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Exact Solutions and Hyers-Ulam Stability of the Nonhomogeneous Riemann-Liouville Fractional Oscillatory Differential Equations with Pure Delay

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Abstract

In this paper, to overcome certain difficulties encountered in previous studies, we first investigate a formula for the Laplace transform of the Riemann-Liouville derivative with a lower limit a. We then obtain the exact solutions of nonhomogeneous Riemann-Liouville fractional differential equations with pure delay. In addition, this paper explores the Hyers-Ulam stability of nonhomogeneous fractional oscillatory differential equations with pure delay.

Keywords

Riemann-Liouville Fractional Derivative, Hyers-Ulam Stability, Laplace Transform

1. Introduction and Preliminaries

There have been many studies on the well-posedness and Hyers-Ulam stability of PDE [1] [2]. In recent decades, the solutions of the Cauchy problem with delay have been studied by many authors [3] [4]. Many researchers have focused on the stability and controllability study of solutions of fractional Caputo differential equations with delay [5]-[11] and fewer researchers have studied fractional Riemann-Liouville differential equations with delay [12]-[15]. The stability theory of fractional order differential equations allows the solution to remain accurate to δ ($\delta = C\varepsilon$) in the presence of a perturbation in the right end term of the equation. For time-lagged systems, this stability avoids the amplifying effect of errors in the time-lag loop. For example, in control system design, it ensures that sensor noise does not trigger violent oscillations in the control commands.

Fan and Pan [16] gave the solutions and the finite time stability of Equation (1) by using the Mittag-Leffler matrix-type functions and the constant variational method in 2023. The authors stated that it was difficult to compute the solutions of system (1) directly by using the Laplace transform, because they didn't find a suitable formula for the Laplace transform of the Riemann-Liouville fractional derivative with the lower limit a. Motivated to the difficulty left over from the literature [16], we further study the nonhomogeneous Riemann-Liouville fractional oscillatory differential equations with pure delay of the following form

$$\begin{cases} {}^{R}D^{\alpha}_{-\tau^{+}} y(t) = \beta y(t-\tau) + f(t), & t \in J := (0,T], \tau > 0, \\ y(t) = \varphi(t), & -\tau < t \le 0, \\ J^{2-\alpha}_{-\tau^{+}} y(-\tau^{+}) = m, \ {}^{R}D^{\alpha-1}_{-\tau^{+}} y(-\tau^{+}) = n, \end{cases} \tag{1}$$

where $\alpha \in (1,2)$, $f: J \to \mathbb{R}^n$ is a continuous function, $T = \mathbf{C}\tau$ for a fixed $\mathbf{C} \in \{1,2,\cdots\}$, β denotes a constant matrix, ${}^{R}D^{\alpha}_{-\tau^{+}}$ and ${}^{R}J^{\alpha}_{-\tau^{+}}$ are, respectively, the Riemann-Liouville fractional derivativeand integral of $\alpha \in (1,2)$ with the lower limit $-\tau^{+}$ defined by

$${}^{R}D^{\alpha}_{-\tau^{+}} f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{-\tau}^{t} (t-s)^{1-\alpha} f(s)\mathrm{d}s, \ t > -\tau.$$

$${}^{R}J^{\alpha}_{-\tau^{+}} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\tau}^{t} (t-s)^{\alpha-1} f(s)\mathrm{d}s, \ t > -\tau.$$

To obtain the exact solutions of system (1), we firstly study the exact solutions of the homogeneous fractional delay differential equation

$$\begin{cases} {}^{R}D^{\alpha}_{-\tau^{+}} y(t) = \beta y(t-\tau), & t \in J := (0,T], \tau > 0, \\ y(t) = \varphi(t), & -\tau < t \le 0, \\ J^{2-\alpha}_{-\tau^{+}} y(-\tau^{+}) = m, \ {}^{R}D^{\alpha-1}_{-\tau^{+}} y(-\tau^{+}) = n. \end{cases} \tag{2}$$

Then, by means of the generalized Mittag-Leffler type matrix functions and the Laplace transform, we will solve the solutions of Equation (1) and derive its Hyers-Ulam stability.

Currently, we introduce two delayed Mittag-Leffler type matrix functions. We use the norm $\|y\| = \sum_{i=1}^{n} |y_i|$, $y \in \mathbb{R}^{n\times 1}$ and the matrix norm $\|\beta\| = \max_{1\le j\le n} \sum_{i=1}^{n} |\beta_{ij}|$, $\beta \in \mathbb{R}^{n\times n}$.

Definition 1. (See [16]) The delayed Mittag-Leffler type matrix function $\mathcal{Q}^{\tau}_{\alpha}(t): \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined as

$$\mathcal{Q}^{\tau}_{\alpha}(t) = \begin{cases} \theta, & -\infty < t \le -\tau, \\ I\dfrac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)}, & -\tau < t \le 0, \\ I\dfrac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)} + \beta\dfrac{t^{2\alpha-1}}{\Gamma(2\alpha)}, & 0 < t \le \tau, \\ \vdots & \vdots \\ I\dfrac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)} + \beta\dfrac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \beta^2\dfrac{(t-\tau)^{3\alpha-1}}{\Gamma(3\alpha)} + \cdots + \beta^k\dfrac{(t-(k-1)\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}, & (k-1)\tau < t \le k\tau. \end{cases}$$

Definition 2. (See [16]) The delayed Mittag-Leffler type matrix function $\mathcal{W}_{\alpha}^{\tau}(t):\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined as

$$\mathcal{W}_{\alpha}^{\tau}(t)=\begin{cases}\theta, & -\infty<t\le-\tau,\\ I\dfrac{(t+\tau)^{\alpha-2}}{\Gamma(\alpha-1)}, & -\tau<t\le 0,\\ I\dfrac{(t+\tau)^{\alpha-2}}{\Gamma(\alpha-1)}+\beta\dfrac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}, & 0<t\le\tau,\\ \vdots & \vdots\\ I\dfrac{(t+\tau)^{\alpha-2}}{\Gamma(\alpha-1)}+\beta\dfrac{t^{2\alpha-2}}{\Gamma(2\alpha-1)}+\beta^{2}\dfrac{(t-\tau)^{3\alpha-2}}{\Gamma(3\alpha-1)}+\cdots+\beta^{k}\dfrac{(t-(k-1)\tau)^{(k+1)\alpha-2}}{\Gamma((k+1)\alpha-1)}, & (k-1)\tau<t\le k\tau.\end{cases}$$

Lemma 1. (See [16]) For $\mathcal{Q}_{\alpha}^{\tau}(t)$ and $\mathcal{W}_{\alpha}^{\tau}(t)$, we have

i) $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}_{\alpha}^{\tau}(t)=\mathcal{W}_{\alpha}^{\tau}(t)$ for all $t\in\mathbb{R}\setminus\{-\tau\}$.

ii) $\mathcal{Q}_{\alpha}^{\tau}(t)$ is a solution of Eq.(2), which meets starting conditions
$\mathcal{Q}_{\alpha}^{\tau}(t)=I\dfrac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $-\tau<t\le 0$, $\mathcal{J}_{-\tau^{+}}^{2-\alpha}\mathcal{Q}_{\alpha}^{\tau}(-\tau^{+})=0$. and ${}^{R}\mathcal{D}_{-\tau^{+}}^{\alpha-1}\mathcal{Q}_{\alpha}^{\tau}(-\tau^{+})=1$.

iii) $\mathcal{W}_{\alpha}^{\tau}(t)$ is a solution of Eq. (2), which meets starting conditions
$\mathcal{W}_{\alpha}^{\tau}(t)=I\dfrac{(t+\tau)^{\alpha-2}}{\Gamma(\alpha-1)}$, $-\tau<t\le 0$, $\mathcal{J}_{-\tau^{+}}^{2-\alpha}\mathcal{W}_{\alpha}^{\tau}(-\tau^{+})=1$ and ${}^{R}\mathcal{D}_{-\tau^{+}}^{\alpha-1}\mathcal{W}_{\alpha}^{\tau}(-\tau^{+})=0$.

Lemma 2. (See [13]) If the Laplace transform of $f_1(t)=f(t-a)$ exists for $t\ge a$. Then we find

$$\mathcal{L}(f_1(t))(s)=\mathrm{e}^{-as}\int_{-a}^{0}\mathrm{e}^{-ts}f(t)\mathrm{d}t+\mathrm{e}^{-as}\mathcal{L}(f(t))(s).$$

Lemma 3. When $f(t)$ is integrable on $[a,+\infty)$, the Laplace transform of $f(t)$ is defined by $F(s)=\mathcal{L}(f(t))(s)=\int_{a}^{\infty}\mathrm{e}^{-st}f(t)\mathrm{d}t$, we obtain

$$\mathcal{L}\left(f^{(n)}(t)\right)(s)=s^{n}\mathcal{L}(f(t))(s)-\sum_{k=0}^{n-1}s^{k}\mathrm{e}^{-sa}f^{(n-k-1)}(a). \tag{3}$$

Proof. We prove it with the help of mathematical induction method. When $n=1$, we get

$$\mathcal{L}(f'(t))(s)=\int_{a}^{\infty}\mathrm{e}^{-st}f'(t)\mathrm{d}t=s\mathcal{L}(f(t))(s)-\mathrm{e}^{-sa}f(a).$$

Assuming Equation (1.3) holds when $n=k$, that is
$\mathcal{L}\left(f^{(k)}(t)\right)(s)=s^{k}\mathcal{L}(f(t))(s)-s^{k-1}\mathrm{e}^{-sa}f(a)-\cdots-s\mathrm{e}^{-sa}f^{(k-2)}(a)-\mathrm{e}^{-sa}f^{(k-1)}(a)$.
Let $n=k+1$, we have

$$\begin{aligned}\mathcal{L}\left(f^{(k+1)}(t)\right)(s)&=\mathcal{L}\left(\frac{\mathrm{d}}{\mathrm{d}t}f^{(k)}(t)\right)(s)=s\mathcal{L}\left(f^{k}(t)\right)(s)-\mathrm{e}^{-sa}f(a)\\&=s^{k+1}\mathcal{L}(f(t))(s)-s^{k}\mathrm{e}^{-sa}f(a)-\cdots-s\mathrm{e}^{-sa}f^{(k-1)}(a)-\mathrm{e}^{-sa}f(a).\end{aligned}$$

Thus, Lemma 3 is proved.

Utilizing the Lemma 3 and the method of proof in Section 2.8.2 of literature [17], we get

Lemma 4. The Laplace integral transform of Riemann-Liouville fractional derivative with the lower limit a is given by

$$\mathcal{L}\left({}^{R}D_{a}^{\alpha}f(t)\right)(s)=s^{\alpha}\mathcal{L}\left(f(t)\right)(s)-\sum_{k=0}^{n-1}s^{k}\mathrm{e}^{-sa}\left[{}^{R}D_{a}^{\alpha-k-1}f(a)\right]. \tag{4}$$

Proof. The Laplace transform of the function $t^{\alpha-1}$ is

$$G(s)=\mathcal{L}\left(t^{\alpha-1}\right)(s)=\Gamma(\alpha)s^{-\alpha}, \tag{5}$$

then, by applying the Laplace transform formula of the convolution equation, we derive the Laplace transform of the Riemann-Liouville fractional integral

$$\mathcal{L}\left(J_{a}^{\alpha}\right)(s)=s^{-\alpha}F(s). \tag{6}$$

Next we explore for the Laplace transform of the Riemannian-Liouville fractional order derivative, and for this purpose we write the following form

$$J_{a}^{n}D_{a}^{\alpha}f(t)=J_{a}^{n}g^{(\alpha)}(t),$$

$$g(t)=J_{a}^{n-\alpha}f(t)=\frac{1}{n-\alpha}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau, n-1\le\alpha<n.$$

According to the Lemma 3, we obtain

$$\mathcal{L}\left(D_{a}^{\alpha}f(t)\right)(s)=s^{n}G(s)-\sum_{k=0}^{n-1}s^{k}g(n-\alpha-1)(a), \tag{7}$$

the Laplace transform of $g(t)$ follows from (6)

$$G(s)=s^{-(n-\alpha)}F(s). \tag{8}$$

Furthermore, based on the definition of Riemann-Liouville fractional derivative, we derive the following result

$$g^{(n-k-1)}(t)=\frac{\mathrm{d}^{n-k-1}}{\mathrm{d}t^{n-k-1}}J_{a}^{n-\alpha}f(t)=D_{a}^{\alpha-k-1}f(t), \tag{9}$$

then, substituting (8) and (9) into (7), we have

$$\mathcal{L}\left({}^{R}D_{a}^{\alpha}f(t)\right)(s)=s^{\alpha}\mathcal{L}\left(f(t)\right)(s)-\sum_{k=0}^{n-1}s^{k}\mathrm{e}^{-sa}\left[{}^{R}D_{a}^{\alpha-k-1}f(t)\right]_{t=a}.$$

Lemma 5. (See [16]) Assuming $1<\alpha<2, \tau>0$, the solution of system (2) can be expressed as

$$u(t)=\mathcal{Q}_{\alpha}^{\tau}(t)n+\mathcal{W}_{\alpha}^{\tau}(t)m+\int_{-\tau}^{0}\mathcal{Q}_{\alpha}^{\tau}(t-\tau-\xi)\left({}^{R}D_{-\tau^{+}}^{\alpha}\varphi\right)(\xi)\mathrm{d}\xi. \tag{10}$$

Lemma 6. (Gronwall's inequality) (See [18]) Let $y(t)$ be a nonnegative summable function on $(0,Q]$, and $y(t)\le C_{1}\int_{0}^{t}y(s)\mathrm{d}s+C_{2}$ for a.e. $t\in(0,Q]$ and constants $C_{1},C_{2}\ge0$. Then $y(t)\le C_{2}\left(1+C_{1}t\mathrm{e}^{C_{1}t}\right)$, for a.e. $0<t\le Q$.

Definition 3. The system (1) is Hyers-Ulam stable on $(0,T]$ if there is, for a given constant $\varepsilon>0$, a function $\overline{y}(t)$ satisfying the inequality

$$\left\|{}^{R}D_{-\tau^{+}}^{\alpha}\overline{y}(t)-\beta\overline{y}(t-\tau)-f(t)\right\|\le\varepsilon,\ t\in(0,T], \tag{11}$$

and satisfying conditions of system (1), there exists a solution $y(t)$ of problem (1) and a number $C>0$ such that $\left\|\overline{y}(t)-y(t)\right\|\le C\varepsilon$ for every $t\in(0,T]$.

2. Exact solutions

Lemma 7. The solution of the problem (1) satisfies

$$\mathcal{L}(y(t))(s)=\left(s^{\alpha}-\beta e^{-\tau s}\right)^{-1}\left(\mathcal{A}_{\tau}(s)+\mathcal{L}(f(t))(s)\right),$$

where $\mathcal{A}_{\tau}(s)=\beta e^{-\tau s}\int_{-\tau}^{0}e^{-sz}\varphi(z)dz$.

Proof. Performing the Laplace transform on both sides of the system (1), according to (4) and Lemma 2, we get

$$\mathcal{L}\left({}^{R}D_{-\tau^{+}}^{\alpha}y(t)\right)(s)=\beta\mathcal{L}(y(t-\tau))(s)+\mathcal{L}(f(t))(s),$$

then

$$s^{\alpha}\mathcal{L}(y(t))(s)=\beta\left(e^{-\tau s}\int_{-\tau}^{0}e^{-sz}y(z)dz+e^{-s\tau}\mathcal{L}(y(t))(s)\right)+\mathcal{L}(f(t))(s),$$

hence

$$\mathcal{L}(y(t))(s)=\left(s^{\alpha}-\beta e^{-\tau s}\right)^{-1}\left(\mathcal{A}_{\tau}(s)+\mathcal{L}(f(t))(s)\right).$$

The proof is completed.

Corollary 1. The solution of problem (2) satisfies

$$\mathcal{L}(u(t))(s)=\left(s^{\alpha}-\beta e^{-\tau s}\right)^{-1}\left(\mathcal{A}_{\tau}(s)\right).$$

Lemma 8. Let $1<\alpha<2,\tau>0$, we have $\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t-\tau)\right)(s)=\left(s^{\alpha}I-\beta e^{-\tau s}\right)^{-1}I$.

Proof. According to the Lemma 1 (ii) and Corollary 1, we find

$$\begin{aligned}\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t)\right)(s)&=\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\cdot\beta e^{-s\tau}\int_{-\tau}^{0}e^{-sz}\frac{(z+\tau)^{\alpha-1}}{\Gamma(\alpha)}dz\\&=\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\cdot\beta\int_{-\tau}^{0}e^{-s(z+\tau)}\frac{(z+\tau)^{\alpha-1}}{\Gamma(\alpha)}dz\\&=\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\cdot\frac{\beta}{\Gamma(\alpha)}\int_{-\tau}^{0}e^{-s(z+\tau)}(z+\tau)^{\alpha-1}dz\\&=\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\cdot\frac{\beta}{s^{\alpha}}I.\end{aligned}$$

Combining this with the Lemma 2, we have

$$\begin{aligned}\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t-\tau)\right)(s)&=e^{-s\tau}\int_{-\tau}^{0}e^{-sz}\frac{(z+\tau)^{\alpha+1}}{\Gamma(\alpha)}dz+e^{-s\tau}\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t)\right)(s)\\&=\frac{I}{\Gamma(\alpha)}e^{-s\tau}\int_{-\tau}^{0}e^{-sz}(z+\tau)^{\alpha-1}dz+e^{-s\tau}\cdot\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\cdot\frac{\beta}{s^{\alpha}}\\&=\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}I.\end{aligned}$$

This proof is completed.

Theorem 1. The fractional delay differential equation (1) has a unique solution $y(t)$, and $y(t)$ can be written in this form

$$y(t)=\mathcal{Q}_{\alpha}^{\tau}(t)n+\mathcal{W}_{\alpha}^{\tau}(t)m+\int_{-\tau}^{0}\mathcal{Q}_{\alpha}^{\tau}(t-\tau-\xi)\left({}^{R}D_{-\tau+}^{\alpha}\varphi\right)(\xi)d\xi+\mathcal{Q}_{\alpha}^{\tau}(t-\tau)*f(t).\quad(12)$$

Proof. The implicit form of this solution is presented by the lemma 6 by using the Laplace transformation. Now all that is needed is to solve the Laplace inverse transformation of this

$$\mathcal{L}^{-1}\left(\left(s^{\alpha}I-\beta e^{-sz}\right)^{-1}\left(\beta e^{-s\tau}\int_{-\tau}^{0}e^{-sz}\varphi(z)\mathrm{d}z+\mathcal{L}(f(t))(s)\right)\right)(t).$$

By Corollary 1, we need only discuss $\mathcal{L}^{-1}\left(\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\mathcal{L}(f(t))(s)\right)(t)$

$$\begin{aligned}\mathcal{L}^{-1}\left(\left(s^{\alpha}I-\beta e^{-s\tau}\right)^{-1}\mathcal{L}(f(t))(s)\right)(t)&=\mathcal{L}^{-1}\left(\mathcal{L}\left(\mathcal{Q}_{\tau}^{\alpha}(t-\tau)\right)(s)\mathcal{L}(f(t))(s)\right)(t)\\&=\mathcal{L}^{-1}\left(\mathcal{L}\left(\mathcal{Q}_{\tau}^{\alpha}(t-\tau)*f(t)\right)(s)\right)(t)\\&=\mathcal{Q}_{\alpha}^{\tau}(t-\tau)*f(t).\end{aligned}\tag{13}$$

Hence

$$y(t)=\mathcal{Q}_{\alpha}^{\tau}(t)n+\mathcal{W}_{\alpha}^{\tau}(t)m+\int_{-\tau}^{0}\mathcal{Q}_{\alpha}^{\tau}(t-\tau-\xi)\left({}^{R}D_{-\tau+}^{\alpha}\varphi\right)(\xi)\mathrm{d}\xi+\mathcal{Q}_{\alpha}^{\tau}(t-\tau)*f(t).$$

This proof is completed.

3. Hyers-Ulam stability

Theorem 2. Let $\beta\in\mathbb{R}^{n\times n}$, $\alpha\in(1,2)$ and $f(t)$ is a function defined on $\mathbb{R}^{n}$, then the system (1) is Hyers-Ulam stable on $(0,T]$.

Proof. Let $\overline{y}(t)$ satisfy the inequality (11) and the conditions of system (1). Putting

$$\mathcal{K}(t)={}^{R}D_{-\tau^{+}}^{\alpha}\overline{y}(t)-\beta\overline{y}(t-\tau)-f(t),t\in(0,T].\tag{14}$$

From the Definition 3, it follows that $\|\mathcal{K}(t)\|\le\varepsilon$. Using the Laplace transform for the left and right sides of (14), we have

$$\mathcal{L}(\mathcal{K}(t))(s)=\left(s^{\alpha}I-\beta e^{-\tau s}\right)\mathcal{L}(\overline{y}(t))(s)-\beta e^{-\tau s}\int_{-\tau}^{0}e^{-sz}\varphi(z)\mathrm{d}z-\mathcal{L}(f(t))(s).$$

Thus

$$\mathcal{L}(\overline{y}(t))(s)=\frac{\beta e^{-s\tau}\int_{-\tau}^{0}e^{-sz}\varphi(z)\mathrm{d}z+\mathcal{L}(f(t))(s)}{s^{\alpha}I-\beta e^{-\tau s}}+\frac{\mathcal{L}(\mathcal{K}(t))(s)}{s^{\alpha}I-\beta e^{-\tau s}}.$$

According to Theorem 1, we find a solution $y(t)$ of equation (1). By the Lemma 6, we have

$$\mathcal{L}(\overline{y}(t)-y(t))(s)=\frac{\mathcal{L}(\mathcal{K}(t))(s)}{s^{\alpha}I-\beta e^{-s\tau}}.$$

Using the convolution theorem and formula (13), we obtain

$$\frac{\mathcal{L}(\mathcal{K}(t))(s)}{s^{\alpha}I-\beta e^{-s\tau}}=\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t-\tau)\right)(s)\cdot\mathcal{L}(\mathcal{K}(t))(s)=\mathcal{L}\left(\mathcal{Q}_{\alpha}^{\tau}(t-\tau)*\mathcal{K}(t)\right)(s).$$

Hence, we have $\overline{y}(t)-y(t)=\mathcal{Q}_{\alpha}^{\tau}(t-\tau)*\mathcal{K}(t)$. For any $t\in(k\tau,(k+1)\tau]$, $k=0,1,2,\cdots,N-1$, we get

$$\begin{aligned}\left\|\bar{y}(t)-y(t)\right\| &= \left\|\int_0^t \mathcal{K}(t-\xi)Q_\alpha^\tau(\xi-\tau)\mathrm{d}\xi\right\| \\ &\le \varepsilon\int_0^t \left\|Q_\alpha^\tau(\xi-\tau)\right\|\mathrm{d}\xi \\ &= \varepsilon\int_0^t \left\|\sum_{j=1}^{k}\frac{\beta^j(\xi-k\tau)^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}\right\|\mathrm{d}\xi.\end{aligned}$$

Let $C=\max_{0<t\le T}\int_0^t\left\|Q_\alpha^\tau(\xi-\tau)\right\|\mathrm{d}\xi$, then $\left\|\bar{y}(t)-y(t)\right\|\le C\varepsilon$. Thus the system (1) has Hyers-Ulam stability.

Hypothesis 1. Assume that $g(t)$ is a non-negative function and belongs to $L(0,T]$. Then there exists $W_g>0$, such that

$$\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s)}{(t-s)^{1-\alpha}}\mathrm{d}s\le W_g g(t), t\in J.$$

Theorem 3. Let $\left\|{}^R D_{-\tau^+}^{\alpha}\bar{y}(t)-\beta\bar{y}(t-\tau)-f(t)\right\|\le\varepsilon g(t)$, $t\in(0,T]$, where $\varepsilon>0$. Assuming that there exists a number $M>0$ such that $\beta(t-s)^{\alpha-1}\le M$, then $\exists$ a solution $z(t)\in J$ of (1) with

$$\left|y(t)-z(t)\right|\le\varepsilon K(t),$$

then the system (1) is the generalized Ulam-Hyers stable.

Proof. Since

$$-\varepsilon g(t)\le {}^R D_{-\tau^+}^{\alpha}y(t)-\beta y(t-\tau)-f(t)\le\varepsilon g(t),$$

then, we get

$$-\varepsilon\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s)}{(t-s)^{1-\alpha}}\mathrm{d}s\le y(t)-\beta J^\alpha y(t-\tau)-J^\alpha f(t)\le\varepsilon\frac{1}{\Gamma(\alpha)}\int_0^t\frac{g(s)}{(t-s)^{1-\alpha}}\mathrm{d}s.$$

For the following equation ${}^R D_{-\tau^+}^{\alpha}z(t)=\beta z(t-\tau)+f(t)$, we obtain $z(t)=\beta J^\alpha z(t-\tau)+J^\alpha f(t)$. Therefore,

$$\begin{aligned}\left|y(t)-z(t)\right| &= \left|y(t)-\beta J^\alpha y(t-\tau)-J^\alpha f(t)+\beta J^\alpha(y-z)(t-\tau)\right| \\ &\le \varepsilon W_g g(t)+\left|\beta J^\alpha(y-z)(t-\tau)\right| \\ &\le \varepsilon W_g g(t)+\frac{1}{\Gamma(\alpha)}\int_0^t\left|\beta(t-s)^{\alpha-1}\left[y(s)-z(s)\right]\right|\mathrm{d}s \\ &\le \varepsilon W_g g(t)+\frac{M}{\Gamma(\alpha)}\int_0^t\left|y(s)-z(s)\right|\mathrm{d}s.\end{aligned}$$

Based on the Gronwall's inequality, we have the following estimates $\left|y(t)-z(t)\right|\le\varepsilon C_2\left(1+C_1\mathrm{e}^{C_1 t}\right)$, where $C_2=W_g g(t)$, $C_1=\dfrac{M}{\Gamma(\alpha)}>0$. Thus, the system (1) is the generalized Ulam-Hyers stable.

4. Example

This section will confirm the soundness of the theoretical results with a concrete example.

Example. Assume that $\alpha=1.8,\tau=0.15,k=4$ and $T=0.6$. Consider

$$\begin{cases} {}^{R}D_{-0.15^{+}}^{1.8}y(t)=\beta y(t-0.15)+f(t), & 0<t\le 0.6, \tau>0, \\ \varphi(t)=\left((t+0.15)^{2}, \dfrac{(t+0.15)^{3}}{2}\right)^{\mathrm{T}}, & 0.15<t\le 0, \\ J_{-0.15^{+}}^{0.2}Y(-0.15^{+})=m=0, \\ {}^{R}D_{-0.15^{+}}^{0.8}Y(-0.15^{+})=n=0, \end{cases} \tag{15}$$

where $y(t)=(y_1(t),y_2(t))^{\mathrm{T}}$, $\beta=\begin{bmatrix}0.45 & 0\\ 0 & 0.5\end{bmatrix}$, $f(t)=\left(\dfrac{t^2}{2},t^3\right)^{\mathrm{T}}$.

According to Theorem 3, the solution of system (7) will exhibit a particular form when the time parameter t is in the interval $(0,0.6]$.

$$y(t)=Q_{1.8}^{0.15}(t)\cdot n+W_{1.8}^{0.15}(t)\cdot m+\int_{-0.15}^{0}Q_{1.8}^{0.15}(t-0.15-\xi)\left({}^{R}D_{-0.15+}^{1.8}\varphi\right)(\xi)\,\mathrm{d}\xi$$
$$+Q_{1.8}^{0.15}(t-0.15)*f(t),$$

where

$$\int_{-0.15}^{0}Q_{1.8}^{0.15}(t-0.15-\xi)\left({}^{R}D_{-0.15+}^{1.8}\varphi\right)(\xi)\,\mathrm{d}\xi$$
$$=\int_{-0.15}^{0}Q_{1.8}^{0.15}(t-0.15-\xi)\cdot\begin{pmatrix}\dfrac{2.64}{\Gamma(0.2)}(t+0.15)^{0.2}B[3,0.2]\\ \dfrac{3.52}{\Gamma(0.2)}(t+0.15)^{1.2}B[4,0.2]\end{pmatrix}\mathrm{d}\xi.$$

Let $C=\max_{0<t\le 0.6}\int_0^t\left\|Q_{1.8}^{0.15}(\xi-0.15)\right\|\mathrm{d}\xi=0.002207$, then

$$|y(t)-z(t)|\le\varepsilon\int_0^t\left\|Q_{\alpha}^{\tau}(\xi-\tau)\right\|\mathrm{d}\xi\le 0.002207\varepsilon.$$

That is to say, the system (1) is Hyers-Ulam stable on $(0,0.6]$.

5. Conclusion

In this paper, we discuss the Laplace transform of the higher-order derivative that is integrable on $[a,+\infty)$ and the Riemann-Liouville derivatives with a lower limit a. Based on these transforms and the generalized delayed Mittag-Leffler type matrix functions, we study the exact solutions of the nonhomogeneous Riemann-Liouville fractional oscillatory differential equation of order $\alpha\in(1,2)$ by using the Laplace transform technique. And explores the Hyers-Ulam stability of the nonhomogeneous fractional oscillatory differential equation with pure delay. We successfully solved the difficulty left over from the literature[16].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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