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The Decay Property of Cauchy Problem for Viscoelastic Hyperbolic Systems with Dissipation

Jing Hu

School of Mathematics and Key Laboratory of Mathematical MIIT, Nanjing University of Aeronautics and Astronautics, Nanjing, China
Email: hujing999@nuaa.edu.cn

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Abstract

This paper investigates the decay properties of solutions to the Cauchy problem for viscoelastic nonlinear hyperbolic dissipative systems on $\mathbb{R}^3$. Due to the weak dissipation of the system, the decay estimate of solutions exhibits a loss of regularity, implying that a higher regularity of initial data is required for optimal decay rates compared to the global existence. The aim is to reduce the initial regularity to the lowest possible level to achieve the optimal decay rate. Based on the global existence, we employ energy methods, L^p - L^q - L^r estimates, and harmonic analysis tools to obtain the optimal decay result of solutions.

Keywords

Viscoelastic Systems, Regularity-Loss, Optimal Decay Estimates, L^p - L^q - L^r Estimates

1. Introduction

We will consider the following viscoelastic nonlinear hyperbolic system:

$$u_{tt} - \sum_{j=1}^{3} b^j \left(\partial_x u\right)_{x_j} + \left(1-\Delta\right)^{-1} \left(\sum_{j,k=1}^{3} K^{jk} * u_{x_j x_k} + L u_t \right) = 0, \tag{1.1}$$

with initial data

$$u\left(x,0\right) = u_0\left(x\right), \quad u_t\left(x,0\right) = u_1\left(x\right). \tag{1.2}$$

Here, u is an m-vector function with respect to the variable $x = \left(x_1, x_2, x_3\right) \in \mathbb{R}^3$ and $t \geq 0$; $b^j\left(v\right)$ $\left(j = 1,2,3\right)$ are smooth m-vector functions of $v=(v_1,v_2,v_3)$, where $v_j=u_{x_j}\in\mathbb{R}^m$; Δ is the Laplace operator, $K^{jk}\ (j,k=1,2,3)$ are $m\times m$ smooth matrix functions of $t\ge 0$ satisfying $\left(K^{jk}(t)\right)^\top=K^{kj}(t)\ (j,k=1,2,3)$ for any $t\ge 0$; L is an $m\times m$ real symmetric constant matrix. The superscript $\top$ denotes the transposed, and the symbol $*$ denotes convolution with respect to t, that is,

$$K^{jk}*u_{x_jx_k}=\int_0^t K^{jk}(t-\tau)u_{x_jx_k}(\tau)\mathrm{d}\tau.$$

The system (1.1) is a set of equations describing the motion of viscoelastic materials, where u and $v(=\partial_x u)$ represent displacement and deformation during the motion of viscoelastic materials, respectively, and $K^{jk}*u_{x_jx_k}$ and Lu_t represent the memory and damping terms during motion, respectively, which together constitute the dissipative part of the system, thereby ensuring the decay of the solution.

Regarding the elastic term $b^j(v)$, we assume that the system (1.1) has a free energy $\phi(v)$, which is a smooth scalar function of v and satisfies

$$b^j(v)=D_{v_j}\phi(v),$$

where $D_{v_j}\phi(v)$ is the *Fréchet* derivative of $\phi(v)$ with respect to v_j, define

$$B^{jk}(v)=D_{v_k}b^j(v)=D_{v_k}D_{v_j}\phi(v).$$

Clearly, for each j,k and $v\in\mathbb{R}^{3m}$, we have $B^{jk}(v)^\top=B^{kj}(v)$. Then Equation (1.1) can be written as the quasi-linear form:

$$u_{tt}-\sum_{j=1}^{3}B^{jk}(\partial_x u)u_{x_jx_k}+(1-\Delta)^{-1}\left(\sum_{j,k=1}^{3}K^{jk}*u_{x_jx_k}+Lu_t\right)=0. \tag{1.3}$$

Thus, the corresponding linearized system around $\partial_x u=0$ is given as follows:

$$u_{tt}-\sum_{j=1}^{3}B^{jk}(0)u_{x_jx_k}+(1-\Delta)^{-1}\left(\sum_{j,k=1}^{3}K^{jk}*u_{x_jx_k}+Lu_t\right)=0. \tag{1.4}$$

To ensure the hyperbolicity and well-posedness of system (1.1), define the following real symmetric matrices

$$B_\omega=\sum_{j,k=1}^{3}B^{jk}\omega_j\omega_k,\quad K_\omega(t)=\sum_{j,k=1}^{3}K^{jk}(t)\omega_j\omega_k,$$

for $\omega=(\omega_1,\omega_2,\omega_3)\in\mathcal{S}^2$. Then the following conditions are imposed as:

[A1] For any $\omega\in\mathcal{S}^2$, $B_\omega(0)$ is positive definite, and $K_\omega(t)\ (t\ge 0)$ is non-negative definite; and L is a real symmetric non-negative definite.

[A2] $B_\omega(0)-\mathcal{K}_\omega(t)$ is positive definite for each $\omega\in\mathcal{S}^2$ uniformly in $t\ge 0$, where $\mathcal{K}_\omega(t)=\int_0^t K_\omega(s)\mathrm{d}s$.

[A3] For each $\omega\in\mathcal{S}^2$, $K_\omega(0)+L$ is real symmetric and positive definite.

[A4] There exist positive constants C_0 and c_0 such that for each $\omega\in\mathcal{S}^2$, $t\ge 0$, we have $-C_0K_\omega(t)\le\dot{K}_\omega(t)\le -c_0K_\omega(t)$ and $-C_0K_\omega(t)\le\ddot{K}_\omega(t)\le C_0K_\omega(t)$ hold, where $\dot{K}_\omega(t)=\partial_t K_\omega(t)$, $\ddot{K}_\omega(t)=\partial_{tt}K_\omega(t)$.

In condition [A4], for real symmetric matrices A and B, $A \geq B$ or $B \leq A$ denotes that $A-B$ is non-negative definite; and $A > B$ or $B < A$ denotes that $A-B$ is a positive definite.

Extensive investigation has been conducted on the decay properties of viscoelastic equations. Dharmawardane, Nakamura and Kawashima [1] employed the classical energy method to examine the decay rates of solutions for the Cauchy problem associated with linear viscoelastic equations on $\mathbb{R}^n$, deriving the standard decay estimates. Moreover, they explored the case involving the operator $(1-\Delta)^{-\theta/2}$, and established the following decay estimate:

$$\begin{aligned}\left\|\left(\partial_x^k u_t, \partial_x^{k+1} u\right)(t)\right\|_{L^2} \leq C(1+t)^{-l/\theta}\left\|\left(\partial_x^{k+l} u_1, \partial_x^{k+l+1} u_0\right)\right\|_{L^2} \\ + C(1+t)^{-n/4-k/2}\left\|\left(u_1, \partial_x u_0\right)\right\|_{L^1},\end{aligned} \tag{1.5}$$

where $0 \leq k+l \leq s$, and C is a positive constant. Inequality (1.5) reveals that to achieve the decay rate of $(1+t)^{-l/\theta}$, it is necessary to assume that the initial data possess an additional l-th order regularity, which makes the initial value need to satisfy a higher regularity index to obtain the optimal decay rate of solutions. Similar situations of regularity loss also appear in the Euler-Maxwell system in [2], the dissipative Timoshenko system in [3], and the hyperbolic-elliptic systems of radiating gas in [4].

We will study the nonlinear problem of the viscoelastic hyperbolic system. When there is no the operator $(1-\Delta)^{-1}$ in Equation (1.1), Dharmawardane, Nakamura and Kawashima [5] [6] have studied the global existence of solutions to the Cauchy problem in n-dimensional space and calculated the $L^1(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ type decay rate of solutions; for problem (1.1) - (1.2), due to the weakening of the dissipative term by $(1-\Delta)^{-1}$, the decay exhibits regularity loss-type. To overcome the difficulties caused by regularity-loss in dealing with nonlinear terms, Dharmawardane has combined time-weighted energy methods and semigroup methods in [7] to calculate the optimal decay estimates of the solutionfor the nonlinear equation, which effectively controls the influence of nonlinear terms and obtains the same decay rate $(1+t)^{-n/4}$ as the linear heat equation $L^1(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ type decay estimate.

Due to the weaker dissipative mechanism, in order to obtain the optimal decay estimate of smooth solutions, compared with the regularity required for the global existence of smooth solutions, we have to impose higher regularity on the initial value. For example, for the Cauchy problem of the three-dimensional compressible Euler-Maxwell equation, although only the regularity $s \geq 3$ of the initial value is needed to prove the global existence of solutions in [8], in order to obtain the optimal decay rate of solutions, the regularity of the initial value needs to reach $s \geq 6$ in [2]. We note that the minimal regularity 6 required for the decay estimate exceeds the minimal regularity 3 required for the global existence, because the decay estimate of the compressible Euler-Maxwell equation is of regularity-loss type. This phenomenon prompts us to think deeply: whether it is possible to achieve the L^1-L^2 optimal decay while keeping the lowest regularity index of the initial value? Research shows that this is feasible. Xu, Mori and Kawashima in [9] constructed a special energy estimate, that is the $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ estimate, and used it to reduce the regularity of the initial value to 3 for obtaining the optimal decay estimate of the compressible Euler-Maxwell equation; Xu and Kawashima also applied this estimate to prove the minimal decay regularity of smooth solutions to the two-fluid Euler-Maxwell equation in [10]; in addition, Cao and Xu also applied this estimate in [11] to study the minimal decay regularity of the Timoshenko-Fourier system in thermoelasticity. Similarly, we can apply this estimate to the three-dimensional viscoelastic nonlinear hyperbolic system and calculate the decay estimate of solutions under the premise of satisfying the minimal decay regularity index of the initial value. Dharmawardane has already proved that when the regularity of the initial value $s \ge 6$ in [7], the Cauchy problem for the smooth solution of the viscoelastic nonlinear hyperbolic system on $\mathbb{R}^3$ has global existence. We will discuss the minimal decay regularity index of smooth solutions to the viscoelastic nonlinear hyperbolic system.

This paper is organized as follows: in Section 2, the main theorems of this paper are given; some basic inequalities and $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ estimates are given in Section 3; in Section 4, energy estimates in the Fourier space and the proof of the main theorems are shown.

Notations. In this paper, the Fourier transform of a function u is denoted as

$$\hat{u} = \mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) \mathrm{e}^{-i\xi\cdot x} \mathrm{d}x,$$

and its inverse transform by

$$\mathcal{F}^{-1}[u](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\xi) \mathrm{e}^{i\xi\cdot x} \mathrm{d}\xi.$$

We denote by $H^{m,p}(\Omega)$ the classical Sobolev space on Ω, when $p=2$, $\Omega = \mathbb{R}^n$, use the abbreviation H^m. $L^p(1 \le p \le \infty)$ denotes the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_{L^p}$. For a non-negative l, ∂^l denotes the totality of all the lth order derivatives with respect to $x \in \mathbb{R}^n$; $f \lesssim g$ denotes $f \le Cg$.

Throughout the paper, C and c denote various generic positive constants.

2. Main Results

For convenience of calculation, define the following norms: Initial energy norms:

$$E_0^2 = \left\|(u_1, \partial_x u_0)\right\|_{H^s}^2, \quad E_1^2 = \left\|(u_1, \partial_x u_0)\right\|_{H^s}^2 + \left\|(u_1, \partial_x u_0)\right\|_{L^1}^2.$$

And time-weighted energy and dissipation norms:

$$E(t)^2 = \sum_{m=0}^{\left[\frac{s-1}{2}\right]} E_m(t)^2, \quad D(t)^2 = \sum_{m=-1}^{\left[\frac{s-1}{2}\right]-1} D_m(t)^2,$$

$$E_m(t)^2 = \sup_{0\le\tau\le t} (1+\tau)^{m-1/2} \left\|\left(\partial_x^m u_t, \partial_x^{m+1} u\right)(\tau)\right\|_{H^{s-2m}}^2,$$

$$D_{m-1}(t)^2 = \int_0^t (1+\tau)^{m-3/2} \left\| \left(\partial_x^m u_t, \partial_x^{m+1} u \right)(\tau) \right\|_{H^{s-2m}}^2 \mathrm{d}\tau.$$

Because of the regularity-loss, classical energy methods fail to provide a priori estimates of solutions to the problem (1.1) - (1.2), thus preventing the establishment of global existence. Therefore, this paper investigates the minimal decay regularity of the Cauchy problem (1.1) - (1.2) based on the time-weighted priori estimate in [7] for $n=3$. We present the result [R1] without the proof, which can be found in [7].

Proposition 2.1 Suppose conditions [A1] - [A4] hold and the initial data $(u_1, \partial_x u_0) \in H^s$ and $s \geq 6$. Then there exists a positive constant δ such that if $\left\| (u_1, \partial_x u_0) \right\|_{H^s} \leq \delta$, the problem (1.1) - (1.2) has a unique global solution and satisying

$$E(t) + D(t) \leq CE_0, \tag{2.1}$$

where E_0 is the initial energy and C is a positive constant. In particular, the solution exhibits decay estimates:

$$\left\| \left(\partial_x^m u_t, \partial_x^{m+1} u \right)(t) \right\|_{H^{s-2m}} \leq CE_0 (1+t)^{1/4-m/2},$$

for $m \geq 0$ and $s \geq 2m$.

Remark 2.1 The inequality (2.1) still holds even without the integral term involving the dissipation.

The following presents the main result of this paper.

Theorem 2.1 Suppose conditions [A1] - [A4] hold and the initial data $(u_1, \partial_x u_0) \in H^6(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Then there exists a positive constant δ_0 such that if $\left\| (u_1, \partial_x u_0) \right\|_{H^6 \cap L^1} \leq \delta_0$, the solution to problem (1.1) - (1.2) satisfies

$$\left\| (u_t, \partial_x u)(t) \right\|_{L^2} \leq C (1+t)^{-3/4} \left\| (u_1, \partial_x u_0) \right\|_{H^6 \cap L^1}, \tag{2.2}$$

where C is a positive constant.

3. Preliminaries

This section primarily provides some basic inequalities and $L^p(\mathbb{R}^n)$ - $L^q(\mathbb{R}^n)$ - $L^r(\mathbb{R}^n)$ estimates, which will be employed in the proofs of pointwise estimates and decay estimates.

Let $\mathcal{X}^m$ be the totality of $m \times m$ real matrices, $\langle \cdot, \cdot \rangle$ be the standard inner product in $\mathbb{C}^m$. We introduce the operator norm of $A \in \mathcal{X}^m$ by

$$|A| = \sup_{\substack{\psi \in \mathbb{C}^m \\ \psi \neq 0}} \frac{|A\psi|}{|\psi|}.$$

Let $\mathcal{S}^m$ be the totality of $m \times m$ real symmetric matrices. For $A \in \mathcal{S}^m$, $A \geq O$ indicates that A is non-negative definite. Also, $A \geq B$ or $B \leq A$ means $A - B$ is non-negative definite. If $A \in \mathcal{S}^m$ and $A \geq O$, then for $\psi \in \mathbb{C}^m$, it holds

$$|A\psi|^2 \leq |A| \langle A\psi, \psi \rangle.$$

For $A(t)\in\mathcal{X}^m$ and $\psi(t)\in\mathbb{C}^m$, we define the convolution

$$(A*\psi)(t)=\int_0^t A(t-\tau)\psi(\tau)\,\mathrm{d}\tau.$$

Furthermore, we introduce related operators for the quadratic form of convolution.

Definition 3.1 For any $\psi(t),\zeta(t)\in\mathbb{C}^m$, define

$$(A\lozenge\psi)(t)=\int_0^t A(t-\tau)\big(\psi(t)-\psi(\tau)\big)\mathrm{d}\tau,$$

$$A[\psi,\zeta](t)=\int_0^t\Big\langle A(t-\tau)\big(\psi(t)-\psi(\tau)\big),\big(\zeta(t)-\zeta(\tau)\big)\Big\rangle\mathrm{d}\tau.$$

From Definition 3.1, we obtain the following relationships involving convolution:

$$A*\psi=\mathcal{A}\psi-A\lozenge\psi, \tag{3.3}$$

where $\mathcal{A}(t)=\int_0^t A(s)\mathrm{d}s$. Taking the derivative with respect to t gives

$$(A*\phi)_t=A(0)\psi+\dot{A}*\psi=A\psi-\dot{A}\lozenge\psi,\ (A*\phi)_{tt}=A(0)\psi_t+\big(\dot{A}*\psi\big)_t \tag{3.4}$$

where $\dot{A}=\dfrac{\mathrm{d}A(t)}{\mathrm{d}t}$.

The following lemmas provide inequalities for controlling memory terms, with detailed proofs available in [1].

Lemma 3.1 Let $\psi(t)\in\mathcal{C}^m$, and assume conditions [A1] and [A4] hold. Then

$$|K_\omega\lozenge\psi|^2\le CK_\omega[\psi,\psi], \tag{3.5a}$$

$$|K_\omega*\psi|^2\le C\big(|\psi|^2+K_\omega[\psi,\psi]\big), \tag{3.5b}$$

$$\left|(K_\omega*\psi)_t\right|^2\le C\big(K_\omega[\psi,\psi]+K_\omega[\psi,\psi]\big), \tag{3.5c}$$

$$\left|(\dot{K}_\omega*\psi)_t\right|^2\le C\big(K_\omega[\psi,\psi]+K_\omega[\psi,\psi]\big). \tag{3.5d}$$

The following part introduces the $L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ estimates, which play a crucial role in proving the main theorem 2.1.

Lemma 3.2 ($L^p(\mathbb{R}^n)$-$L^q(\mathbb{R}^n)$-$L^r(\mathbb{R}^n)$ estimates [9]) Let $\eta(\xi)$ be a positive, continuous and real-valued function in $\mathbb{R}^n$ satisfying

$$\eta(\xi)\sim\begin{cases}|\xi|^{\sigma_1}, & \text{as } |\xi|\to 0;\\ |\xi|^{-\sigma_2}, & \text{as } |\xi|\to\infty;\end{cases}$$

for $\sigma_1,\sigma_2>0$. For $\phi\in\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[|\xi|^k\mathrm{e}^{-\eta(\xi)t}\left|\hat{\phi}(\xi)\right|\right]\right\|_{L^p}\\ &\lesssim(1+t)^{-\gamma_{\sigma_1}(q,p)-\frac{k-j}{\sigma_1}}\left\|\partial_x^j\phi\right\|_{L^q}+(1+t)^{-\frac{l}{\sigma_2}+\gamma_{\sigma_2}(r,p)}\left\|\partial_x^{k+l}\phi\right\|_{L^r},\end{aligned} \tag{3.6}$$

for $l>n\left(\frac{1}{r}-\frac{1}{p}\right),1\le q,r\le 2\le p\le\infty,0\le j\le k$, where $\gamma_\sigma(q,p)=\frac{n}{\sigma}\left(\frac{1}{q}-\frac{1}{p}\right)$ $(\sigma>0)$ and when $p=r=2$, $l\ge 0$.

Finally, this section introduces several inequalities for dealing nonlinear terms.

Lemma 3.3 (Gagliardo-Nirenberg inequality [12]) Let $1 \le q, r \le \infty$, k be a positive integer, $u \in L^q(\mathbb{R}^n)$, and $D^k u \in L^r(\mathbb{R}^n)$. Then $D^j u$ $(0 \le j \le k)$ satisfies

$$\left\|D^j u\right\|_{L^p} \le C\left\|D^k u\right\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{3.7}$$

where $\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{k}{n}\right) + (1-\alpha)\frac{1}{q}$, with $\frac{j}{k} \le \alpha \le 1$. The constant C depends on n, k, j, q, r, α.

Lemma 3.4 [4] [13] Let $1 \le p, q, r \le \infty$ and $1/p = 1/q + 1/r$. Then we have

$$\left\|\partial_x^l(uv)\right\|_{L^p} \le C\left(\|u\|_{L^q}\left\|\partial_x^l v\right\|_{L^r} + \|v\|_{L^q}\left\|\partial_x^l u\right\|_{L^r}\right) \ (l \ge 0), \tag{3.8}$$

$$\left\|\left[\partial_x^l, u\right]\partial_x v\right\|_{L^p} \le C\left(\left\|\partial_x u\right\|_{L^q}\left\|\partial_x^l v\right\|_{L^r} + \left\|\partial_x v\right\|_{L^q}\left\|\partial_x^l u\right\|_{L^r}\right) \ (l \ge 1), \tag{3.9}$$

where $\left[\partial_x^l, u\right]v := \partial_x^l(uv) - u\partial_x^l v$ is a commutator.

Lemma 3.5 [4] [13] Let $f(u)$ be smooth function of u, and there exists a positive constant $M \ge 0$ such that $\|u\|_{L^\infty} \le M$. Then

$$\left\|\partial_x^l f(u)\right\|_{L^p} \le C(M)\left(1 + \|u\|_{L^\infty}\right)^{l-1}\left\|\partial_x^l u\right\|_{L^p} \tag{3.10}$$

for $1 \le p \le \infty$ and $l \ge 1$, where $C(M)$ is a constant depending on M.

4. The proof of Theorem 2.1

This section first employs the energy method in Fourier space to establish pointwise estimates for solutions of the nonlinear problem (1.1)-(1.2). Then, L^p - L^q - L^r estimates are used to prove the optimal L^1 - L^2 type decay estimates of solutions. The proof is divided into two parts.

4.1. The Energy Estimates in Fourier Space

We first rewrite (1.1) as the following form. Assuming $\phi(0) = 0, b^j(0) = 0$ $(j = 1, 2, 3)$, by Taylor's theorem, we have

$$u_{tt} - \sum_{j,k=1}^{3} B^{jk}(0)u_{x_j x_k} + (1-\Delta)^{-1}\left(\sum_{j,k=1}^{3} K^{jk} * u_{x_j x_k} + Lu_t\right) = \sum_{j=1}^{3} g^j(\partial_x u)_{x_j} \triangleq \partial_x g, \tag{4.11}$$

where $g^j(\partial_x u) = b^j(\partial_x u) - \sum_{k=1}^{3} B^{jk}(0)u_{x_k} = O\left(\left|\partial_x u\right|^2\right)$. Applying the Fourier transform, we obtain

$$\hat{u}_{tt} + |\xi|^2 B_\omega(0)\hat{u} - |\xi|^2\langle\xi\rangle^{-2}\left(K_\omega * \hat{u}\right) + \langle\xi\rangle^{-2} L\hat{u}_t = \xi \cdot \hat{g}. \tag{4.12}$$

$$\hat{u}(\xi, 0) = \hat{u}_0(\xi), \quad \hat{u}_t(\xi, 0) = \hat{u}_1(\xi). \tag{4.13}$$

where $\xi \in \mathbb{R}^3, \langle\xi\rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$.

Dharmawardane [1] and others have studied pointwise estimates for viscoelastic linear systems using the energy method in Fourier space. Building upon this approach, this paper will subsequently estimate the nonlinear terms and finally apply Gronwall's inequality to obtain pointwise estimates for solutions of the nonlinear system.

Proposition 4.1 Assuming that conditions [A1] - [A4] hold, the solution to (4.12) - (4.13) satisfies

$$\begin{aligned}&\left|\hat{u}(\xi,t)\right|^2+|\xi|^2\left|\hat{u}(\xi,t)\right|^2\\&\le C\mathrm{e}^{-c\rho(\xi)t}\left(\left|\hat{u}_1(\xi)\right|^2+|\xi|^2\left|\hat{u}_0(\xi)\right|^2\right)+C\int_0^t \mathrm{e}^{-c\rho(\xi)(t-\tau)}\tilde{G}(\tau)\,\mathrm{d}\tau,\end{aligned}\tag{4.15}$$

for $t\ge 0,\xi\in\mathbb{R}^3$, where $\rho(\xi)=|\xi|^2\Big/\left(1+|\xi|^2\right)^2$, and C,c are positive constants.

Proof. First, we construct the Lyapunov functional E for equation (4.12), which is equivalent to

$$E_0=\left|\hat{u}_t\right|^2+|\xi|^2\left|\hat{u}\right|^2+\langle\xi\rangle^{-2}|\xi|^2K_\omega[\hat{u},\hat{u}],$$

where $K_\omega[\hat{u},\hat{u}]$ is defined as in 3.1. The proof is similar to the pointwise estimate proof in Dharmawardane [1], with the distinction that this paper includes an additional treatment of the nonlinear terms.

Step 1: Performing the inner product of (4.12) with $\hat{u}_t$ and taking the real part to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_1+\langle\xi\rangle^{-2}F_1+\langle\xi\rangle^{-2}\langle L\hat{u}_t,\hat{u}_t\rangle=G_1,\tag{4.15}$$

where

$$E_1=\left|\hat{u}_t\right|^2+|\xi|^2\left\langle\left(B_\omega(0)-\mathcal{K}_\omega\right)\hat{u},\hat{u}\right\rangle+\langle\xi\rangle^{-2}|\xi|^2K_\omega[\hat{u},\hat{u}]+|\xi|^4\langle\xi\rangle^{-2}\langle\mathcal{K}_\omega\hat{u},\hat{u}\rangle,$$

$$F_1=\frac{1}{2}|\xi|^2\left(-\dot{K}_\omega[\hat{u},\hat{u}]+\langle K_\omega\hat{u},\hat{u}\rangle\right),$$

$$G_1=\mathrm{Re}\langle\xi\cdot\hat{g},\hat{u}_t\rangle.$$

By conditions [A2] and [A4], which leads to

$$\begin{aligned}&cE_0\le E_1\le CE_0,\ \ F_1\ge\frac{1}{2}|\xi|^2\left(c_0K_\omega[\hat{u},\hat{u}]+\langle K_\omega\hat{u},\hat{u}\rangle\right)\ge c|\xi|^2F_0,\\&G_1=\mathrm{Re}\langle\xi\cdot\hat{g},\hat{u}_t\rangle\le|\xi||\hat{g}||\hat{u}_t|.\end{aligned}$$

Here, $F_0=K_\omega[\hat{u},\hat{u}]+\langle K_\omega\hat{u},\hat{u}\rangle$.

Step 2: Performing the inner product of (4.12) with $-\left(K_\omega*\hat{u}\right)_t$ and taking the real part yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_2+\langle K_\omega(0)\hat{u}_t,\hat{u}_t\rangle=R_2+G_2,\tag{4.16}$$

where

$$E_2=|\xi|^2\langle\xi\rangle^{-2}\left|K_\omega*\hat{u}\right|^2-2\,\mathrm{Re}\left\langle\hat{u}_t,\left(K_\omega*\hat{u}\right)_t\right\rangle,$$

$$R_2=-\mathrm{Re}\left\langle\hat{u}_t,\left(\dot{K}_\omega*\hat{u}\right)_t\right\rangle+|\xi|^2\,\mathrm{Re}\left\langle B_\omega(0)\hat{u},\left(K_\omega*\hat{u}\right)_t\right\rangle+\langle\xi\rangle^{-2}\,\mathrm{Re}\left\langle L\hat{u}_t,\left(K_\omega*\hat{u}\right)_t\right\rangle,$$

$$G_2=\mathrm{Re}\left\langle\xi\cdot\hat{g},-\left(K_\omega*\hat{u}\right)_t\right\rangle.$$

For E_2, applying Lemma 3.1 and the condition that $K_\omega(t)$ is symmetric and positive semi-definite give

$$\begin{aligned}|E_2|&\le C\left|\hat{u}_t\right|^2+C\left(1+|\xi|^2\langle\xi\rangle^{-2}\right)\left(\left|\hat{u}\right|^2+K_\omega[\hat{u},\hat{u}]\right)\\&\le C\left|\hat{u}_t\right|^2+C\left(\left|\hat{u}\right|^2+K_\omega[\hat{u},\hat{u}]\right).\end{aligned}$$

For any real numbers $\epsilon, \delta > 0$, applying condition [A1], Young's inequality, and Lemma 3.1 to estimate R_2, then

$$\begin{aligned}|R_2| &\le \epsilon |\hat{u}_t|^2 + C_\epsilon \left(K_\omega[\hat{u},\hat{u}] + \langle K_\omega \hat{u},\hat{u}\rangle\right) + \delta |\xi|^2 |\hat{u}|^2 + C_\delta |\xi|^2 \left(K_\omega[\hat{u},\hat{u}] + \langle K_\omega \hat{u},\hat{u}\rangle\right)\\ &\quad + \epsilon \langle \xi\rangle^{-2} |\hat{u}_t|^2 + C_\epsilon \langle \xi\rangle^{-2}\left(K_\omega[\hat{u},\hat{u}] + \langle K_\omega \hat{u},\hat{u}\rangle\right)\\ &\le \epsilon\left(1+\langle\xi\rangle^{-2}\right)|\hat{u}_t|^2 + \delta|\xi|^2|\hat{u}|^2 + C_{\epsilon,\delta}\left(\langle\xi\rangle^2 + \langle\xi\rangle^{-2}\right)F_0,\end{aligned}$$

Similarly, estimating the nonlinear term G_2 gives

$$|G_2| \le \delta|\hat{g}|^2 + C_\delta|\xi|^2\left(K_\omega[\hat{u},\hat{u}] + \langle K_\omega\hat{u},\hat{u}\rangle\right) = \delta|\hat{g}|^2 + C_\delta|\xi|^2 F_0.$$

Step 3: Performing the inner product of (4.12) with $\hat{u}$ and taking the real part yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_3 + |\xi|^2\left\langle\left(B_\omega(0) - \mathcal{K}_\omega\right)\hat{u},\hat{u}\right\rangle + |\xi|^4\langle\xi\rangle^{-2}\langle\mathcal{K}_\omega\hat{u},\hat{u}\rangle = R_3 + G_3, \tag{4.17}$$

where

$$E_3 = \langle\xi\rangle^{-2}\langle L\hat{u},\hat{u}\rangle + 2\,\mathrm{Re}\langle\hat{u}_t,\hat{u}\rangle,$$

$$R_3 = |\hat{u}_t|^2 - |\xi|^2\langle\xi\rangle^{-2}\,\mathrm{Re}\langle K_\omega \Diamond \hat{u},\hat{u}\rangle,$$

$$G_3 = \mathrm{Re}\langle\xi\cdot\hat{g},\hat{u}\rangle.$$

Since L is positive definite, we get

$$\begin{aligned}|E_3| &\le \langle\xi\rangle^{-2}\left|\langle L\hat{u},\hat{u}\rangle\right| + \left|2\,\mathrm{Re}\langle\hat{u}_t,\hat{u}\rangle\right|\\ &\le C\left(|\hat{u}_t|^2 + |\hat{u}|^2\right).\end{aligned}$$

For any real number $\gamma > 0$, with the aid of Young's inequality and Lemma 3.1, we estimate R_3 and G_3 as follows:

$$|R_3| \le |\hat{u}_t|^2 + \gamma|\xi|^2\langle\xi\rangle^{-2}|\hat{u}|^2 + C_\gamma|\xi|^2\langle\xi\rangle^{-2}K_\omega[\hat{u},\hat{u}],$$

$$|G_3| \le |\xi||\hat{g}||\hat{u}| \le \gamma|\xi|^2|\hat{u}|^2 + C_\gamma|\hat{g}|^2.$$

Step 4: Let $\alpha > 0, \beta > 0$, and define $\rho(\xi) = |\xi|^2/\langle\xi\rangle^4$. By adding $\rho(\xi)$ (α (4.16) + β (4.17)) to (4.15), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E + F = R + G, \tag{4.18}$$

where

$$\begin{aligned}E &= E_1 + \rho(\xi)\left(\alpha E_2 + \beta E_3\right),\\ F &= \langle\xi\rangle^{-2}F_1 + \langle\xi\rangle^{-2}\langle L\hat{u}_t,\hat{u}_t\rangle + \rho(\xi)\alpha\langle K_\omega(0)\hat{u}_t,\hat{u}_t\rangle\\ &\quad + \rho(\xi)\beta\left[|\xi|^2\left\langle\left(B_\omega(0)-\mathcal{K}_\omega\right)\hat{u},\hat{u}\right\rangle + |\xi|^4\langle\xi\rangle^{-2}\langle\mathcal{K}_\omega\hat{u},\hat{u}\rangle\right],\\ R &= \rho(\xi)\left(\alpha R_2 + \beta R_3\right),\\ G &= G_1 + \rho(\xi)\left(\alpha G_2 + \beta G_3\right).\end{aligned}$$

Using the above estimates and conditions [A2] and [A3], we have the following inequalities for E and F:

$$cE_0 \le E \le CE_0 + C(\alpha+\beta)E_0,$$

$$F \ge \rho(\xi)\left(c\alpha|\hat{u}_t|^2 + c\beta|\xi|^2|\hat{u}|^2 + c\langle\xi\rangle^{-2}|\xi|^2 F_0\right).$$

Finally, estimating $|R+G|$ as

$$\begin{aligned}|R+G| &\le \rho(\xi)\left(\alpha|R_2|+\beta|R_3|\right)+G_1+\rho(\xi)\left(\alpha|G_2|+\beta|G_3|\right)\\ &\le \rho(\xi)\left\{(2\alpha\epsilon+2\beta)|\hat{u}_t|^2+(\alpha\delta+2\beta\gamma)|\xi|^2|\hat{u}|^2\right\}+(\alpha+\beta)C_{\epsilon,\delta,\gamma}\\ &\quad\times|\xi|^2\langle\xi\rangle^{-2}F_0+C_{\beta,\gamma}\left(\langle\xi\rangle^4+\alpha\delta+\beta\right)|\hat{g}|^2.\end{aligned}$$

By choosing appropriate $\epsilon,\beta,\gamma,\alpha$, we can ensure that

$$cE_0 \le E \le CE_0,\ F \ge c\rho(\xi)E_0,\ |R+G| \le \frac{1}{2}F+\tilde{G},$$

where $\tilde{G} = C_{\beta,\gamma}\left(\langle\xi\rangle^4+\alpha\delta+\beta\right)|\hat{g}|^2$. The parameters must be selected such that

$$2\alpha\epsilon+2\beta \le \frac{1}{2\alpha c},\ \alpha\delta+2\beta\gamma \le \frac{1}{2}\beta c,\ C_{\epsilon,\delta,\gamma}(\alpha+\beta) \le \frac{c}{2},\ 1-\alpha|\xi|^2\langle\xi\rangle^{-2} \ge 0.$$

This is achievable, for example, by taking $\epsilon=\gamma=\frac{c}{8}$, $\beta=\frac{\alpha c}{8}$, $\delta=\left(\frac{c}{4\sqrt{2}}\right)^2$, and choosing $0<\alpha\le\frac{1}{2}$ such that

$$C_{\epsilon,\delta,\gamma}\left(\alpha+\frac{\alpha c}{8}\right) \le \frac{c}{2}.$$

Combining with (4.18), gives the differential inequality

$$\frac{\mathrm{d}E}{\mathrm{d}t}+c\rho(\xi)E \le C\tilde{G},$$

where the nonlinear term $\tilde{G} \le C\left(|\hat{g}|^2+|\xi|^2|\hat{g}|^2+|\xi|^4|\hat{g}|^2\right)$. Applying Gronwall's inequality to the above equation gives

$$\begin{aligned}&E(\xi,t) \le \mathrm{e}^{-c\rho(\xi)t}E(\xi,0)+C\int_0^t \mathrm{e}^{-c\rho(\xi)(t-\tau)}\tilde{G}(\tau)\,\mathrm{d}\tau\\ &\Rightarrow E_0(\xi,t) \le C\mathrm{e}^{-c\rho(\xi)t}E_0(\xi,0)+C\int_0^t \mathrm{e}^{-c\rho(\xi)(t-\tau)}\tilde{G}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Finally, by the definition of E_0, the proposition 4.1 is proved.

4.2. Optimal Decay Rate

In the following part, we will calculate the minimal decay regularity of solutions to (1.1) - (1.2) based on Propositions 2.1 and 4.1. To this end, define new time-weighted energy functionals:

$$N(t) = \sup_{0\le\tau\le t}(1+\tau)^{\frac{3}{4}}\left\|(u_t,\partial_x u)(\tau)\right\|_{L^2}.$$

Proposition 4.2 Let $u(x,t)$ be the global classical solutions to (1.1) - (1.2), if $(u_1,\partial_x u_0)(t)\in L^1$, then

$$N(t) \lesssim \left\|(u_1,\partial_x u_0)\right\|_{H^6\cap L^1}+N(t)D_1(t)+N(t)^2. \tag{4.19}$$

Proof. From Proposition 4.1, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(\left|\hat{u}(\xi,t)\right|^2+|\xi|^2\left|\hat{u}(\xi,t)\right|^2\right)\mathrm{d}\xi\\
&\lesssim\int_{\mathbb{R}^3}\mathrm{e}^{-c\rho(\xi)t}\left(\left|\hat{u}_1(\xi)\right|^2+|\xi|^2\left|\hat{u}_0(\xi)\right|^2\right)\mathrm{d}\xi+\int_{\mathbb{R}^3}\int_0^t\mathrm{e}^{-c\rho(\xi)(t-\tau)}\tilde{G}(\tau)\,\mathrm{d}\tau\mathrm{d}\xi\\
&\triangleq I_1+I_2+I_3.
\end{aligned}
$$

The right-hand side of the inequality is divided into three parts to estimate. First,

$$
\begin{aligned}
I_1&=\int_{\mathbb{R}^3}\mathrm{e}^{-c\rho(\xi)t}\left|\hat{u}_1(\xi)\right|^2\mathrm{d}\xi\\
&\lesssim(1+t)^{-\frac{3}{2}}\left\|u_1\right\|_{L^1}^2+(1+t)^{-2}\left\|\partial_x^2u_1\right\|_{L^2}^2.
\end{aligned}
\tag{4.20}
$$

The inequality is obtained from the L^p - L^q - L^r estimate with $\sigma_1=\sigma_2=2$, $k=j=0$, $p=2$, $q=1$, $l=r=2$.

For I_2, with $\sigma_1=\sigma_2=2,k=j=1,p=2,q=1,l=r=2$, we obtain

$$
\begin{aligned}
I_2&=\int_{\mathbb{R}^3}\mathrm{e}^{-c\rho(\xi)t}|\xi|^2\left|\hat{u}_0(\xi)\right|^2\mathrm{d}\xi\\
&\lesssim(1+t)^{-\frac{3}{2}}\left\|\partial_xu_0\right\|_{L^1}^2+(1+t)^{-2}\left\|\partial_x^3u_0\right\|_{L^2}^2.
\end{aligned}
\tag{4.21}
$$

Regarding I_3, we further divide it into three parts:

$$
\begin{aligned}
I_3&=\int_{\mathbb{R}^3}\int_0^t\mathrm{e}^{-c\rho(\xi)(t-\tau)}\tilde{G}(\tau)\,\mathrm{d}\tau\mathrm{d}\xi\\
&\lesssim\int_{\mathbb{R}^3}\int_0^t\mathrm{e}^{-c\rho(\xi)(t-\tau)}\left(|\hat{g}|^2+|\xi|^2|\hat{g}|^2+|\xi|^4|\hat{g}|^2\right)\mathrm{d}\tau\mathrm{d}\xi\\
&\triangleq I_{31}+I_{32}+I_{33}.
\end{aligned}
$$

where I_{31} is split into low frequency $(|\xi|\le 1)$ and high frequency $(|\xi|\ge 1)$ parts, as follows:

$$
\begin{aligned}
I_{31}&=\int_0^t\int_{\mathbb{R}^3}\mathrm{e}^{-c\rho(\xi)(t-\tau)}|\hat{g}|^2\,\mathrm{d}\xi\mathrm{d}\tau\\
&=\int_0^t\left(\int_{|\xi|\le1}+\int_{|\xi|\ge1}\right)(\cdots)\,\mathrm{d}\xi\mathrm{d}\tau\\
&\triangleq I_{31L}+I_{31H}.
\end{aligned}
$$

For the low-frequency part, with $\sigma_1=\sigma_2=2,p=2,q=1,k=j=0$, we get

$$
\begin{aligned}
I_{31L}&\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\left\|g\right\|_{L^1}^2\mathrm{d}\tau\\
&\lesssim\int_0^t(1+t-\tau)^{-\frac{3}{2}}\left\|\partial_xu\right\|_{L^2}^4\mathrm{d}\tau\\
&\lesssim N(t)^4\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-3}\mathrm{d}\tau\\
&\lesssim N(t)^4(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{4.22}
$$

The second inequality in the above expression is derived from $g^j(\partial_xu)=O\left(|\partial_xu|^2\right)$. For the high-frequency part, we divide it into two parts as follows:

$$
I_{31H}=\left(\int_0^{t/2}+\int_{t/2}^t\right)(\cdots)\,\mathrm{d}\tau\triangleq I_{31H}^1+I_{31H}^2.
$$

For I_{31H}^1, taking $\sigma_1=\sigma_2=2,p=2,r=l=2,k=0$ yields

$$
\begin{aligned}
I_{31H}^1 &\lesssim \int_0^{t/2}(1+t-\tau)^{-2}\left\|\partial_x^2 g\right\|_{L^2}^2 \mathrm{d}\tau \\
&\lesssim \int_0^{t/2}(1+t-\tau)^{-2}\left\|\partial_x u\right\|_{L^\infty}^2\left\|\partial_x^3 u\right\|_{L^2}^2 \mathrm{d}\tau \\
&\lesssim E(t)^2 \sup_{0\le\tau\le\frac{t}{2}}\left\{(1+t-\tau)^{-2}(1+\tau)^{\frac{1}{2}}\right\}\int_0^t(1+\tau)^{-\frac{3}{2}}\left\|\partial_x^3 u(\tau)\right\|_{L^2}^2 \mathrm{d}\tau \\
&\lesssim (1+t)^{-\frac{3}{2}} E(t)^2 D_{-1}(t)^2 \\
&\lesssim (1+t)^{-\frac{3}{2}} E_0^2.
\end{aligned}
\tag{4.23}
$$

Regarding the nonlinear term, it follows from Lemmas 3.4 and 3.5 that

$$
\left\|\partial_x^{m+1} g\right\|_{L^2} \lesssim \left\|\partial_x u\right\|_{L^\infty}\left\|\partial_x^{m+2} u\right\|_{L^2},
$$

which ensures the second inequality. Applying the Gagliardo-Nirenberg inequality to the third inequality, we have

$$
\left\|\partial_x u\right\|_{L^\infty}^2 \lesssim E(t)^2(1+t)^{-1} \quad (s\ge 4),
$$

and the last inequality is based on Proposition 2.1, which states that

$$
E(t)^2 + D_{-1}(t)^2 \lesssim E_0^2,
$$

where

$$
D_{-1}(t)^2 = \int_0^t (1+\tau)^{-3/2}\left\|(u_t,\partial_x u)\right\|_{H^s}^2 \quad \text{with } (s\ge 2).
$$

For I_{31H}^2, taking $\sigma_1=\sigma_2=2, p=2, r=1, l=2, k=0$ gives

$$
\begin{aligned}
I_{31H}^2 &\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\left\|\partial_x^2 g\right\|_{L^1}^2 \mathrm{d}\tau \\
&\lesssim \int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\left\|\partial_x u\right\|_{L^2}^2\left\|\partial_x^3 u\right\|_{L^2}^2 \mathrm{d}\tau \\
&\lesssim N(t)^2 \sup_{\frac{t}{2}\le\tau\le t}\left\{(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-2}\right\}\int_0^t(1+\tau)^{\frac{1}{2}}\left\|\partial_x^3 u(\tau)\right\|_{L^2}^2 \mathrm{d}\tau \\
&\lesssim (1+t)^{-2} N(t)^2 D_1(t)^2,
\end{aligned}
\tag{4.24}
$$

where, with Lemmas 3.4 and 3.5 again it holds

$$
\left\|\partial_x^2 g\right\|_{L^1} \lesssim \left\|\partial_x u\right\|_{L^2}\left\|\partial_x^3 u\right\|_{L^2}.
$$

Here, $s\ge 4$ because

$$
D_1(t)^2 = \int_0^t \left\|\left(\partial_x^2 u_t, \partial_x^3 u\right)\right\|_{H^{s-4}}^2.
$$

For I_{32}, similarly to I_{31}, we estimate it by dividing into low and high frequencies:

$$
\begin{aligned}
I_{32} &= \int_0^t \int_{\mathbb{R}^3} \mathrm{e}^{-c\rho(\xi)(t-\tau)}|\xi|^2|\hat{g}|^2 \mathrm{d}\xi \mathrm{d}\tau \\
&= \int_0^t\left(\int_{|\xi|\le 1} + \int_{|\xi|\ge 1}\right)(\cdots)\mathrm{d}\xi\mathrm{d}\tau \\
&\triangleq I_{32L} + I_{32H}.
\end{aligned}
$$

For the low-frequency part I_{32L}, taking $\sigma_1=\sigma_2=2, p=2, k=1, j=0, q=1$, we get

$$
\begin{aligned}
I_{32L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|g\|_{L^1}^2 \,\mathrm{d}\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{5}{2}} \|\partial_x u(\tau)\|_{L^2}^4 \,\mathrm{d}\tau \\
&\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-3} \,\mathrm{d}\tau \\
&\lesssim N(t)^4 (1+t)^{-\frac{5}{2}}.
\end{aligned} \tag{4.25}
$$

For the high-frequency part, dividing it into two parts, namely,

$$
I_{32H} = \left(\int_0^{t/2} + \int_{t/2}^t\right)(\cdots)\,\mathrm{d}\tau \triangleq I_{32H}^1 + I_{32H}^2.
$$

For I_{32H}^1, taking $\sigma_1=\sigma_2=2, p=2, r=l=2, k=1$ yields

$$
\begin{aligned}
I_{32H}^1 &\lesssim \int_0^{t/2} (1+t-\tau)^{-2} \|\partial_x^3 g\|_{L^2}^2 \,\mathrm{d}\tau \\
&\lesssim \int_0^{t/2} (1+t-\tau)^{-2} \|\partial_x u\|_{L^\infty}^2 \|\partial_x^4 u\|_{L^2}^2 \,\mathrm{d}\tau \\
&\lesssim (1+t)^{-\frac{3}{2}} E(t)^2 D_{-1}(t)^2 \\
&\lesssim (1+t)^{-\frac{3}{2}} E_0^2,
\end{aligned} \tag{4.26}
$$

where, it holds by Lemmas 3.4 and 3.5

$$
\|\partial_x^3 g\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\partial_x^4 u\|_{L^2}.
$$

For the estimate of I_{32H}^2, taking $\sigma_1=\sigma_2=2, p=2, r=1, l=2, k=1$ gives

$$
\begin{aligned}
I_{32H}^2 &\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^3 g\|_{L^1}^2 \,\mathrm{d}\tau \\
&\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x u\|_{L^2}^2 \|\partial_x^4 u\|_{L^2}^2 \,\mathrm{d}\tau \\
&\lesssim N(t)^2 \sup_{\frac{t}{2}\le\tau\le t} \left\{(1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-2}\right\} \int_0^t (1+\tau)^{\frac{1}{2}} \|\partial_x^4 u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\
&\lesssim (1+t)^{-2} N(t)^2 D_1(t)^2,
\end{aligned} \tag{4.27}
$$

where

$$
\|\partial_x^3 g\|_{L^1} \lesssim \|\partial_x u\|_{L^2} \|\partial_x^4 u\|_{L^2},
$$

and the last inequality holds when the L^2 norm of the forth derivative of u is controlled by $D_1(t)$, which requires regularity $s-4\ge 1$, i.e., $s\ge 5$. The calculation process for I_{33} is similar, namely,

$$
\begin{aligned}
I_{33} &= \int_0^t \int_{\mathbb{R}^3} \mathrm{e}^{-c\rho(\xi)(t-\tau)} |\xi|^4 |\hat{g}|^2 \,\mathrm{d}\xi \mathrm{d}\tau \\
&= \int_0^t \left\| \mathrm{e}^{-c\rho(\xi)(t-\tau)} |\xi|^2 |\hat{g}| \right\|_{L^2(\mathbb{R}_\xi^3)} \mathrm{d}\tau \\
&\triangleq I_{33L} + I_{33H}.
\end{aligned}
$$

For I_{33L}, similarly to I_{31L}, taking $\sigma_1 = \sigma_2 = 2, p = 2, q = 1, k = 2, j = 0$ gives

$$\begin{aligned} I_{33L} &\lesssim \int_0^t (1+t-\tau)^{-\frac{7}{2}} \|g\|_{L^1}^2 \,\mathrm{d}\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\frac{7}{2}} \|\partial_x u(\tau)\|_{L^2}^4 \,\mathrm{d}\tau \\ &\lesssim N(t)^4 \int_0^t (1+t-\tau)^{-\frac{7}{2}} (1+\tau)^{-3} \,\mathrm{d}\tau \\ &\lesssim N(t)^4 (1+t)^{-3}. \end{aligned} \tag{4.28}$$

For the high-frequency part I_{33H}, we divide it into two parts, namely,

$$I_{33H} = \left(\int_0^{t/2} + \int_{t/2}^t \right)(\cdots)\,\mathrm{d}\tau \triangleq I_{33H}^1 + I_{33H}^2.$$

For the first part I_{33H}^1, taking $\sigma_1 = \sigma_2 = 2, p = 2, r = l = 2, k = 2$, we obtain

$$\begin{aligned} I_{33H}^1 &\lesssim \int_0^{t/2} (1+t-\tau)^{-2} \|\partial_x^4 g\|_{L^2}^2 \,\mathrm{d}\tau \\ &\lesssim \int_0^{t/2} (1+t-\tau)^{-2} \|\partial_x u\|_{L^\infty}^2 \|\partial_x^5 u\|_{L^2}^2 \,\mathrm{d}\tau \\ &\lesssim (1+t)^{-\frac{3}{2}} E(t)^2 D_{-1}(t)^2 \\ &\lesssim (1+t)^{-\frac{3}{2}} E_0^2, \end{aligned} \tag{4.29}$$

where

$$\|\partial_x^4 g\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\partial_x^5 u\|_{L^2}.$$

For the second part I_{33H}^2, taking $\sigma_1 = \sigma_2 = 2, p = 2, r = 1, l = k = 2$, we are led to

$$\begin{aligned} I_{33H}^2 &\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x^4 g\|_{L^1}^2 \,\mathrm{d}\tau \\ &\lesssim \int_{t/2}^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_x u\|_{L^2}^2 \|\partial_x^5 u\|_{L^2}^2 \,\mathrm{d}\tau \\ &\lesssim N(t)^2 \sup_{\frac{t}{2}\le\tau\le t} \left\{ (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-2} \right\} \int_0^t (1+\tau)^{\frac{1}{2}} \|\partial_x^5 u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\ &\lesssim (1+t)^{-2} N(t)^2 D_1(t)^2, \end{aligned} \tag{4.30}$$

where

$$D_1(t)^2 = \int_0^t (1+\tau)^{1/2} \left\|\left(\partial_x^2 u_t, \partial_x^3 u\right)(\tau)\right\|_{H^{s-4}}^2 \,\mathrm{d}\tau,$$

and the last inequality holds when the L^2 norm of $\partial_x^5 u$ is controlled by $D_1(t)$, which requires regularity $s-4 \ge 2$, i.e., $s \ge 6$.

Therefore, combining the above inequalities (4.20) - (4.30) and taking $s = 6$, we obtain

$$\begin{aligned} \|(u_t, \partial_x u)(t)\|_{L^2}^2 \lesssim{}& (1+t)^{-\frac{3}{2}} \|(u_1, \partial_x u_0)\|_{H^6}^2 + (1+t)^{-\frac{3}{2}} N(t)^4 \\ &+ (1+t)^{-\frac{3}{2}} N(t)^2 D_1(t)^2, \end{aligned}$$

where $E_0 \le E_1$, then

$$N(t) \lesssim \|(u_1, \partial_x u_0)\|_{L^1 \cap H^6} + N(t)^2 + N(t) D_1(t).$$

□

When $\left\|\left(u_1,\partial_x u_0\right)\right\|_{L^1\cap H^6}$ is sufficiently small, according to inequality (2.1), the dissipative norm $D_1(t)\lesssim E_0\lesssim\left\|\left(u_1,\partial_x u_0\right)\right\|_{L^1\cap H^6}$, hence

$$N(t)\lesssim\left\|\left(u_1,\partial_x u_0\right)\right\|_{L^1\cap H^6}+N(t)^2\Rightarrow N(t)\lesssim E_1.$$

Consequently, it holds

$$\left\|\left(u_t,\partial_x u\right)(t)\right\|_{L^2}\lesssim(1+t)^{-\frac{3}{4}}\left\|\left(u_1,\partial_x u_0\right)\right\|_{H^6\cap L^1}.$$

The proof of Theorem 2.1 is completed.

We reduce the regularity requirements for the initial data when establishing the optimal decay estimate of solutions to the Cauchy problem in viscoelastic hyperbolic dissipative systems. Since it is often difficult for initial data to meet the necessary smoothness in the real world, lowering the regularity of the initial values implies that the initial conditions are more in line with actual situations. Furthermore, this aids in a better understanding of the dynamic behavior of physical systems, particularly when dealing with complex or imperfect initial conditions.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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