equations, there are considered the 8 dimensions: the three spatial dimensions (vectors), the three planes (bivectors), the scalars and the trivector (which they turn in the representation of the time and/or volume depending on the context).

The same way, the state of the particles is not represented by 4-vectors but by the 8 parameters of the wave equation (that includes its state in the 8 dimensions).

9.4. The Gravitational Bivectors Create Magnetic-Like Effects That Explain the Speed of Rotation of the Galaxies without the Need of Dark Matter

As commented in point 9.3, the equations in Chapters 9.1 to 9.3 consider the 8 dimensions: the three spatial dimensions (vectors), the three planes (bivectors), the scalars and the trivector. We will focus on this chapter in the effect of the bivectors. These bivectors are not considered in the standard Einstein field equations, that only consider the three spatial dimensions and the time, so they do not consider the planes or bivectors-.

In **Annex A6**, you can find an explanation of what would mean that the gravitational fields are also defined by bivectors. The bivectors create a “magnetic-like” effect in whatever field they are acting on.

This means, consider two moving particles interacting one to each other. The main interaction/force (represented by vectors) normally appears in the direction of the line of sight of the two particles.

But if there are also bivectors acting, it will appear also another interaction whose direction will depend both on the line of sight and on the line of direction of the speed of the particles (in general, on the plane they form and the angle between them).

This effect is called magnetic-like because historically the first interaction that was discovered to have this effect was the magnetism. But it is the effect of whatever interaction that is provoked by bivectors instead of vectors. As commented, gravitation in this paper is also defined by bivectors that should create a magnetic-like effect.

The curious thing is that it has been studied if this effect happens in gravitation [26] [27] and it seems that it really happens.

In reference [26], they study the speed of rotation of the following galaxies:

NGC 1560;

NGC 3198;

NGC 3115.

Leading to the conclusion that the speed of rotation of these galaxies can be perfectly explained by magnetic-like effects without the need of exotic dark-matter.

The important difference of paper [26] compared to the present paper is the following. In paper [26] and, in general, for all the defenders of gravitomagnetism, the gravitomagnetism has to be added as an ad-hoc addition to gravitational theory to match certain effects. As in [26], where it is added to the equations to match the rotation of galaxies.

But in the present paper, the gravitomagnetism effects appear directly in the equations of gravitation because of the existence of gravitational bivectors, not needing to add the gravitomagnetism ad-hoc. It is just inherent in the equations of Chapters 9.1 to 9.3 just because the bivectors are directly part of them.

Anyhow, all the conclusions commented in [26] regarding the accurate matching of measured rotation of speed compared to calculated one using gravitomagnetism for the above-mentioned galaxies would apply also for the present paper. Leading to the non-necessity of dark matter to explain the speed of rotation of the galaxies.

See **Annex A6** and [26] [27] for more information.

9.5. The Energy-Momentum Relation Create a Negative Energy Whose Value Is Exactly the One Expected for the Dark Energy

In **Annex A7**, we make a study regrading dark energy. I will make here a summary. Please, check it for details.

The most identifiable effect of the dark energy is the existence of the cosmological constant Λ [28]. Which value [28] is in the order of:

$$\Lambda = 1.1056E - 52 \text{ m}^{-2}$$

It is to be remarked that although being an effect of energy, the dimensions are Length^{-2} .

In the paper [29], we get to the following energy-momentum relation coming from the equations (Chapters 7 to 9) of the present paper:

$$E^2 = m^2 c^4 + p^2 c^2 - R \hbar^2 c^2$$

You can check that there is a new term that reduces the energy of each particle depending on the Ricci scalar (the space-time curvature in the place where the particle is lying). This term has the units of energy squared as it has to be coherent with the rest of the equation.

If we want this term to have units of Length^{-2} , we have to make the following escalation. I call it escalation, because we are not modifying at all the equation. We are just dividing by constants, not adding new variables. The equation is the same. In fact, as if some authors do, I used all the constants equal to 1, the following operation would not even be necessary:

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^4}{\hbar^2 c^2} + \frac{p^2 c^2}{\hbar^2 c^2} - \frac{R \hbar^2 c^2}{\hbar^2 c^2}$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

The equation above has the same meaning as the original but in another units. Here the energy squared has the units Length^{-2} . So, the last term ($-R$) represents the square of the reduction of the energy (a negative energy) that apply to every particle in the units Length^{-2} .

What we do in **Annex A7** is to calculate the value of this $-R$ considering different possible metrics of the universe (Interior Schwarzschild metric and exterior Schwarzschild metric via the Kretschmann scalar. As commented this R represents a reduction of energy (a negative energy) in the units Length^{-2} .

We get to the following results:

$$R_{intSchw} = 1.603E - 52 \text{ m}^{-2}$$

$$R_{intSchw1/3} = 0.5345E - 52 \text{ m}^{-2}$$

$$R_{extSchw} \rightarrow \sqrt{K} = 1.852E - 52 \text{ m}^{-2}$$

While the cosmological constant [28] is in the order of:

$$\Lambda = 1.1056E - 52 \text{ m}^{-2}$$

You can see that the value of R is (even considering different metrics) is in the order of Λ . This means, this equation obtained applying geometric algebra to Klein-Gordon equation and Einstein Field equations and in [29]:

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^4}{\hbar^2 c^2} + \frac{p^2 c^2}{\hbar^2 c^2} - \frac{R \hbar^2 c^2}{\hbar^2 c^2}$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Has led (not added ad-hoc, it has appeared directly in the equations) to an element ($-R$) that reduces the energy of the particles in the exact same value as expected to be considered a candidate for the Dark Energy.

10. Influence of Ricci Scalar in the Energy of a Particle

We have seen in Chapter 9.1, the following equation:

$$E_{particle} = mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}}$$

But what is the influence of the second element? Let's check the influence in a proton at the surface of Earth

We know:

$$m_{proton} = 1.6726E - 27 \text{ kg}$$

$$\hbar = 1.05457E - 34 \text{ J} \cdot \text{s}$$

$$c = 299792458 \text{ m/s}$$

Calculating the Ricci scalar R is more complicated. If we use the Schwarzschild metric would be zero. What we can do is to calculate the Kretschmann scalar [30] considering Schwarzschild metric in the surface of Earth (related to the Ricci scalar curvature) and take its square root (its dimensions are L^{-4} and the Ricci scalar is L^{-2} . As commented, this is just a reference:

$$G = 6.6743E - 11 \frac{\text{N} \cdot \text{m}^2}{\text{kg}}$$

$$M_{earth} = 5.9722E24 \text{ kg}$$

$$r = r_{earth} = 6.371E6 \text{ m}$$

$$\begin{aligned} \sqrt{\text{Kretschmann scalar}} &= \sqrt{\frac{48G^2 M^2}{c^4 r^6}} \sqrt{\frac{48 \cdot (6.6743E - 11)^2 (5.9722E24)^2}{299792458^4 (6371E3)^6}} \\ &= 1.18821E - 22 \text{ m}^{-2} \end{aligned}$$

Coming back here, now considering a proton:

$$\begin{aligned} E_{particle} &= mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}} \\ &= 1.6726E - 27 \cdot 299792458^2 \sqrt{1 - \frac{(1.05457E - 34)^2}{1.6726E - 27} \cdot 1.18821E - 22} \\ &= 1.503257E - 10 \sqrt{1 - 7.9E - 64} \end{aligned}$$

We can see that the square root factor effect is several orders of magnitude lower than the original energy. Even if we consider $R = 1$ (an example), we would be in a similar situation:

$$\begin{aligned} E_{particle} &= mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}} \\ &= 1.6726E - 27 \cdot 299792458^2 \sqrt{1 - \frac{(1.05457E - 34)^2}{1.6726E - 27} \cdot 1} \\ &= 1.503257E - 10 \sqrt{1 - 6.651E - 42} \end{aligned}$$

We can see that that the square root factor effect is neglectable in general. And only in very big gravitational fields (with R very high), the second element could start having an effect.

Anyhow, this last point is important. As commented in Chapter 9.1, the higher the mass, the Ricci scalar increases due to gravitational effects. As the Ricci scalar is being subtracted to the energy depending on the mass, the system will arrive to a balance avoiding singularities.

11. Conclusions

In this paper, we have used Geometric Algebra to be able to embed the Klein-Gordon equation for a particle in a non-Euclidean field (vacuum solution in a gravitational field), getting the following equation:

$$e^\beta \nabla_\beta \left(\nabla_\alpha (\psi^\dagger \psi) e^\alpha \right) = \frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right) \psi^\dagger \psi$$

$$e^\beta \nabla_\beta \left(\nabla_\alpha (\psi^\dagger \psi) e^\alpha \right) = \frac{m^2 c^2}{\hbar^2} \psi^\dagger \psi - R \psi^\dagger \psi$$

which is similar to the Klein-Gordon equation but with an extra term involving the Ricci scalar R .

$\psi^\dagger \psi$ is the wavefunction collapsed (multiplied by its reverse), this way:

$$\begin{aligned} \psi^\dagger \psi &= (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7) (\psi^0 e_0 \\ &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\ &= \rho + \vec{j} \end{aligned}$$

where ρ and \vec{j} are the probability density and the fermionic current, respectively.

The equation above can be factored to be simplified into:

$$\nabla_\alpha \psi = \sqrt{\frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R \right)} \psi e_\alpha$$

$$\nabla_\alpha \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} \psi e_\alpha$$

which again, is similar to the Dirac equation but with an extra term involving the Ricci scalar R .

Meaning that the energy of a particle is somehow decreased by a factor that depends on the Ricci scalar (the curvature of the space where it lies in):

$$E_{particle} = mc^2 \sqrt{1 - \frac{R\hbar^2}{m^2 c^2}}$$

This reduction is in general negligible, being several orders of magnitude below the normal energy. Anyhow, as the mass increases, the Ricci scalar also increases due to gravitational effects. As the Ricci scalar is being subtracted to the energy depending on the mass, the system will arrive at a balance before becoming a sin-

gularity.

This is summed up in the following equation that imposes a limit to the Ricci scalar depending on the mass (not the mass density), highly reducing the possibilities of arriving at singularities:

$$R < \frac{m^2 c^2}{\hbar^2}$$

Even considering the Dirac equation in standard tensor notation:

$$i\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar} \psi$$

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2}} \psi$$

We could adapt it, just adding that element to the equation:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} \psi$$

In a similar way, we obtain a variation of the Einstein equation with this form:

$$\frac{8\pi G}{c^4} T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R \right) = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

Following another path, we found another equation:

$$\frac{1}{2} \frac{\hbar^2}{m} g_{\mu\nu} \left(e^\beta \nabla_\beta \left(\nabla_\alpha (\psi^\dagger \psi) e^\alpha \right) \right) + \frac{1}{2} g_{\mu\nu} \left(\frac{\hbar^2}{m} R - mc^2 \right) \psi^\dagger \psi$$

$$- \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0$$

This equation (that are in fact 8 embedded equations) has 14 or 15 unknown variables: 8 coefficients of the wavefunction ψ^0 to ψ^7 and 6 metric elements g_{ij} (i, j from 1 to 3) with a possible added g_{00} .

The rest of the equations (8 equations more) come from the continuity equation:

$$\nabla_\lambda T^{\lambda\rho} = 0$$

Being:

$$T^{\lambda\rho} = g^{\lambda\mu} g^{\rho\nu} T_{\mu\nu} = \frac{1}{2} \left(\frac{\hbar^2}{m} R - mc^2 \right) e^\lambda \psi^\dagger \psi e^\rho + \frac{1}{2} \frac{\hbar^2}{m} e^\lambda \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e^\rho$$

So, the equation is in fact solvable.

Also, we have commented how the magnetic-like effects of the gravitation appearing from the equations can explain the speed of rotation of the galaxies (studied NGC 1560, NGC 3198 and NGC 3115) without the need for Dark Matter.

The last point studied is how the obtained equation:

$$E^2 = m^2 c^4 + p^2 c^2 - R \hbar^2 c^2$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

has an element ($-R$, representing the Ricci scalar curvature) that here is acting to reduce the energy of the particles. It represents a reduction of the energy but in units of Length^{-2} in the latter equation. This value of R is calculated (considering different metrics for the universe) to find that it corresponds almost exactly with the expected value of the cosmological constant (the effects created by the Dark Energy):

$$\begin{aligned} R_{intSchw} &= 1.603E - 52 \text{ m}^{-2} \\ R_{intSchw^{1/3}} &= 0.5345E - 52 \text{ m}^{-2} \\ R_{extSchw} &\rightarrow \sqrt{K} = 1.852E - 52 \text{ m}^{-2} \end{aligned}$$

While the cosmological constant is in the order of:

$$\Lambda = 1.1056E - 52 \text{ m}^{-2}$$

So, making this $-R$ a perfect candidate for the Dark Energy. But not as an added element to match the observations, on the contrary, an element that appears directly in the equations results in a match to the observations.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Annex A1. Bra-Ket Product in Euclidean Metric

The bra-ket product of a reversed spinor (in orthogonal metrics is the same as reverse) can be calculated as:

$$\begin{aligned}
 \psi^\dagger \psi &= \psi^\mu e_\mu^\dagger \psi^\nu e_\nu \\
 &= (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger) (\psi^0 e_0 \\
 &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\
 &= \psi^* \psi \\
 &= (\psi^0 e_0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_4 - \psi^5 e_5 - \psi^6 e_6 - \psi^7 e_7) (\psi^0 e_0 \\
 &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\
 &= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 - \psi^4 e_2 e_3 - \psi^5 e_3 e_1 - \psi^6 e_1 e_2 - \psi^7 e_1 e_2 e_3) (\psi^0 \\
 &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) \\
 &= (\psi^0)^2 + \psi^0 \psi^1 e_1 + \psi^0 \psi^2 e_2 + \psi^0 \psi^3 e_3 + \psi^0 \psi^4 e_2 e_3 + \psi^0 \psi^5 e_3 e_1 + \psi^0 \psi^6 e_1 e_2 \\
 &\quad + \psi^0 \psi^7 e_1 e_2 e_3 + \psi^1 \psi^0 e_1 + (\psi^1)^2 + \psi^1 \psi^2 e_1 e_2 - \psi^1 \psi^3 e_3 e_1 + \psi^1 \psi^4 e_1 e_2 e_3 \\
 &\quad - \psi^1 \psi^5 e_3 + \psi^1 \psi^6 e_2 + \psi^1 \psi^7 e_2 e_3 + \psi^2 \psi^0 e_2 - \psi^2 \psi^1 e_1 e_2 + (\psi^2)^2 + \psi^2 \psi^3 e_2 e_3 \\
 &\quad + \psi^2 \psi^4 e_3 + \psi^2 \psi^5 e_1 e_2 e_3 - \psi^2 \psi^6 e_1 + \psi^2 \psi^7 e_3 e_1 + \psi^3 \psi^0 e_3 + \psi^3 \psi^1 e_3 e_1 \\
 &\quad - \psi^3 \psi^2 e_2 e_3 + (\psi^3)^2 - \psi^3 \psi^4 e_2 + \psi^3 \psi^5 e_1 + \psi^3 \psi^6 e_1 e_2 e_3 + \psi^3 \psi^7 e_1 e_2 \\
 &\quad - \psi^4 \psi^0 e_2 e_3 - \psi^4 \psi^1 e_1 e_2 e_3 + \psi^4 \psi^2 e_3 - \psi^4 \psi^3 e_2 + (\psi^4)^2 + \psi^4 \psi^5 e_1 e_2 \\
 &\quad - \psi^4 \psi^6 e_3 e_1 + \psi^4 \psi^7 e_1 - \psi^5 \psi^0 e_3 e_1 - \psi^5 \psi^1 e_3 - \psi^5 \psi^2 e_1 e_2 e_3 + \psi^5 \psi^3 e_1 \\
 &\quad - \psi^5 \psi^4 e_1 e_2 + (\psi^5)^2 + \psi^5 \psi^6 e_2 e_3 + \psi^5 \psi^7 e_2 - \psi^6 \psi^0 e_1 e_2 + \psi^6 \psi^1 e_2 - \psi^6 \psi^2 e_1 \\
 &\quad - \psi^6 \psi^3 e_1 e_2 e_3 + \psi^6 \psi^4 e_3 e_1 - \psi^6 \psi^5 e_2 e_3 + (\psi^6)^2 + \psi^6 \psi^7 e_3 - \psi^7 \psi^0 e_1 e_2 e_3 \\
 &\quad - \psi^7 \psi^1 e_2 e_3 - \psi^7 \psi^2 e_3 e_1 - \psi^7 \psi^3 e_1 e_2 + \psi^7 \psi^4 e_1 + \psi^7 \psi^5 e_2 + \psi^7 \psi^6 e_3 + (\psi^7)^2
 \end{aligned}$$

Please, take into account that for simplification I have considered directly $e_0 = 1$. If in the end, it has another value, it has just to be considered in the operations.

Continuing with the operation. If we separate from the result above only the scalars, we have:

$$(\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

We will call this sum ρ (probability density):

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

If we separate the components that multiply by e_1 , we get:

$$\begin{aligned}
 &\psi^0 \psi^1 + \psi^1 \psi^0 - \psi^2 \psi^6 + \psi^3 \psi^5 + \psi^4 \psi^7 + \psi^5 \psi^3 - \psi^6 \psi^2 + \psi^7 \psi^4 \\
 &= 2(\psi^1 \psi^0 - \psi^2 \psi^6 + \psi^3 \psi^5 + \psi^4 \psi^7)
 \end{aligned}$$

In e_2 , we get:

$$\begin{aligned} & \psi^0\psi^2 + \psi^1\psi^6 + \psi^2\psi^0 - \psi^3\psi^4 - \psi^4\psi^3 + \psi^5\psi^7 + \psi^6\psi^1 + \psi^7\psi^5 \\ & = 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7) \end{aligned}$$

In e_3 , we get:

$$\begin{aligned} & \psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^3\psi^0 + \psi^4\psi^2 - \psi^5\psi^1 + \psi^6\psi^7 + \psi^7\psi^6 \\ & = 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7) \end{aligned}$$

In e_2e_3 :

$$\psi^0\psi^4 + \psi^1\psi^7 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6 - \psi^6\psi^5 - \psi^7\psi^1 = 0$$

In e_3e_1 :

$$\psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi_{xyz} + \psi^3\psi^1 - \psi^4\psi^6 - \psi^5\psi^0 + \psi^6\psi^4 - \psi^7\psi^2 = 0$$

In e_1e_2 :

$$\psi^0\psi^6 + \psi^1\psi^2 - \psi^2\psi^1 + \psi^3\psi^7 + \psi^4\psi^5 - \psi^5\psi^4 - \psi^6\psi^0 - \psi^7\psi^3 = 0$$

In $e_1e_2e_3$:

$$\psi^0\psi^7 + \psi^1\psi^4 + \psi^2\psi^5 + \psi^3\psi^6 - \psi^4\psi^1 - \psi^5\psi^2 - \psi^6\psi^3 - \psi^7\psi^0 = 0$$

If we call vector \vec{j} (fermionic current) the sum in e_1 , e_2 and e_3 , we get:

$$\begin{aligned} \vec{j} &= 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 \\ &+ 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 \\ &+ 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3 \end{aligned}$$

So, in total, we have:

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j}$$

With:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

And:

$$\begin{aligned} \vec{j} &= 2(\psi^1\psi^0 - \psi^2\psi^6 + \psi^3\psi^5 + \psi^4\psi^7)e_1 \\ &+ 2(\psi^0\psi^2 + \psi^1\psi^6 - \psi^3\psi^4 + \psi^5\psi^7)e_2 \\ &+ 2(\psi^0\psi^3 - \psi^1\psi^5 + \psi^2\psi^4 + \psi^6\psi^7)e_3 \end{aligned}$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi = j^\mu e_\mu$$

where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

Annex A2. Bra-Ket Product in Non-Euclidean Metric (Orthogonal but Not Orthonormal)

We apply the following relations, when performing the multiplication:

$$(e_0)^2 = \|e_0\|^2 = g_{00} \quad (e_1)^2 = \|e_1\|^2 = g_{11} \quad (e_2)^2 = \|e_2\|^2 = g_{22} \quad (e_3)^2 = \|e_3\|^2 = g_{33}$$

$$e_0 e_i = e_i e_0 \quad e_2 e_3 = -e_3 e_2 \quad e_3 e_1 = -e_1 e_3 \quad e_1 e_2 = -e_2 e_1$$

For simplification, we will consider directly $e_0 = 1$. If in the end, it has another value, it just will have to be considered in the operations.

$$\begin{aligned} \psi^\dagger \psi &= \psi^\mu e_\mu^\dagger \psi^\nu e_\nu \\ &= (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger) (\psi^0 e_0 \\ &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\ &= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_1) (\psi^0 \\ &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) \\ &= (\psi^0)^2 + \psi^0 \psi^1 e_1 + \psi^0 \psi^2 e_2 + \psi^0 \psi^3 e_3 + \psi^0 \psi^4 e_2 e_3 + \psi^0 \psi^5 e_3 e_1 + \psi^0 \psi^6 e_1 e_2 \\ &\quad + \psi^0 \psi^7 e_1 e_2 e_3 + \psi^1 \psi^0 e_1 + (\psi^1)^2 \|e_1\|^2 + \psi^1 \psi^2 e_1 e_2 - \psi^1 \psi^3 e_3 e_1 + \psi^1 \psi^4 e_1 e_2 e_3 \\ &\quad - \psi^1 \psi^5 \|e_1\|^2 e_3 + \psi^1 \psi^6 \|e_1\|^2 e_2 + \psi^1 \psi^7 \|e_1\|^2 e_2 e_3 + \psi^2 \psi^0 e_2 - \psi^2 \psi^1 e_1 e_2 \\ &\quad + (\psi^2)^2 \|e_2\|^2 + \psi^2 \psi^3 e_2 e_3 + \psi^2 \psi^4 \|e_2\|^2 e_3 + \psi^2 \psi^5 e_1 e_2 e_3 - \psi^2 \psi^6 \psi_{xy} e_1 e_2 e_3 \\ &\quad + \psi^3 \psi^7 \|e_3\|^2 e_1 e_2 - \psi^4 \psi^0 e_2 e_3 - \psi^4 \psi^1 e_1 e_2 e_3 + \psi^4 \psi^2 \|e_2\|^2 e_3 - \psi^4 \psi^3 \|e_3\|^2 e_2 \\ &\quad + (\psi^4)^2 \|e_2\|^2 \|e_3\|^2 + \psi^4 \psi^5 \|e_3\|^2 e_1 e_2 - \psi^4 \psi^6 \|e_2\|^2 e_3 e_1 + \psi^4 \psi^7 \|e_2\|^2 \|e_3\|^2 e_1 \\ &\quad - \psi^5 \psi^0 e_3 e_1 - \psi^5 \psi^1 \|e_1\|^2 e_3 - \psi^5 \psi^2 e_1 e_2 e_3 + \psi^5 \psi^3 \|e_3\|^2 e_1 - \psi^5 \psi^4 \|e_3\|^2 e_1 e_2 \\ &\quad + (\psi^5)^2 \|e_3\|^2 \|e_1\|^2 + \psi^5 \psi^6 \|e_1\|^2 e_2 e_3 + \psi^5 \psi^7 \|e_3\|^2 \|e_1\|^2 e_2 - \psi^6 \psi^0 e_1 e_2 \\ &\quad + \psi^6 \psi^1 \|e_1\|^2 e_2 - \psi^6 \psi^2 \|e_2\|^2 e_1 - \psi^6 \psi^3 e_1 e_2 e_3 + \psi^6 \psi^4 \|e_2\|^2 e_3 e_1 \\ &\quad - \psi^6 \psi^5 \|e_1\|^2 e_2 e_3 + (\psi^6)^2 \|e_1\|^2 \|e_2\|^2 + \psi^6 \psi^7 \|e_1\|^2 \|e_2\|^2 e_3 - \psi^7 \psi^0 e_1 e_2 e_3 \\ &\quad - \psi^7 \psi^1 \|e_1\|^2 e_2 e_3 - \psi^7 \psi^2 \|e_2\|^2 e_3 e_1 - \psi^7 \psi^3 \|e_3\|^2 e_1 e_2 + \psi^7 \psi^4 \|e_2\|^2 \|e_3\|^2 e_1 \\ &\quad + \psi^7 \psi^5 \|e_1\|^2 \|e_3\|^2 e_2 + \psi^7 \psi^6 \|e_1\|^2 \|e_2\|^2 e_3 + (\psi^7)^2 \|e_1\|^2 \|e_2\|^2 \|e_3\|^2 \end{aligned}$$

If we separate from the result above only the scalars, we have:

$$\begin{aligned} \rho &= (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22} g_{33} \\ &\quad + (\psi^5)^2 g_{33} g_{11} + (\psi^6)^2 g_{11} g_{22} + (\psi^7)^2 g_{11} g_{22} g_{33} \end{aligned}$$

We will call above sum ρ (probability density).

Now, if we separate by e_1 :

$$\begin{aligned} &\psi^0 \psi^1 + \psi^1 \psi^0 - \psi^2 \psi^6 \|e_2\|^2 + \psi^3 \psi^5 \|e_3\|^2 + \psi^4 \psi^7 \|e_2\|^2 \|e_3\|^2 + \psi^5 \psi^3 \|e_3\|^2 \\ &\quad - \psi^6 \psi^2 \|e_2\|^2 + \psi^7 \psi^4 \|e_2\|^2 \|e_3\|^2 \\ &\quad 2(\psi^0 \psi^1 - \psi^2 \psi^6 \|e_2\|^2 + \psi^3 \psi^5 \|e_3\|^2 + \psi^4 \psi^7 \|e_2\|^2 \|e_3\|^2) \end{aligned}$$

$$\begin{aligned} & \psi^0\psi^1 + \psi^1\psi^0 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33} + \psi^5\psi^3 g_{33} \\ & - \psi^6\psi^2 g_{22} + \psi^7\psi^4 g_{22}g_{33} \\ & 2\left(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33}\right) \end{aligned}$$

By e_2 :

$$\begin{aligned} & + \psi^0\psi^2 + \psi^1\psi^6 \|e_1\|^2 + \psi^2\psi^0 - \psi^3\psi^4 \|e_3\|^2 - \psi^4\psi^3 \|e_3\|^2 + \psi^5\psi^7 \|e_3\|^2 \|e_1\|^2 \\ & + \psi^6\psi^1 \|e_1\|^2 + \psi^7\psi^5 \|e_1\|^2 \|e_3\|^2 \\ & 2\left(+\psi^0\psi^2 + \psi^1\psi^6 \|e_1\|^2 - \psi^3\psi^4 \|e_3\|^2 + \psi^5\psi^7 \|e_3\|^2 \|e_1\|^2\right) \\ & + \psi^0\psi^2 + \psi^1\psi^6 g_{11} + \psi^2\psi^0 - \psi^3\psi^4 g_{33} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11} \\ & + \psi^6\psi^1 g_{11} + \psi^7\psi^5 g_{11}g_{33} \\ & 2\left(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11}\right) \end{aligned}$$

By e_3 :

$$\begin{aligned} & + \psi^0\psi^3 - \psi^1\psi^5 \|e_1\|^2 + \psi^2\psi^4 \|e_2\|^2 + \psi^3\psi^0 + \psi^4\psi^2 \|e_2\|^2 - \psi^5\psi^1 \|e_1\|^2 \\ & + \psi^6\psi^7 \|e_1\|^2 \|e_2\|^2 + \psi^7\psi^6 \|e_1\|^2 \|e_2\|^2 \\ & 2\left(+\psi^0\psi^3 - \psi^1\psi^5 \|e_1\|^2 + \psi^2\psi^4 \|e_2\|^2 + \psi^6\psi^7 \|e_1\|^2 \|e_2\|^2\right) \\ & + \psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^3\psi^0 + \psi^4\psi^2 g_{22} - \psi^5\psi^1 g_{11} \\ & + \psi^6\psi^7 g_{11}g_{22} + \psi^7\psi^6 g_{11}g_{22} \\ & 2\left(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11}g_{22}\right) \end{aligned}$$

In e_2e_3 plane:

$$\begin{aligned} & + \psi^0\psi^4 + \psi^1\psi^7 \|e_1\|^2 + \psi^2\psi^3 - \psi^3\psi^2 - \psi^4\psi^0 + \psi^5\psi^6 \|e_1\|^2 \\ & - \psi^6\psi^5 \|e_1\|^2 - \psi^7\psi^1 \|e_1\|^2 = 0 \end{aligned}$$

In e_3e_1 plane:

$$\begin{aligned} & + \psi^0\psi^5 - \psi^1\psi^3 + \psi^2\psi^7 \|e_2\|^2 + \psi^3\psi^1 - \psi^4\psi^6 \|e_2\|^2 - \psi^5\psi^0 \\ & + \psi^6\psi^4 \|e_2\|^2 - \psi^7\psi^2 \|e_2\|^2 = 0 \end{aligned}$$

In e_1e_2 plane:

$$\begin{aligned} & + \psi^0\psi^6 + \psi^1\psi^2 - \psi^2\psi^1 + \psi^3\psi^7 \|e_3\|^2 + \psi^4\psi^5 \|e_3\|^2 - \psi^5\psi^4 \|e_3\|^2 \\ & - \psi^6\psi^0 - \psi^7\psi^3 \|e_3\|^2 = 0 \end{aligned}$$

In $e_1e_2e_3$ plane:

$$+ \psi^0\psi^7 + \psi^1\psi^4 + \psi^2\psi^5 + \psi^3\psi^6 - \psi^4\psi^1 - \psi^5\psi^2 - \psi^6\psi^3 - \psi^7\psi^0 = 0$$

So, in this case, we can sum up the result as:

$$\psi^\dagger \psi = \rho + \vec{j}$$

Being:

$$\begin{aligned} \rho = & (\psi^0)^2 + (\psi^1)^2 g_{11} + (\psi^2)^2 g_{22} + (\psi^3)^2 g_{33} + (\psi^4)^2 g_{22}g_{33} \\ & + (\psi^5)^2 g_{33}g_{11} + (\psi^6)^2 g_{11}g_{22} + (\psi^7)^2 g_{11}g_{22}g_{33} \end{aligned}$$

$$\begin{aligned} \bar{j} = & 2(\psi^0\psi^1 - \psi^2\psi^6 g_{22} + \psi^3\psi^5 g_{33} + \psi^4\psi^7 g_{22}g_{33})e_1 \\ & + 2(+\psi^0\psi^2 + \psi^1\psi^6 g_{11} - \psi^4\psi^3 g_{33} + \psi^5\psi^7 g_{33}g_{11})e_2 \\ & + 2(+\psi^0\psi^3 - \psi^1\psi^5 g_{11} + \psi^2\psi^4 g_{22} + \psi^6\psi^7 g_{11}g_{22})e_3 \end{aligned}$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^\dagger\psi = j^\mu e_\mu$$

where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

Annex A3. Bra-Ket Product between the Reverse of a Spinor and a Spinor in Non-Euclidean Metric (Non-Orthogonal and Non-Orthonormal)

We should do the following operation again:

$$\begin{aligned} \psi^\dagger\psi &= \psi^\mu e_\mu^\dagger \psi^\nu e_\nu \\ &= (\psi^0 e_0^\dagger + \psi^1 e_1^\dagger + \psi^2 e_2^\dagger + \psi^3 e_3^\dagger + \psi^4 e_4^\dagger + \psi^5 e_5^\dagger + \psi^6 e_6^\dagger + \psi^7 e_7^\dagger)(\psi^0 e_0 \\ &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_4 + \psi^5 e_5 + \psi^6 e_6 + \psi^7 e_7) \\ &= (\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_3 e_2 + \psi^5 e_1 e_3 + \psi^6 e_2 e_1 + \psi^7 e_3 e_2 e_1)(\psi^0 \\ &\quad + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3) \end{aligned}$$

But using the following rules commented in Chapter 3.3,

$$\begin{aligned} (e_i)^2 &= e_i e_i = \|e_i\|^2 = g_{ii} \\ e_i e_j &= 2g_{ij} - e_j e_i = 2g_{ji} - e_j e_i \\ e_i \cdot e_j &= e_j \cdot e_i = g_{ij} = g_{ji} \\ e_i e_j &= e_i \cdot e_j + e_i \wedge e_j = g_{ij} + e_i \wedge e_j \\ (e_1)^2 &= e_1 e_1 = \|e_1\|^2 = g_{11} \\ (e_2)^2 &= e_2 e_2 = \|e_2\|^2 = g_{22} \\ (e_3)^2 &= e_3 e_3 = \|e_3\|^2 = g_{33} \\ e_1 e_2 &= 2g_{12} - e_2 e_1 = 2g_{21} - e_2 e_1 \\ e_2 e_3 &= 2g_{23} - e_3 e_2 = 2g_{32} - e_3 e_2 \\ e_3 e_1 &= 2g_{31} - e_1 e_3 = 2g_{13} - e_1 e_3 \end{aligned}$$

I am not going to do it (you have a start of these calculations in [14]), but anyhow, you can understand that the result, whatever it is, will have this form:

$$\psi^\dagger\psi = j^\mu e_\mu$$

where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

Annex A4. Bra-Ket Product between the Inverse of a Spinor and a Spinor in Non-Euclidean Metric (Orthogonal but Not Orthonormal)

If instead of multiplying by the reverse, we multiply by the inverse (in orthogonal but not orthonormal metric), we should use the following rules from previous chapters:

$$\begin{aligned}(e_0)^2 &= \|e_0\|^2 = g_{00} & (e_1)^2 &= \|e_1\|^2 = g_{11} & (e_2)^2 &= \|e_2\|^2 = g_{22} & (e_3)^2 &= \|e_3\|^2 = g_{33} \\ e_0 e_i &= e_i e_0 & e_2 e_3 &= -e_3 e_2 & e_3 e_1 &= -e_1 e_3 & e_1 e_2 &= -e_2 e_1 \\ (e_i)^{-1} &= e^i = \frac{e_i}{g_{ii}} = \frac{e_i}{\|e_i\|^2} & (e_i e_j)^{-1} &= \frac{e_j e_i}{\|e_j\|^2 \|e_i\|^2} = \frac{e_j e_i}{g_{jj} g_{ii}}\end{aligned}$$

where all the above relations we have seen in previous chapters.

Operating:

$$\begin{aligned}\psi^{-1}\psi &= \left(\psi^0 + \psi^1 \frac{e_1}{\|e_1\|^2} + \psi^2 \frac{e_2}{\|e_2\|^2} + \psi^3 \frac{e_3}{\|e_3\|^2} + \psi^4 \frac{e_3 e_2}{\|e_2\|^2 \|e_3\|^2} + \psi^5 \frac{e_1 e_3}{\|e_3\|^2 \|e_1\|^2} \right. \\ &\quad \left. + \psi^6 \frac{e_2 e_1}{\|e_1\|^2 \|e_2\|^2} + \psi^7 \frac{e_3 e_2 e_1}{\|e_1\|^2 \|e_2\|^2 \|e_3\|^2} \right) \left(\psi^0 + \psi^1 e_1 + \psi^2 e_2 + \psi^3 e_3 \right. \\ &\quad \left. + \psi^4 e_2 e_3 + \psi^5 e_3 e_1 + \psi^6 e_1 e_2 + \psi^7 e_1 e_2 e_3 \right) \\ &= (\psi^0)^2 + \psi^1 \psi^0 \frac{e_1}{\|e_1\|^2} + \psi^2 \psi^0 \frac{e_2}{\|e_2\|^2} + \psi^3 \psi^0 \frac{e_3}{\|e_3\|^2} - \psi^4 \psi^0 \frac{e_2 e_3}{\|e_2\|^2 \|e_3\|^2} \\ &\quad - \psi^5 \psi^0 \frac{e_3 e_1}{\|e_3\|^2 \|e_1\|^2} - \psi^6 \psi^0 \frac{e_1 e_2}{\|e_1\|^2 \|e_2\|^2} - \psi^7 \psi^0 \frac{e_1 e_2 e_3}{\|e_1\|^2 \|e_2\|^2 \|e_3\|^2} + \psi^0 \psi^1 e_1 \\ &\quad + (\psi^1)^2 - \psi^2 \psi^1 e_1 \frac{e_2}{\|e_2\|^2} + \psi^3 \psi^1 \frac{e_3}{\|e_3\|^2} e_1 - \psi^4 \psi^1 e_1 \frac{e_2 e_3}{\|e_2\|^2 \|e_3\|^2} - \psi^5 \psi^1 \frac{e_3}{\|e_3\|^2} \\ &\quad + \psi^6 \psi^1 \frac{e_2}{\|e_2\|^2} - \psi^7 \psi^1 \frac{e_2 e_3}{\|e_2\|^2 \|e_3\|^2} + \psi^0 \psi^2 e_2 + \psi^1 \psi^2 \frac{e_1}{\|e_1\|^2} e_2 + (\psi^2)^2 \\ &\quad - \psi^3 \psi^2 e_2 \frac{e_3}{\|e_3\|^2} + \psi^4 \psi^2 \frac{e_3}{\|e_3\|^2} - \psi^5 \psi^2 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^6 \psi^2 \frac{e_1}{\|e_1\|^2} \\ &\quad - \psi^7 \psi^2 \frac{e_3}{\|e_3\|^2} \frac{e_1}{\|e_1\|^2} + \psi^0 \psi^3 e_3 - \psi^1 \psi^3 e_3 \frac{e_1}{\|e_1\|^2} + \psi^2 \psi^3 \frac{e_2}{\|e_2\|^2} e_3 + (\psi^3)^2 \\ &\quad - \psi^4 \psi^3 \frac{e_2}{\|e_2\|^2} + \psi^5 \psi^3 \frac{e_1}{\|e_1\|^2} - \psi^6 \psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2} e_3 - \psi^7 \psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2} \\ &\quad + \psi^0 \psi^4 e_2 e_3 + \psi^1 \psi^4 \frac{e_1}{\|e_1\|^2} e_2 e_3 + \psi^2 \psi^4 e_3 - \psi^3 \psi^4 e_2 + (\psi^4)^2 - \psi^5 \psi^4 \frac{e_1}{\|e_1\|^2} e_2 \\ &\quad + \psi^6 \psi^4 e_3 \frac{e_1}{\|e_1\|^2} + \psi^7 \psi^4 \frac{e_1}{\|e_1\|^2} + \psi^0 \psi^5 e_3 e_1 - \psi^1 \psi^5 e_3 + \psi^2 \psi^5 e_1 \frac{e_2}{\|e_2\|^2} e_3 \\ &\quad + \psi^3 \psi^5 e_1 + \psi^4 \psi^5 e_1 \frac{e_2}{\|e_2\|^2} + (\psi^5)^2 - \psi^6 \psi^5 \frac{e_2}{\|e_2\|^2} e_3 + \psi^7 \psi^6 \frac{e_2}{\|e_2\|^2} + \psi^0 \psi^6 e_1 e_2 \\ &\quad + \psi^1 \psi^6 e_2 - \psi^2 \psi^6 e_1 + \psi^3 \psi^6 e_1 e_2 \frac{e_3}{\|e_3\|^2} - \psi^4 \psi^6 \frac{e_3}{\|e_3\|^2} e_1 + \psi^5 \psi^6 e_2 \frac{e_3}{\|e_3\|^2} + (\psi^6)^2 \\ &\quad + \psi^7 \psi^6 \frac{e_3}{\|e_3\|^2} + \psi^0 \psi^7 e_1 e_2 e_3 + \psi^1 \psi^7 e_2 e_3 + \psi^2 \psi^7 e_3 e_1 + \psi^3 \psi^7 e_1 e_2 + \psi^4 \psi^7 e_1 \\ &\quad + \psi^5 \psi^7 e_2 + \psi^6 \psi^7 e_3 + (\psi^7)^2\end{aligned}$$

The scalar part is the same as the one multiplying by the reverse in a Euclidean orthonormal metric:

$$\rho = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 + (\psi^4)^2 + (\psi^5)^2 + (\psi^6)^2 + (\psi^7)^2$$

This could be a hint, that probably this is the real operation that has to be done in general, instead of the reverse. The issue is that in orthonormal metric, the inverse and the reverse are the same operation. But this is not true in general, in non-orthonormal metrics.

If continuing with the operation, for example, we separate by e_1 , we can see that the result is not as compact and in orthonormal or orthogonal solutions.

$$\begin{aligned} &\psi^1\psi^0 \frac{e_1}{\|e_1\|^2} + \psi^0\psi^1 e_1 - \psi^6\psi^2 \frac{e_1}{\|e_1\|^2} + \psi^5\psi^3 \frac{e_1}{\|e_1\|^2} + \psi^7\psi^4 \frac{e_1}{\|e_1\|^2} \\ &+ \psi^3\psi^5 e_1 - \psi^2\psi^6 e_1 + \psi^4\psi^7 e_1 \end{aligned}$$

Even we can see that the result in the planes is not zero. Example e_2e_3 :

$$\begin{aligned} &-\psi^4\psi^0 \frac{e_2e_3}{\|e_2\|^2\|e_3\|^2} - \psi^7\psi^1 \frac{e_2e_3}{\|e_2\|^2\|e_3\|^2} - \psi^3\psi^2 e_2 \frac{e_3}{\|e_3\|^2} + \psi^2\psi^3 \frac{e_2}{\|e_2\|^2} e_3 \\ &+ \psi^0\psi^4 e_2e_3 - \psi^6\psi^5 \frac{e_2}{\|e_2\|^2} e_3 + \psi^5\psi^6 e_2 \frac{e_3}{\|e_3\|^2} + \psi^1\psi^7 e_2e_3 \end{aligned}$$

Or $e_1e_2e_3$, also different from zero:

$$\begin{aligned} &-\psi^7\psi^0 \frac{e_1e_2e_3}{\|e_1\|^2\|e_2\|^2\|e_3\|^2} - \psi^4\psi^1 e_1 \frac{e_2e_3}{\|e_2\|^2\|e_3\|^2} - \psi^5\psi^2 e_1e_2 \frac{e_3}{\|e_3\|^2} \\ &-\psi^6\psi^3 \frac{e_1}{\|e_1\|^2} \frac{e_2}{\|e_2\|^2} e_3 + \psi^1\psi^4 \frac{e_1}{\|e_1\|^2} e_2e_3 + \psi^2\psi^5 e_1 \frac{e_2}{\|e_2\|^2} e_3 \\ &+ \psi^3\psi^6 e_1e_2 \frac{e_3}{\|e_3\|^2} + \psi^0\psi^7 e_1e_2e_3 \end{aligned}$$

Anyhow, in general we can always say that whatever the final result is, the product will have the following shape:

$$\psi^{-1}\psi = j^\mu e_\mu$$

where j^μ are just scalar coefficients (or functions that output a scalar) and the e_μ are the basis vectors as they have been defined throughout the paper.

In case that we perform this operation (multiplying by the inverse) in an orthonormal metric, we will get the same result as in **Annex A1** (as the inverse is the same as the reverse in this case).

In case that we perform this operation in a non-orthogonal (and therefore non-orthogonal case), we will have to follow the rules in Chapter 3.3.

Anyhow, the result will always have this form:

$$\psi^{-1}\psi = j^\mu e_\mu$$

Annex A5. Other Considerations regarding Chapter 9

In Chapter 9.1, we have made a modification in the standard tensor/matrix nota-

tion in the Dirac equation based on the results of this paper. From here:

$$i\gamma^\mu \partial_\mu \psi = \frac{mc}{\hbar} \psi$$

To here:

$$i\gamma^\mu \partial_\mu \psi = \sqrt{\frac{m^2 c^2}{\hbar^2} - R} \psi$$

Why not make similar changes in other equations? For example, as we have reduced the factor that involves the mass in above equation, why not make the same in the stress energy tensor for example?

If we divide the factor: $\frac{m^2 c^2}{\hbar^2} - R$ by $\frac{m^2 c^2}{\hbar^2}$,

we get a per unit factor of:

$$1 - \frac{\hbar^2}{m^2 c^2} R$$

This is the factor to use in equations that are quadratic in ψ (like the ones involving Stress-Energy tensor or Ricci tensor. And the following one that are linear with ψ , like the Dirac equation above.

$$\sqrt{1 - \frac{\hbar^2}{m^2 c^2} R}$$

So, for example the Einstein equation with this modifier should read something like:

$$T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R \right) = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Going, even further, we have used in Chapter 8, a step where we converted the Ricci tensor in Ricci scalar in a not very rigorous way. We can see that there is no problem with that as we could put it directly in the equation this way:

$$T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R_{\mu\nu} \right) = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Going even further, to assure that the divergence of the stress energy tensor keeps being zero, we could add the subtraction by the half of the Ricci scalar, this way:

$$T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \right) = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Or even include the cosmological constant:

$$T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) \right) = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Of all these possibilities, the most possible (or the one most coherent with the paper) is the one already commented:

$$T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R \right) = \frac{c^4}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right)$$

Or in the typical form:

$$\frac{8\pi G}{c^4} T_{\mu\nu} \left(1 - \frac{\hbar^2}{m^2 c^2} R \right) = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

The same way, if consider that instead of the Ricci scalar we should use the Ricci tensor, the Klein-Gordon equation should read:

$$e_\mu \left(e^\beta \nabla_\beta \psi^\dagger \psi \nabla_\alpha^\dagger e^\alpha \right) e_\nu = \frac{m}{\hbar^2} \left(mc^2 - \frac{\hbar^2}{m} R_{\mu\nu} \right) e_\mu \psi^\dagger \psi e_\nu$$

With all the different variations as commented above regarding the stress-energy tensor.

We cannot take a “Dirac equation” from here as we cannot take the “square root” (or factorization in two factors) of $R_{\mu\nu}$.

Annex A6. Considerations about the Gravitational Bivector (Gravitomagnetism-Like)

You can find in [32] a simplification of how magnetic effects work.

Consider two charged particles moving in parallel in the direction of the x axis. They are separated a distance r in the direction of the y axis. This means, they are moving in parallel in the xy plane with a separation of r . And the speed is in the direction of x and their separation is the axis y (both lines are perpendicular).

Considering standard algebra, the magnetic field will be in the z direction:

$$B \propto \hat{v} \times \hat{r} = \hat{x} \times \hat{y} = \hat{z}$$

$$B \propto \hat{z}$$

And the force will be (we can obviate the signs; we just need the direction axes):

$$F \propto \hat{v} \times \hat{B} = \hat{x} \times \hat{z} \propto \hat{y}$$

So, we can check that the magnetic interaction when the line of sight and the speed of the particles is perpendicular, it is acting in the line of sight.

Now, let’s do the same calculation of the direction of the magnetic interaction, but using geometric algebra:

$$B \propto \hat{v} \hat{r} = e_1 e_2$$

$$F \propto \hat{v} \hat{B} = (e_1)(e_1 e_2) = e_1 e_1 e_2 = (e_1)^2 e_2 = e_2$$

The juxtaposition between vectors, means geometric product. And we are considering an orthonormal basis where e_1 corresponds to x , e_2 to y and e_3 to z .

You can see that the direction of the force obtained is the same (y or e_2).

The issue with signs (in the first equation the result should be $-y$) comes because a minus sign should be added when converting the traditional cross product to the geometric or wedge product (see [1] [2] for further explanations) that I have not included for simplification.

The message of above equations can be summed up as:

$$F \propto (e_1)(e_1 e_2) = e_2$$

When we have a field defined by a bivector, in this case the bivector $e_1 e_2$, the interaction depends on a vector defined by the affected particle (in this case speed, in the direction of e_1). But the effect of the bivector will not be in the direction of that vector but in the one defined between the product of the bivector and the vector (in this example the result is e_2 that in fact is perpendicular to the speed vector e_1).

As commented, the gravitation/tensor equations defined in Chapters 7, 8 and 9 will go from indices 0 to 7. This means, they include, not only the 3 vectors (representing the 3 spatial directions), but also the 3 bivectors (representing the 3 planes) and the scalars and the trivector (representing time and/or volume depending on context).

The issue is that if gravitation includes also bivectors, magnetic-like effects would be expected as explained before.

The thing is that in fact these effects seem to exist [26] [27], as we will see now. And they have been already studied. But with one main difference. Normally, these effects were considered historically as an ad-hoc addition to gravitation to explain certain phenomena (as speed of rotation of galaxies) [26]. But in this paper, they are not an ad-hoc added interaction, they come directly from the equations, where these bivectors (and consequently their magnetic-like effects) are included.

But, in the present paper, the magnetic-like effects do not appear as specific solutions to the equations for certain situations. On the contrary, they are directly included in the gravitational equations themselves (even without solving them), just because the existence of the bivectors in the gravitational equations in the first place.

If we consider the classical gravity [33] and Coulomb force [34], we have:

$$\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r}; \quad \vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

Now, if try to make an “equivalence” of the constants in both equations, we arrive to:

$$-G \rightarrow \frac{1}{4\pi\epsilon_0}; \quad \epsilon_0 \rightarrow -\frac{1}{4\pi G}$$

Also, if we consider the equation that relates the permittivity and the permeability in vacuum, we have:

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = c; \quad \frac{1}{\epsilon_0 \mu_0} = c^2; \quad \mu_0 = \frac{1}{\epsilon_0 c^2}$$

So, following the “equivalence” commented before, we have:

$$\mu_0 \rightarrow \frac{1}{-\frac{1}{4\pi G} c^2} = -\frac{4\pi G}{c^2}$$

You can find that these relations are correct in [26] and [27].

In reference [26], they study the speed of rotation of the following galaxies: NGC 1560, NGC 3198, NGC 3115.

Leading to the conclusion that the speed of rotation of these galaxies can be perfectly explained by magnetic-like effects without the need of exotic dark-matter.

The important difference of paper [26] compared to the present paper is the following. In paper [26] and, in general, for all the defenders of gravitomagnetism, the gravitomagnetism has to be added as an ad-hoc addition to gravitational theory to match certain effects. As in [26] where it is added to the equations to match the rotation of galaxies.

But in the present paper, the gravitomagnetism effects appear directly in the equations of gravitation because of the existence of gravitational bivectors, not needing to add the gravitomagnetism ad-hoc. It is just inherent in the equations of Chapters 9.1 to 9.3 just because the bivectors are directly part of them.

Anyhow, all the conclusions commented in [26] regarding the accurate matching of measured rotation of speed compared to calculated one using gravitomagnetism for the above-mentioned galaxies would apply also for the present paper. Leading to the non-necessity of dark matter to explain the speed of rotation of the galaxies.

It is also to be noted that there are other authors [35] that consider that the magnetic-like effects could appear just because of the dragging-effect [36] or the non-linear effects of general relativity, when solving Einstein Field equations for certain cases [35]. I just wanted to add this, so you can work with all the information.

Annex A7. Considerations about Dark Energy

The most identifiable effect of the dark energy is the existence of the cosmological constant Λ [28]. Which value [28] is in the order of:

$$\Lambda = 1.1056E - 52 \text{ m}^{-2}$$

It is to be remarked that although being an effect of energy, the dimensions are Length^{-2} .

In the paper [29], we get to the following energy-momentum relation coming from the equations (Chapters 7 to 9) of the present paper:

$$E^2 = m^2 c^4 + p^2 c^2 - R \hbar^2 c^2$$

You can check that there is a new term that reduces the energy of each particle depending on the Ricci scalar (the space-time curvature in the place where the particle is lying). This term has the units of energy squared as it has to be coherent with the rest of the equation.

If we want this term to have units of Length^{-2} , we have to make the following escalation. I call it escalation, because we are not modifying at all the equation. We are just dividing by constants, not adding new variables. The equation is the

same. In fact, as if some authors do, I used all the constants equal to 1, the following operation would not even be necessary:

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^4}{\hbar^2 c^2} + \frac{p^2 c^2}{\hbar^2 c^2} - \frac{R \hbar^2 c^2}{\hbar^2 c^2}$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

The equation above has the same meaning as the original but in another units. Here the energy squared has the units Length⁻². So, the last term (−R) represents the square of the reduction of the energy (a negative energy) that apply to every particle in the units Length⁻².

So, let’s calculate this value and check if it corresponds to the value of the cosmological constant Λ. This means, let’s check if this term can be the origin of the “dark energy”.

To calculate the Ricci scalar, we have to decide which metric to apply to the universe. The most appropriate for an isotropic universe would be the interior Schwarzschild metric [37].

According to some authors the Ricci scalar of the interior Schwarzschild metric is:

$$R = \frac{1}{3} \frac{8\pi G}{c^2} (\rho + 3p + \dots)$$

where ρ is the Energy density (in kg/m³) and p is the pressure. The three points represent some other elements that normally are considered small compared to the previous ones.

In other references [38], probably depending on different assumptions or definitions, the Ricci scalar of the interior Schwarzschild metric does not include the 1/3 factor [38].

$$R = \frac{8\pi G}{c^2} (\rho + 3p + \dots)$$

As it is just a factor, and what we want is to calculate an order of magnitude, let’s use the latter equation that does not include it. Anyhow, as the relation between both equations is pretty straight forward, it can be added if it is discovered that the one with 1/3 is the correct one.

For simplification, let’s consider that the pressure is zero—or very low compared to energy density—and the points elements can be neglectable. This leads to:

$$R = \frac{8\pi G \rho}{c^2}$$

So, to solve the equation, we need to know the value of the energy density ρ.

The value of ρ (considering that corresponds to the critical density) is calculated using the Hubble constant [28] [39] [40]. Its value is:

$$H = \frac{67.66 \frac{\text{km}}{\text{s}}}{1 \text{ Mpc}} = \frac{67.66 \frac{\text{km}}{\text{s}}}{3.086 \text{E}19 \text{ km}} = 2.192 \text{E} -18 \text{ s}^{-1}$$

Using the equation [40], we can calculate the density as:

$$\rho = \frac{3H^2}{8\pi G} = \frac{3(2.192E-18 \text{ s}^{-1})^2}{8\pi(6.67E-11 \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2})} = 8.598E-27 \frac{\text{kg}}{\text{m}^3}$$

And now, we can introduce it in:

$$R = \frac{8\pi G \rho}{c^2}$$

$$R = \frac{8\pi(6.67E-11 \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}) \left(8.598E-27 \frac{\text{kg}}{\text{m}^3}\right)}{\left(299792458 \frac{\text{m}}{\text{s}}\right)^2} = 1.603E-52 \text{ m}^{-2}$$

That you can see that is very near (or at least of the same magnitude) as the cosmological constant Λ :

$$\Lambda = 1.1056E-52 \text{ m}^{-2}$$

So, this R representing an energy to be subtracted to the particles according to:

$$E^2 = m^2 c^4 + p^2 c^2 - R \hbar^2 c^2$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

It is a perfect candidate for this dark energy.

$$R = 1.603E-52 \text{ m}^{-2}$$

$$\Lambda = 1.1056E-52 \text{ m}^{-2}$$

Not much more to say here, the numbers speak by themselves. An equation that has come from the equations of this paper (from Chapters 7 to 9), has led to an appearance of a reduction of energy of the particles (a negative energy) that is a perfect candidate to correspond to the dark energy.

Other Calculations for R That Lead to Similar Results

In A.7, we have calculated:

$$R = 1.603E-52 \text{ m}^{-2}$$

Using the equation of R that does not include the 1/3 factor. Just for info, if we had used the equation for R that includes the 1/3 factor, we would have gotten:

$$R = \frac{1}{3}(1.603E-52 \text{ m}^{-2}) = 0.5345E-52 \text{ m}^{-2}$$

$$\Lambda = 1.1056E-52 \text{ m}^{-2}$$

That is also in the order of Λ although not so precise. But the idea that the concept that this negative R in the following equation could be the candidate for the dark energy keeps being the same:

$$E^2 = m^2 c^4 + p^2 c^2 - R \hbar^2 c^2$$

$$\frac{E^2}{\hbar^2 c^2} = \frac{m^2 c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R$$

Another thing we could do is to use another metric, to see if we obtain a similar result or not. Instead of using the interior Schwarzschild metric, let's use the exterior Schwarzschild metric.

In the exterior Schwarzschild metric, the value of the Ricci scalar is zero. But we can use the Kretschmann scalar instead [30].

$$K = \frac{48G^2 M^2}{c^4 r^6}$$

The Kretschmann scalar is a quadratic element. Its units are Length⁻⁴. The Ricci scalar is a square element (Length⁻²). So, to convert the Kretschmann scalar to an equivalent Ricci scalar, we have to take the square root:

$$R \rightarrow \sqrt{K} = \frac{\sqrt{48GM}}{c^2 r^3}$$

Now, considering spherical symmetry (as in fact it is the Schwarzschild metric) we can make some operations to obtain the equation as dependent of ρ (the energy density in kg/m³).

$$\begin{aligned} R &= \frac{\sqrt[3]{48G} M}{c^2 r^3} = \frac{\sqrt[3]{48G} M \left(\frac{4}{3}\pi\right)}{c^2 r^3 \left(\frac{4}{3}\pi\right)} = \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \frac{M}{\frac{4}{3}\pi r^3} \\ &= \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \frac{M}{V} = \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \rho \end{aligned}$$

So,

$$R = \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \rho$$

Now, we apply the value of ρ we have calculated in **Annex A7**:

$$\begin{aligned} R &= \frac{\sqrt[3]{48G} M}{c^2 r^3} = \frac{\sqrt[3]{48G} M \left(\frac{4}{3}\pi\right)}{c^2 r^3 \left(\frac{4}{3}\pi\right)} = \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \frac{M}{\frac{4}{3}\pi r^3} = \frac{\sqrt[3]{48}\left(\frac{4}{3}\pi\right)G}{c^2} \rho \\ &= \frac{29.020 \left(6.672E-11 \frac{\text{m}^3}{\text{s}^2 \cdot \text{kg}} \right)}{\left(299792458 \frac{\text{m}}{\text{s}} \right)^2} 8.598E-27 \frac{\text{kg}}{\text{m}^3} = 1.852E-52 \text{ m}^{-2} \end{aligned}$$

So, we have:

$$\begin{aligned} R &= 1.852E-52 \text{ m}^{-2} \\ \Lambda &= 1.1056E-52 \text{ m}^{-2} \end{aligned}$$

That again is in the order of Λ, even having an assumption of a different metric.

In fact, the Ricci scalar in the exterior and the interior Schwarzschild metric are in fact very similar (see **Annex A7**):

$$\begin{aligned}
 R_{extSchw} &= 1.852E - 52 \text{ m}^{-2} \\
 R_{intSchw} &= 1.603E - 52 \text{ m}^{-2} \\
 R_{intSchw1/3} &= 0.5345E - 52 \text{ m}^{-2} \\
 \Lambda &= 1.1056E - 52 \text{ m}^{-2}
 \end{aligned}$$

One point to be commented also is the possibility that these equations are “circular”. In **Annex A7**, it is calculated the critical density. This density is defined as the density needed to have a near flat space and is calculated using the scape velocity equation. It could be the Ricci scalar curvature was another representation of this “escape velocity”, so we are incurring in some kind of circular reference. This is to be studied.

One last comment is regarding the pressure in the equation of the Ricci scalar for the interior Schwarzschild metric:

$$R = \frac{8\pi G}{c^2}(\rho + 3p + \dots)$$

A study should be done of how it affects when it cannot be considered zero and this could be related to the non-exact equality (although yes in the order of magnitude) between Λ and R in the previous calculations.

Also, we could consider the not at rest particle case with a linear moment different from zero in the following equation:

$$\begin{aligned}
 E^2 &= m^2c^4 + p^2c^2 - R\hbar^2c^2 \\
 \frac{E^2}{\hbar^2c^2} &= \frac{m^2c^2}{\hbar^2} + \frac{p^2}{\hbar^2} - R
 \end{aligned}$$

But the negative element $-R$ will be the same whether the linear momentum (p) is zero or not in this equation. But it could affect indirectly if the linear momenta of the different particles are affecting the pressure value in the equation of R (via the square of momenta) and so really affecting the final value of R .