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The CMB Temperature Is Simply the Geometric Mean: $T_{cmb} = \sqrt{T_{min} T_{max}}$ of the Minimum and Maximum Temperature in the Hubble Sphere

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Abstract

In the Hubble sphere, we assume that the wavelength of pure energy spreads out in all directions. The maximum wavelength in the Hubble sphere is then the circumference of the Hubble sphere. We assume the minimum wavelength occurs in a Planck mass black hole, which is given by $4\pi R_{s,p} = 8\pi l_p$. Here, we build further on the geometric mean CMB approach by Haug and Tatum and based on new analysis given in this paper[1] we conclude that the CMB temperature is simply given as: $T_{cmb} = \sqrt{T_{min} T_{max}}$, which is the geometric mean of the minimum and maximum physically possible temperatures in the Hubble sphere. This again means the CMB temperature simply is the geometric mean of the Hawking temperature of the Hubble sphere (in black hole cosmology) and the Hawking temperature of the Planck mass black hole, se we have also $T_{cmb} = \sqrt{T_{Haw,H} T_{Haw,p}}$.

Keywords

CMB Temperature, Geometric Mean Temperature, Minimum Temperature, Maximum Temperature, Hubble Sphere, Cosmology

1. Introduction

The Λ-CDM model, despite its success in many areas, is not able to predict the CMB temperature today. See, for example, Narlikar and Padmanabhan [2], which

[1]This paper is a strongly improved version of our preprint [1].

states:

"*The textit The present theory is, however, unable to predict the value of* T *at* $t = t_0$. *It is therefore a free parameter in SC* (*Standard Cosmology*)."

The CMB temperature is likely the most precisely measured cosmological parameter [3]-[5], but it is clearly not fully understood within Λ-CDM cosmology. In recent years, however, there has been a breakthrough in understanding the CMB temperature and its connection to the Hubble parameter, which we will soon revisit.

We will be operating within a black hole $R_{H_t} = ct$ cosmology. Although black hole cosmology is much less well-known than Λ-CDM, it is not new; it dates back at least to 1972 with a paper by Pathria (1972) [6]. The topic continues to be actively discussed by various researchers to this day [7]-[16].

There are also multiple variations of $R_{H_t} = ct$ cosmologies, all of which share the common feature that the universe has expanded—or is at least related to—he speed of light; see [17]-[22]. The Melia $R_H = ct$ model is the best known among these, and he has done a tremendous job demonstrating that $R_{H_t} = ct$ cosmology can perform at least as well as, and often better than, the Λ-CDM model. However, in this work, we will focus specifically on black hole $R_{H_t} = ct$ cosmology, as described by Haug and Tatum [23], which is a subcategory within $R_{H_t} = ct$ cosmologies.

2. The CMB Temperature as a Geometric Mean of the Minimum and Maximum Temperature in the Hubble Sphere

The geometric mean plays an important role in thermodynamics and in other areas of physics [24]-[27]. For example, the optimal reheating pressure is given as the geometric mean of the maximum and minimum pressure: $P_{reheating} = \sqrt{P_{max}P_{min}}$, and the optimal intercooling in an ideal two-stage compressor is also given by the geometric mean pressure, $P_{intercooling} = \sqrt{P_{max}P_{min}}$, see [28]. The geometric mean temperature: $T_{gm} = \sqrt{T_{hot}T_{cold}}$ play a central role in Carnot engines where it defines a type of equilibrium, see [29]-[31]. That geometric means could also potentially play an important role in the thermodynamics of cosmic black hole universe temperatures should not come as a surprise.

Haug and Tatum [32] have recently shown that the CMB temperature, at a deeper physical level, is likely linked to the geometric mean of the shortest and longest possible wavelengths in the Hubble sphere. They presented their formula as:

$$T_{cmb} = \hbar \frac{c}{\sqrt{\bar{\lambda}_{min}\bar{\lambda}_{max}}} \frac{1}{4\pi k_b} = \hbar \frac{c}{\bar{\lambda}_{gm}} \frac{1}{4\pi k_b} \tag{1}$$

where k_b is the Boltzmann constant and $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant (the Dirac constant). They assumed the shortest wavelength $\bar{\lambda}_{min} = l_p = \sqrt{\frac{G\hbar}{c^3}}$ was the Planck [33] [34] length and the maximum wavelength was the diameter of the Hubble sphere $\bar{\lambda}_{max} = 2R_H$, they mention also the circumference could be the limiting factor. Their geometric mean wavelength is $\bar{\lambda}_{gm} = \sqrt{\bar{\lambda}_{min}\bar{\lambda}_{max}}$. This, again, they demonstrate to be consistent with the CMB formula heuristically first suggested by Tatum, Seshavatharam and Lakshminarayana [35]. Haug and Wojnow [36] [37] have further demonstrated the CMB formula fully consistent with this can be derived from the Stefan-Boltzmann law. The Stefan-Boltzmann [38] [39] law holds for a perfect black body and the CMB is the closest we likely get to a perfect black body in the real world as stated by for example Muller *et al.* [40]:

"*Observations with the COBE satellite have demonstrated that the CMB corresponds to a nearly perfect black body characterized by a temperature* T_0 *at* $z = 0$, *which is measured with very high accuracy,* $T_0 = 2.72548 \pm 0.00057$ K."

Haug [41] has recently expanded on the geometric mean approach of Haug and Tatum and shown that the CMB formula can even be written directly in the form:

$$T_{cmb} = \frac{\sqrt{T_{max}T_{min}}}{4\pi} = \frac{\sqrt{T_p T_{min}}}{4\pi} \tag{2}$$

where he suggested $T_{max} = T_p = \frac{1}{k_b} = \sqrt{\frac{\hbar c^5}{G}} = \frac{m_p c^2}{k_b} = \frac{E_p}{k_b}$, which is the Planck [33] [34] temperature, and $T_{min} = \hbar \frac{c}{2R_H}\frac{1}{k_b}$. Furthermore k_b is the Boltzmann constant.

The question is: why does the geometric mean temperature have to be multiplied by the constant $\frac{1}{4\pi}$? We now think we have an answer to that. Electromagnetic waves (such as CMB radiation), as well as matter waves, tend to spread out in all directions—like throwing a stone into a lake, where the waves propagate in all directions. If a Planck-mass Schwarzschild black hole is the smallest black hole and the Hubble sphere is a cosmic black hole, then the maximum wavelength is the circumference of the Hubble sphere, $\bar{\lambda}_{max} = 4\pi R_H$, and the minimum wavelength is the circumference of a Schwarzschild Planck-mass black hole, which is $\bar{\lambda}_{min} = 4\pi R_{s,p} = 8\pi l_p$.

The minimum and maximum energies and temperatures in the Hubble sphere are then based on the Planck—Einstein relation:

$$E_{min} = \hbar \frac{c}{4\pi R_{H_t}}, \quad T_{min} = \frac{E_{min}}{k_b} \tag{3}$$

and the maximum energy and maximum temperature is then

$$E_{max} = \hbar \frac{c}{4\pi R_{s,p}} = \hbar \frac{c}{8\pi l_p}, \quad T_{max} = \frac{E_{max}}{k_b} \tag{4}$$

The CMB temperature now is then given by:

$$T_{cmb} = \sqrt{T_{max}T_{min}} \approx 2.725 \text{ K} \tag{5}$$

While the maximum temperature is always constant, $T_{max}=\hbar\frac{c}{8\pi l_p}\frac{1}{k_b}$, the minimum temperature varies as we travel along the cosmic epoch as we assume $R_{H_t}=ct$. It is worth mention that the minimum temperature now always is equal to the Hawking [42] temperature $T_{min}=\frac{\hbar c}{4\pi R_H}=T_{Haw}=\frac{\hbar c}{4\pi R_s}$ when the Hubble radius is equal to the Schwarzschild radius $R_{H_t}=R_{s,t}=ct$ as it will be in a black hole Hubble universe where the equivalent mass is the critical Friedmann mass: $M_{BH}=M_{cr}=\frac{c^2 R_{H_t}}{2G}$.

This means the CMB temperature also can be seen as simply the geometric mean of the Hawking temperature of the Hubble sphere and a Planck mass black hole. The Hawking temperature of the Hubble sphere is given as:

$$T_{min}=T_{Haw,H}=\frac{\hbar c^3}{k_b 8\pi G M_{BH}}=\hbar\frac{c}{4\pi R_s}\frac{1}{k_b}=\hbar\frac{c}{4\pi R_H}\frac{1}{k_b} \tag{6}$$

We here assume that the relevant mass of the Hubble sphere in relation to the CMB temperature is the critical Friedmann [43] mass, $M_{BH}=M_{cr}=\frac{c^2 R_H}{2G}$. If we solve for the Hubble radius in terms of the critical Friedmann mass, we get $R_H=\frac{2GM_{cr}}{c^2}$, and we can see that it must be identical to the Schwarzschild radius of a black hole with mass equal to the critical Friedmann mass: $R_s=\frac{2GM_{BH}}{c^2}$. This is not a new result, but it is important for understanding why we can apply the Hawking temperature to a black hole Hubble sphere universe.

In addition, we have the Hawking—Planck temperature, which is the Hawking temperature of a Schwarzschild black hole, given by:

$$T_{max}=T_{Haw,p}=\hbar\frac{c}{4\pi R_{s,p}}\frac{1}{k_b}=\hbar\frac{c}{4\pi 2 l_p}\frac{1}{k_b}=\hbar\frac{c}{8\pi l_p}\frac{1}{k_b} \tag{7}$$

This is the maximum possible temperature in the black hole, related to a Planck mass black hole that again likely could be a Planck mass particle linked to quantum gravity.

The CMB temperature is then given by:

$$T_{cmb}=\sqrt{T_{max}T_{min}}=\sqrt{T_{Haw,H}T_{Haw,p}}\approx 2.725\text{ K} \tag{8}$$

Based on a $H_0\approx 66.9$ km/s/Mpc as reported by Haug and Tatum [22]. We further assume the Hubble sphere and the CMB temperature follows the $R_{H_t}=ct$ cosmology, where the circumference of the black hole Hubble sphere was smaller in the past.

Alternatively we can express the CMB temperature from energies:

$$T_{cmb}=\sqrt{E_{max}E_{min}}\frac{1}{k_b} \tag{9}$$

where $E_{max}=k_b T_{max}$ and $E_{min}=k_b T_{min}$.

The maximum energy is naturally much smaller than the energy in the Hubble sphere. This can be seen as the maximum possible energy from a single particle or photon (or perhaps even a graviton), which we conjecture is linked to the Planck scale and is actually a Schwarzschild Planck mass black hole (Planck mass particle). It is common for researchers working on quantum gravity to assume that the Planck scale will play an important role; see, for example [44]-[47]. Therefore, it should not be a big surprise that the Planck scale also plays an important role in the CMB temperature.

It is naturally remarkable that, based on recent years of research on the CMB temperature, we can now accurately predict the CMB temperature today—something the Λ-CDM model has not been able to do and still cannot, as it is not compatible with $R_{H_t} = ct$ black hole cosmology. Even more important than predicting the CMB temperature is the fact that the approach developed in recent years has found the exact mathematical relationship between the CMB temperature and the Hubble parameter. Tatum, Haug and Wojnow [48], as well as Haug and Tatum [22], have recently demonstrated that one can predict the Hubble parameter much more precisely than with other methods. This is possible because the CMB temperature can be used to determine the Hubble constant due to these new exact mathematical relationships.

Based on the geometric mean approach above we get the following formula:

$$H_0 = \frac{T_{cmb,0}^2}{T_{max}} \frac{k_b 4\pi}{\hbar} = \frac{T_{cmb,0}^2}{T_{Haw,p}} \frac{k_b 4\pi}{\hbar} = 66.8943 \pm 0.0287 \text{ km/s/Mpc} \tag{10}$$

we have used the Fixsen [49] measured CMB temperature now (at $z = 0$): $T_0 = 2.72548 \pm 0.00057$ K . This is as expected in line with the research just mentioned above.

The most plausible reason the CMB temperature is simply the geometric mean of the lowest and highest possible Hawking temperatures in the Hubble sphere is likely that the Hubble sphere operates as a Carnot [50] engine. The idea that the universe could be a Carnot engine is not new and has been suggested for the Friedmann-Robertson-Walker (FRW) metric-based universe by Debnath [51] [52]. Despite Debnath's highly interesting paper, he did not derive a CMB prediction formula for the current temperature and is it even possible inside the FRW model? To our knowledge, this is the very first paper to predict that $T_{cmb} = \sqrt{T_{max} T_{min}} = \sqrt{T_{Haw,H} T_{Haw,p}}$, which is likely the most elegant way to express the CMB temperature both mathematically and intuitively. Anyone deeply knowledgeable in Carnot theory (heat theory) will recognize it simply as the geometric mean temperature required for an ideal Carnot engine.

We plan to follow up on our findings with an in-depth analysis of why the Hubble sphere is likely a black hole Carnot engine, probably governed by the extremal solution of Reissner-Nordström [53] [54], Kerr [55], or the Haug-Spavieri [56] metric rather than the FRW metric. However, we consider the findings in this paper so important that making them publicly available at this stage could benefit the research community, despite the deeper mathematical foundation will first be laid out in future papers. For considerably deeper mathematical evidence linked to Carnot engine theory, see our follow-up working paper [57].

3. The Link to Cosmological Redshift

It is well known from observations that the CMB temperature in relation to cosmological redshift is given by $T_{cmb,t} = T_{cmb,0}(1+z)$, where z is the cosmological redshift, see [58]-[60]. This means to be consistent with this we must have:

$$T_{cmb,t} = \sqrt{T_{max}T_{min,t}} = \sqrt{T_{max}T_{min,0}}(1+z) \tag{11}$$

and since $T_{max} = T_{Haw,max} = \frac{\hbar c}{k_b 8\pi l_p}$ and $T_{min,t} = T_{Haw,min,t} = \frac{\hbar c}{k_b 4\pi R_{H_t}}$ so we can also write this as:

$$T_{cmb.t} = \sqrt{T_{Haw,max}T_{Haw,min,t}} = \sqrt{T_{Haw,max}T_{Haw,min,0}}(1+z) \tag{12}$$

and this means we also must have:

$$z = \frac{T_{cmb,t}}{\sqrt{T_{max}T_{min}}} - 1 = \frac{T_{cmb,t}}{\sqrt{T_{Haw,max}T_{Haw,min,0}}} - 1 = \sqrt{\frac{R_{H_0}}{R_{H_t}}} - 1 \tag{13}$$

which is fully consistent with the cosmological redshift findings of Haug and Tatum [61]. Interestingly, this also leads to the conclusion that the Hawking temperature of the whole Hubble sphere in relation to cosmological redshift is found by solving the following equation for $T_{Haw,min,0}$, which is the Hawking temperature for the whole Hubble sphere:

$$T_{Haw,H_0} = T_{Haw,min,0} = \frac{T_{cmb.t}^2}{T_{Haw,max,0}(1+z)^2} \tag{14}$$

where, again, $T_{Haw,max,0} = \frac{\hbar c}{k_b 8\pi l_p}$. Since we know from observations that

$T_{cmb,t} = T_{cmb,0}(1+z)$, we must have:

$$T_{Haw,H_0} = \frac{T_{cmb,0}^2(1+z)^2}{T_{Haw,max,0}(1+z)^2} = \frac{T_{cmb,0}^2}{T_{Haw,max,0}} \tag{15}$$

which is naturally consistent with our main findings that $T_{cmb} = \sqrt{T_{Haw,max}T_{Haw,min}}$. Equation (15) also demonstrates that the Hawking temperature now for the hole Hubble sphere is naturally independent of cosmological redshift.

4. Comparison to the Λ-CDM model

The Λ-CDM model cannot predict the CMB temperature now, despite the CMB temperature being the most precisely measured cosmological parameter. However, using the geometric mean approach, we can predict the CMB temperature now. Furthermore, this also link the CMB temperature to the Hubble constant in a veruy powerful way. If we know the CMB temperature we know the Hubble parameter now, or vice versa.

Haug [62] has recently also demonstrated that the CMB radiation density parameter consistent with this way to express the CMB temperature is given by $\Omega_\gamma = \frac{\rho_\gamma}{\rho_{cr}} = \frac{1}{5760\pi} \approx 5.52621330180192\times10^{-5}$, which is well within the 95% confidence interval of the Particle Data Group (PDG)[2]: 5.08×10^{-5} to 5.68×10^{-5}. Our model predicts an exact CMB temperature density parameter.

Remarkably, the Hubble tension also seems to be resolved when applying this CMB temperature relation in addition to a cosmological redshift of $z = \sqrt{\frac{R_{H_0}}{R_{H_t}}} - 1$, as demonstrated by Haug and Tatum [22] [63]. They use a more complicated formula for the CMB that at a deeper level is the same as the one presented in this paper. They demonstrate that one obtains a basically perfect match between predicted and observed supernova data using the full distance ladder of observed SN Ia (the PhantonPlusSH0ES database) with only one H_0 value ($H_0 \approx 66.9$ km/s/Mpc).

It is important to understand that all of this is based on $R_{H_t} = ct$ cosmology. Melia [21] (see also the Melia references in section 1) has demonstrated that $R_{H_t} = ct$ cosmology is outperforming the Λ-CDM model on a long list of testable points. That said, the CMB temperature presented here is consistent with only a subclass of $R_{H_t} = ct$ models, namely black hole $R_{H_t} = ct$ cosmological models, so clearly, much more research is needed here, for example, also in relation to comparison studies with the interesting Melia $R_{H_t} = ct$ model.

The most plausible reason the CMB temperature is simply the geometric mean of the lowest and highest possible Hawking temperatures in the Hubble sphere is that the Hubble sphere likely operates as a Carnot [50] engine. The idea that the universe could be a Carnot engine is intriguing. We plan to follow up on our findings with an in-depth analysis of why the Hubble sphere is likely a black hole Carnot engine, possibly governed by the extremal solution of the Reissner-Nordström, Kerr, or Haug-Spavieri metric rather than the FRW metric. However, we believe that the findings in this paper—that the CMB temperature is simply the geometric mean of the maximum and minimum temperatures in the Hubble sphere given by the very elegant formula first presented in this paper: $T_{cmb} = \sqrt{T_{max}T_{min}} \approx 2.725\text{ K}$ —are so significant that making this result publicly available now could greatly benefit the black hole and cosmological research community. For a considerably deeper mathematical analysis linked to Carnot engine theory and black hole cosmology, see our follow-up working paper [57].

5. Conclusions

Based on years of research on the CMB temperature by several authors, we can now conclude that the CMB temperature, in its simplest and most understandable form, is simply the geometric mean of the minimum and maximum temperatures possible

[2]See https://pdg.lbl.gov/2023/reviews/rpp2023-rev-astrophysical-constants.pdf.

in the Hubble sphere. The CMB temperature is given by $T_{cmb}=\sqrt{T_{max}T_{min}}$. This also means that the CMB temperature is the geometric mean temperature of the Hawking Hubble temperature and the Hawking Planck temperature: $T_{cmb}=\sqrt{T_{max}T_{min}}=\sqrt{T_{Haw,H}T_{Haw,p}}\approx 2.725\text{ K}$.

This has important implications, as it provides a precise mathematical relationship between the CMB temperature and the Hubble parameters, as well as a deeper physical understanding of the CMB temperature. Unlike in the Λ-CDM model, we can now accurately predict the CMB temperature in black hole $R_{H_t}=ct$ cosmology. In addition, we can predict the Hubble parameter much more precisely, as recentyl demonstrated by Tatum, Haug and Wojnow [48] and Haug and Tatum [22].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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