

# A Phase-Field Model Based on Type III Heat Conduction

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## Abstract

Our aim in this article is to study a phase-field system based on type III heat conduction. In particular, we prove the existence and uniqueness of solutions and then the dissipativity of the associated solution operators.

## Keywords

Phase-Field System, Type III Heat Conduction, Neumann Boundary Conditions, Well-Posedness, Dissipativity

## 1. Introduction

We introduce the following equations, see [1] and [2] (see also [3]), such that:

$$\frac{\partial u}{\partial t} + \Delta^2 v - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t}, \quad (1.1)$$

$$u = v - \varepsilon \Delta v, \quad \varepsilon > 0, \quad (1.2)$$

$$\frac{\partial \theta}{\partial t} + \operatorname{div} q = -\frac{\partial u}{\partial t}, \quad q = -\nabla \theta. \quad (1.3)$$

In this context,  $u$  is the order parameter,  $v$  is the microconcentration variable,  $\varepsilon$  is the inverse of a penalty modulus and is expected to be small,  $\theta$  is the relative temperature (defined as  $\theta = \tilde{\theta} - \theta_E$ , where  $\tilde{\theta}$  is the absolute temperature and  $\theta_E$  is the equilibrium melting temperature),  $q$  is the thermal flux vector and  $f$  is the derivative of a double-well potential (a typical choice of the potential is  $F(s) = \frac{1}{4}(s^2 - 1)^2$ , hence the usual cubic nonlinear term  $f(s) = s^3 - s$ ). Moreover, all physical constants have been set equal to one. This system models, e.g., melting-solidification phenomena in certain classes of

materials (see, e.g., [4]-[14]). However, the Fourier law

$$\mathbf{q} = -\nabla\theta \quad (1.4)$$

has drawback which predicts that thermal signals propagate with an infinite speed, which violates causality.

In this paper, in order to correct this unrealistic feature, by reformulating the problem in terms of order parameter  $u$  and the enthalpy  $H$ , recalling that

$$\frac{\partial H}{\partial t} = -\operatorname{div}\mathbf{q}, \quad (1.5)$$

where

$$H = u + \theta, \quad (1.6)$$

replacing the Fourier law  $\mathbf{q} = -\nabla\theta$  with the Maxwell-Cattaneo law

$$\left(1 + \frac{\partial}{\partial t}\right)\mathbf{q} = -\nabla\theta. \quad (1.7)$$

We define the thermal displacement variable  $\alpha$  as

$$\alpha(x, t) = \alpha(x, 0) + \int_0^t \theta(x, \tau) d\tau, \quad (1.8)$$

where

$$\theta = \frac{\partial\alpha}{\partial t}. \quad (1.9)$$

This model can be derived as follows: One introduces the (total Ginzburg-Landau) free energy (see [15] and [16])

$$\Psi = \int_{\Omega} \left( \frac{1}{2\varepsilon} (u - v)^2 + |\nabla v|^2 + F(u) - u\theta - \frac{1}{2}\theta^2 \right) dx, \quad (1.10)$$

where  $\Omega$  is the domain occupied by the system and we assume here that it is a bounded and regular domain of  $\mathbb{R}^n$  ( $n = 1, 2$  and  $3$ ), and the enthalpy equation is written

$$H = u + \theta \left( = u + \frac{\partial\alpha}{\partial t} \right). \quad (1.11)$$

As far as the evolution equation for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)

$$\frac{\partial u}{\partial t} = \Delta \frac{D\Psi}{Du}, \quad (1.12)$$

where  $\frac{D}{Du}$  denotes a variational derivative with respect to  $u$ . Then, we obtain the following phase-field system:

$$\frac{\partial u}{\partial t} + \Delta^2 v - \Delta f(u) = -\Delta \frac{\partial\alpha}{\partial t}, \quad (1.13)$$

$$u = v - \varepsilon\Delta v, \quad (1.14)$$

$$\frac{\partial^2\alpha}{\partial t^2} - \Delta \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t}. \quad (1.15)$$

Our aim in this article is to study the existence and uniqueness of solutions to

this problem. In particular, the existence of a solution is based on proper a priori estimates and a classical Galerkin scheme. We are also interested in the study the dissipativity of the associated solution operators.

### 2. Statement of the Problem

We consider, in a bounded and regular domain  $\Omega \subset \mathbb{R}^n$ ,  $n = 1, 2$  or  $3$ , the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} + \Delta^2 v - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t}, \tag{2.1}$$

$$u = v - \varepsilon \Delta v, \tag{2.2}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \tag{2.3}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial \Delta v}{\partial \eta} = \frac{\partial \alpha}{\partial \eta} = 0 \text{ on } \partial \Omega, \tag{2.4}$$

$$u|_{t=0} = u_0, \alpha|_{t=0} = \alpha_0, \left. \frac{\partial \alpha}{\partial t} \right|_{t=0} = \alpha_1, \tag{2.5}$$

where  $\eta$  denotes the exterior normal to  $\partial \Omega$  and  $\frac{\partial \varphi}{\partial \eta} = \nabla \varphi \cdot \eta$  denotes the normal derivative on  $\partial \Omega$ .

We assume here that

$$0 < \varepsilon \leq \varepsilon_0 < 1. \tag{2.6}$$

As far as the nonlinear term  $f$  is concerned, we assume that

$$f \in C^2(\mathbb{R}), f(0) = 0, \tag{2.7}$$

$$f' \geq -c_0, c_0 \geq 0, \tag{2.8}$$

$$f(s)s \geq c_1 F(s) - c_2 \geq -c_3, c_1 > 0, c_2, c_3 \geq 0, s \in \mathbb{R}, \tag{2.9}$$

$$F(s) \geq c_4 s^4 - c_5, c_4 > 0, c_5 \geq 0, s \in \mathbb{R}, \tag{2.10}$$

$$|f(s)| \leq \varepsilon F(s) + c_\varepsilon, \forall \varepsilon > 0, s \in \mathbb{R}, \tag{2.11}$$

where

$$F(s) = \int_0^s f(\xi) d\xi.$$

In particular, these assumptions are satisfied by the usual cubic nonlinear term, we take, for simplicity,  $f(s) = s^3 - s$ .

#### Notation

We denote by  $(\cdot, \cdot)$  the usual  $L^2(\Omega)$ -scalar product, with associated norm  $\|\cdot\|$ . We set  $\|\cdot\|_{-1} = \left\| (-\Delta)^{-\frac{1}{2}} \cdot \right\|$ , where  $(-\Delta)^{-1}$  denotes the inverse of the minus Laplace operator associated with Neumann boundary conditions. More generally  $\|\cdot\|_X$  denotes the norm in Banach space  $X$ . We also note that

$$\varphi \mapsto \left( \left\| (-\Delta)^{-\frac{1}{2}} \bar{\varphi} \right\|^2 + \langle \varphi \rangle^2 \right)^{\frac{1}{2}}$$

and acting on function with null overage and where it is understood here that

$$\langle \cdot \rangle = \frac{1}{|\Omega|} \langle \cdot, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)},$$

$$\varphi \mapsto \left( \|\bar{\varphi}\|^2 + \langle \varphi \rangle^2 \right)^{\frac{1}{2}},$$

$$\varphi \mapsto \left( \|\nabla \bar{\varphi}\|^2 + \langle \varphi \rangle^2 \right)^{\frac{1}{2}}$$

and

$$\varphi \mapsto \left( \|\Delta \bar{\varphi}\|^2 + \langle \varphi \rangle^2 \right)^{\frac{1}{2}}$$

are norms in  $H^{-1}(\Omega)$ ,  $L^2(\Omega)$ ,  $H^1(\Omega)$  and  $H^2(\Omega)$ , respectively, which are equivalent to the usual ones.

For  $\varphi \in L^1(\Omega)$ , we set

$$\langle \varphi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x) dx$$

and, for  $\varphi \in H^{-1}(\Omega)$ ,

$$\langle \varphi \rangle = \frac{1}{|\Omega|} \langle \varphi, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)},$$

with  $\langle \cdot, \cdot \rangle$  the duality product. Furthermore, we set

$$\varphi = \bar{\varphi} + \langle \varphi \rangle.$$

We introduce the operator  $A$  defined by

$$\langle Au, w \rangle_{H^{-1}(\Omega), H^1(\Omega)} = (\nabla u, \nabla w), \quad \forall \varphi \in \dot{H}^1(\Omega),$$

for  $\varphi \in \dot{H}^1(\Omega)$ ,

$$\dot{H}^1(\Omega) = \{\varphi \in H^1(\Omega), \langle \varphi \rangle = 0\}.$$

We also set

$$\dot{L}^2(\Omega) = \{\varphi \in L^2(\Omega), \langle \varphi \rangle = 0\}.$$

It then follows from elliptic regularity results for linear elliptic operators of order 2 that  $A$  is a strictly positive, selfadjoint and unbounded linear operator with compact inverse and is an isomorphism from  $\dot{H}^1(\Omega)$  onto its dual, with domain

$$\mathcal{D}(A) = \left\{ w \in H^2(\Omega) \cap \dot{H}^1(\Omega), \frac{\partial \varphi}{\partial \eta} = 0 \text{ on } \partial\Omega \right\}$$

and

$$Au = h, \varphi \in \mathcal{D}(A), h \in \dot{L}^2(\Omega),$$

is equivalent to

$$-\Delta \varphi = h, \frac{\partial \varphi}{\partial \eta} = 0 \text{ on } \partial\Omega.$$

We will therefore write  $-\Delta$  instead of  $A$  in what follows.

Throughout this paper, the same letter  $c$  (and, sometimes,  $c'$  and  $c''$ ) denotes constants which may change from line to line, or even in a same line.

### 3. A Priori Dissipative Estimates

Integrating (2.1) over  $\Omega$ , we have, owing to (2.2),

$$\int_{\Omega} \frac{\partial u}{\partial t} dx = 0, \quad (3.1)$$

which yields

$$\frac{d}{dt} \langle u \rangle = 0, \quad (3.2)$$

we obtain, after integration between 0 and  $t$

$$\langle u(t) \rangle = \langle u_0 \rangle, \quad \forall t \geq 0. \quad (3.3)$$

Integrating then (2.2) over  $\Omega$ , we obtain

$$\langle v \rangle = \langle u \rangle, \quad (3.4)$$

so that, also,

$$\frac{d}{dt} \langle v \rangle = 0. \quad (3.5)$$

Now, integrating (2.3) over  $\Omega$ , we have

$$\frac{d}{dt} \left\langle \frac{\partial \alpha}{\partial t} \right\rangle = - \frac{d \langle u \rangle}{dt},$$

which yields, after integration between 0 and  $t$  and owing to (3.2),

$$\left\langle \frac{\partial \alpha}{\partial t} \right\rangle = \langle \alpha_1 \rangle, \quad (3.6)$$

this yields, integrating between 0 and  $t$

$$\langle \alpha(t) \rangle = \langle \alpha_1 \rangle t + \langle \alpha_0 \rangle. \quad (3.7)$$

We assume that

$$|\langle u_0 \rangle| \leq M, \quad |\langle \alpha_1 \rangle| \leq N, \quad (3.8)$$

for fixed positive constants  $M$  and  $N$ , which yields, owing to (3.3), (3.6) and (3.8),

$$|\langle u(t) \rangle| \leq M, \quad \left| \left\langle \frac{\partial \alpha}{\partial t}(t) \right\rangle \right| \leq N, \quad t \geq 0 \quad (3.9)$$

and

$$|\langle \alpha(t) \rangle| \leq |\langle \alpha_0 \rangle| + Nt, \quad t \geq 0. \quad (3.10)$$

Now, it follows (3.2), (3.5) and (3.6) that we can rewrite (2.1)-(2.5) as

$$\frac{\partial \bar{u}}{\partial t} + \Delta^2 \bar{v} - \Delta \overline{f(u)} = -\Delta \frac{\partial \bar{\alpha}}{\partial t}, \quad (3.11)$$

$$\bar{u} = \bar{v} - \varepsilon \Delta \bar{v}, \quad (3.12)$$

$$\frac{\partial^2 \bar{\alpha}}{\partial t^2} - \Delta \frac{\partial \bar{\alpha}}{\partial t} - \Delta \bar{\alpha} = -\frac{\partial \bar{u}}{\partial t}, \quad (3.13)$$

$$\frac{\partial \bar{u}}{\partial \eta} = \frac{\partial \bar{v}}{\partial \eta} = \frac{\partial \Delta \bar{v}}{\partial \eta} = \frac{\partial \bar{\alpha}}{\partial \eta} = 0 \text{ on } \partial \Omega, \quad (3.14)$$

$$\bar{u}|_{t=0} = \bar{u}_0 = u_0 - \langle u_0 \rangle, \quad \bar{\alpha}|_{t=0} = \bar{\alpha}_0 = \alpha_0 - \langle \alpha_0 \rangle, \quad \frac{\partial \bar{\alpha}}{\partial t}|_{t=0} = \bar{\alpha}_1 = \alpha_1 - \langle \alpha_1 \rangle, \quad (3.15)$$

where we set  $\bar{u} = u - \langle u \rangle$ ,  $\bar{v} = v - \langle v \rangle$  and  $\bar{\alpha} = \alpha - \langle \alpha \rangle$ .

We rewrite (3.11) in the following equivalent form:

$$(-\Delta)^{-1} \frac{\partial \bar{u}}{\partial t} - \Delta \bar{v} + \overline{f(u)} = \frac{\partial \bar{\alpha}}{\partial t}. \quad (3.16)$$

Multiplying (3.16) by  $\frac{\partial \bar{u}}{\partial t}$  and integrating over  $\Omega$  and by parts, we have, owing to (3.1),

$$\left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 - \left( \Delta \bar{v}, \frac{\partial \bar{u}}{\partial t} \right) + \frac{d}{dt} \int_{\Omega} F(u) dx = \left( \frac{\partial \bar{\alpha}}{\partial t}, \frac{\partial \bar{u}}{\partial t} \right).$$

Noting that it follows from (3.12) that

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{v}}{\partial t} - \varepsilon \Delta \frac{\partial \bar{v}}{\partial t}, \quad (3.17)$$

we obtain

$$\frac{d}{dt} \left( \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2 + 2 \int_{\Omega} F(u) dx \right) + \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 = 2 \left( \frac{\partial \bar{\alpha}}{\partial t}, \frac{\partial \bar{u}}{\partial t} \right). \quad (3.18)$$

We multiply (3.13) by  $\frac{\partial \bar{\alpha}}{\partial t}$  and find

$$\frac{d}{dt} \left( \|\nabla \alpha\|^2 + \left\| \frac{\partial \bar{\alpha}}{\partial t} \right\|^2 \right) + 2 \left\| \nabla \frac{\partial \bar{\alpha}}{\partial t} \right\|^2 = -2 \left( \frac{\partial \bar{\alpha}}{\partial t}, \frac{\partial \bar{u}}{\partial t} \right). \quad (3.19)$$

Summing (3.18) and (3.19), we find, setting

$$E_1 = \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2 + 2 \int_{\Omega} F(u) dx + \|\nabla \alpha\|^2 + \left\| \frac{\partial \bar{\alpha}}{\partial t} \right\|^2,$$

an equality

$$\frac{dE_1}{dt} + 2 \left\| \frac{\partial \bar{u}}{\partial t} \right\|_{-1}^2 + 2 \left\| \nabla \frac{\partial \bar{\alpha}}{\partial t} \right\|^2 = 0, \quad (3.20)$$

this yields the decay of the total free energy.

We now multiply (3.16) by  $\bar{u}$ , owing to (3.9), (3.10) and the generalized Poincaré inequality

$$\|\bar{\varphi}\| \leq \|\nabla \varphi\|, \quad \forall \varphi \in H^1(\Omega), \quad (3.21)$$

we have

$$\frac{d}{dt} \left\| \bar{u} \right\|_{-1}^2 + \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2 + c \int_{\Omega} F(u) dx \leq c' \left\| \nabla \frac{\partial \bar{\alpha}}{\partial t} \right\|^2 + c_M'', \quad c > 0. \quad (3.22)$$

We multiply (3.13) by  $\bar{\alpha}$  and find, owing to (3.9),

$$\frac{d}{dt} \left( \|\nabla \alpha\|^2 + 2 \left( \bar{\alpha}, \frac{\partial \bar{\alpha}}{\partial t} \right) \right) + \|\nabla \alpha\|^2 \leq \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + c \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2. \tag{3.23}$$

Summing (3.20)  $\delta_1$  times (3.22) and  $\delta_2$  times (3.23), where  $\delta_1, \delta_2 > 0$  are small enough, and we get, setting

$$E_2 = E_1 + \delta_1 \|\bar{u}\|_{-1}^2 + \delta_2 \left( \|\nabla \alpha\|^2 + 2 \left( \bar{\alpha}, \frac{\partial \bar{\alpha}}{\partial t} \right) \right)$$

an inequality of the form

$$\frac{dE_2}{dt} + c \left( E_2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) \leq c'_M, \quad c > 0. \tag{3.24}$$

We finally multiply (3.11) by  $\bar{u}$  to obtain, owing to (3.8), (3.9) and (3.21),

$$\frac{d}{dt} \|\bar{u}\|^2 + \|\Delta v\|^2 + \varepsilon \|\nabla \Delta v\|^2 \leq c \left( \|\nabla u\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right). \tag{3.25}$$

We sum (3.24) and  $\delta_3$  times (3.25), where  $\delta_3 > 0$  is small enough, and find, setting

$$E_3 = E_2 + \delta_3 \|\bar{u}\|^2 + \langle u \rangle^2 + \left\langle \frac{\partial \alpha}{\partial t} \right\rangle^2$$

an inequality of the form

$$\frac{dE_3}{dt} + c \left( E_3 + \varepsilon \|\nabla \Delta v\|^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) \leq c'_{MN}, \quad c > 0 \tag{3.26}$$

where  $E_3$  satisfies, owing to (2.10),

$$E_3 \geq c \left( \|u\|_{L^4(\Omega)}^2 + \int_{\Omega} F(u) dx + \varepsilon \|v\|_{H^2(\Omega)}^2 + \|\bar{\alpha}\|_{H^1(\Omega)}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) - c'_{MN}, \quad c > 0. \tag{3.27}$$

It follows from (3.26) and Growall's lemma that

$$E_3(t) \leq e^{-ct} E_3(0) + c'_{MN}, \quad c > 0, \quad t \geq 0, \tag{3.28}$$

which yields the dissipative inequality

$$E_3(t) \leq ce^{-c't} \left( \|u_0\|_{L^4(\Omega)}^2 + \int_{\Omega} F(u_0) dx + \varepsilon \|v_0\|_{H^2(\Omega)}^2 + \|\bar{\alpha}_0\|_{H^1(\Omega)}^2 + \|\alpha_1\|^2 \right) + c''_{MN}, \quad c' > 0, \quad \forall t \geq 0, \tag{3.29}$$

and

$$\int_0^t \left( \varepsilon \|v\|_{H^3(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|_{H^1(\Omega)}^2 \right) ds \leq ce^{-c't} \left( \|u_0\|_{L^4(\Omega)}^2 + \int_{\Omega} F(u_0) dx + \varepsilon \|v_0\|_{H^2(\Omega)}^2 + \|\bar{\alpha}_0\|_{H^1(\Omega)}^2 + \|\alpha_1\|^2 \right) + c''_{MN}, \quad c' > 0, \quad \forall t \geq 0, \tag{3.30}$$

with  $v_0 = (I - \varepsilon \Delta)^{-1} u_0$ . Together with estimates on  $v \in L^\infty(\mathbb{R}^+; H^1(\Omega))$ ,  $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}^+; H^{-1}(\Omega))$  uniformly with respect to  $\varepsilon$ ,  $\bar{\alpha} \in L^\infty(\mathbb{R}^+; H^1(\Omega))$  and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

$$\varepsilon^{1/2} v \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)) \text{ uniformly with respect to } \varepsilon.$$

Moreover, according to (2.2)-(2.8), we have  $\|u\| \leq \|v\| + \varepsilon_0^{1/2} \varepsilon^{1/2} \|\Delta v\|$ , hence we conclude that  $u \in L^\infty(\mathbb{R}^+, L^4(\Omega)) \cap L^2(0, T; H^1(\Omega))$  uniformly with respect to  $\varepsilon$ .

$$\text{Since } \frac{\partial v}{\partial t} = (I - \varepsilon \Delta)^{-1} \frac{\partial u}{\partial t}, \text{ it also follows that } \frac{\partial v}{\partial t} \text{ belongs to } L^2(0, T; H^1(\Omega))$$

but this estimate is not uniform.

**Remark 3.1.** When  $\varepsilon \rightarrow 0^+$ ,  $v \rightarrow u$ , then the problem (2.1)-(2.5) converges to the generalization of the Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction, namely,

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t}, \quad (3.31)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t}, \quad (3.32)$$

which, in [17] the author proved the well-posedness results, the existence of exponential attractors and, thus, of finite-dimensional global attractors.

#### 4. The Dissipative Semigroup $\bar{S}_\varepsilon(t)$

Here, we consider that  $\varepsilon > 0$  is fixed. We start with the following theorem.

**Theorem 4.1.** We assume that (2.6)-(2.11) hold. Then, for every  $(u_0, \varepsilon u_0, \alpha_0, \alpha_1) \in H^1(\Omega) \times H^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)$  such that  $\frac{\partial u_0}{\partial \eta} = \frac{\partial \alpha_0}{\partial \eta} = \frac{\partial \alpha_1}{\partial \eta} = 0$  on  $\partial \Omega$ , the system (2.1)-(2.4) possesses a unique weak solution  $\left(u, v, \alpha, \frac{\partial \alpha}{\partial t}\right)$  such that,  $\forall T \geq 0$

$$u \in L^\infty(\mathbb{R}^+; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega)), \frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)),$$

$$v \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \frac{\partial v}{\partial t} \in L^2(0, T; H^1(\Omega)),$$

and

$$\bar{\alpha} \in L^\infty(\mathbb{R}^+; H^1(\Omega)), \frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

**Proof. i) Uniqueness:**

Let  $\left(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right)$  and  $\left(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$  be two solutions to (2.1)-(2.4) with initial data  $(u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$  and  $(u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$ , respectively. We set

$$\left(u, v, \alpha, \frac{\partial \alpha}{\partial t}\right) = \left(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right) - \left(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$$

and

$$(u_0, \alpha_0, \alpha_1) = (u_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

Then, we have

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta v + \overline{f(u^{(1)}) - f(u^{(2)})} = \frac{\partial \bar{\alpha}}{\partial t}, \tag{4.1}$$

$$\bar{u} = \bar{v} - \varepsilon \Delta \bar{v}, \tag{4.2}$$

$$\langle u \rangle = \langle v \rangle = 0, \quad \forall t \geq 0 \tag{4.3}$$

$$\frac{\partial^2 \bar{\alpha}}{\partial t^2} - \Delta \frac{\partial \bar{\alpha}}{\partial t} - \Delta \bar{\alpha} = -\frac{\partial \bar{u}}{\partial t}, \tag{4.4}$$

$$\frac{\partial \bar{v}}{\partial \nu} = \frac{\partial \bar{\alpha}}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{4.5}$$

$$u|_{t=0} = u_0, \quad \bar{\alpha}|_{t=0} = \bar{\alpha}_0, \quad \frac{\partial \bar{\alpha}}{\partial t} \Big|_{t=0} = \alpha_1. \tag{4.6}$$

We multiply (4.1) by  $u$  and have, owing to (2.8),

$$\frac{1}{2} \frac{d}{dt} \|u\|_{-1}^2 + (u, -\Delta v) \leq c_0 \|u\|^2 + \left(u, \frac{\partial \bar{\alpha}}{\partial t}\right). \tag{4.7}$$

Note that it follows from (4.2) that

$$(u, -\Delta v) = \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2 \tag{4.8}$$

and

$$\|u\|^2 = \|v\|^2 + 2\varepsilon \|\nabla v\|^2 + \varepsilon^2 \|\Delta v\|^2. \tag{4.9}$$

We thus deduce that

$$\frac{d}{dt} \|u\|_{-1}^2 + \|\nabla v\|^2 + \varepsilon \|\Delta v\|^2 \leq c \left(\|v\|^2 + 2\varepsilon \|\nabla v\|^2 + \varepsilon^2 \|\Delta v\|^2\right) + 2 \left(u, \frac{\partial \bar{\alpha}}{\partial t}\right) \tag{4.10}$$

and (4.10) yields, by the Poincare-Wirtinger inequality

$$\frac{d}{dt} \|u\|_{-1}^2 + (1 - \varepsilon_0) \varepsilon \|\Delta v\|^2 \leq c \|\nabla v\|^2 + 2 \left(u, \frac{\partial \bar{\alpha}}{\partial t}\right). \tag{4.11}$$

Writing next

$$\|u\|_{-1}^2 = \left\| (-\Delta)^{-1} (v - \varepsilon \Delta v) \right\|^2 = \|v\|_{-1}^2 + 2\varepsilon \|v\|^2 + \varepsilon^2 \|\nabla v\|^2, \tag{4.12}$$

it follows that

$$\frac{d}{dt} \|u\|_{-1}^2 + (1 - \varepsilon_0) \varepsilon \|\Delta v\|^2 \leq \frac{c}{\varepsilon^2} \|u\|_{-1}^2 + 2 \left(u, \frac{\partial \bar{\alpha}}{\partial t}\right). \tag{4.13}$$

Now, we rewrite (4.4) as

$$(-\Delta)^{-1} \left( \frac{\partial^2 \bar{\alpha}}{\partial t^2} + \frac{\partial \bar{u}}{\partial t} \right) + \frac{\partial \bar{\alpha}}{\partial t} + \bar{\alpha} = 0. \tag{4.14}$$

Multiplying (4.14) by  $\frac{\partial \bar{\alpha}}{\partial t} + u$  and integrating over  $\Omega$ , we obtain

$$\frac{d}{dt} \left( \left\| u + \frac{\partial \bar{\alpha}}{\partial t} \right\|_{-1}^2 + \|\bar{\alpha}\|^2 \right) + \left\| \frac{\partial \bar{\alpha}}{\partial t} \right\|^2 \leq -2 \left( \frac{\partial \bar{\alpha}}{\partial t}, u \right) + \|\bar{\alpha}\|^2 + \|u\|^2. \tag{4.15}$$

Summing (4.13) and (4.15), owing to (4.9), we have in particular, setting

$$E_4 = \|u\|_{-1}^2 + \left\| u + \frac{\partial \bar{\alpha}}{\partial t} \right\|_{-1}^2 + \|\bar{\alpha}\|^2,$$

an inequality of the form

$$\frac{dE_4}{dt} \leq c_\varepsilon E_4 \tag{4.16}$$

and hence

$$E_4(t) \leq e^{c_\varepsilon t} E_4(0). \tag{4.17}$$

Noting also that

$$\langle u(t) \rangle = \langle v(t) \rangle = \langle u_0 \rangle = 0 \text{ and } |\langle \alpha(t) \rangle| \leq |\langle \alpha_0 \rangle| + |\langle \alpha_1 \rangle| t, \quad \forall t \geq 0, \tag{4.18}$$

we deduce from (4.17) and (4.18) the uniqueness as well as the continuous dependence with respect to the initial data.

ii) **Existence:** The proof of existence is done as follows

**First of all: Approximated problems**

Let  $e_1, e_2, \dots$  be an orthonormal in  $L^2(\Omega)$  and orthigonal in  $\dot{H}^1(\Omega)$  family associated with the eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  of the operator A associated with Neumann boundary conditions,

$$Ae_i = \lambda_i e_i$$

We set, for  $m \in \mathbb{N}$

$$E_m = span\{e_1, \dots, e_m\}$$

and

$$\beta_m = \frac{\partial \alpha_m}{\partial t} \left( \text{resp. } \beta = \frac{\partial \alpha}{\partial t} \right).$$

We introduce the following approximated problems:

Find  $(u_m, v_m, \alpha_m, \beta_m) : [0, T] \rightarrow E_m, \quad T > 0$  given, such that

$$\frac{d}{dt} \left( (-\Delta)^{-1} \bar{u}_m, \varphi \right) + (\nabla \bar{v}_m, \nabla \varphi) + \left( \overline{f(u_m)}, \varphi \right) = \left( \bar{\beta}_m, \varphi \right) \text{ in } \mathcal{D}'(0, T), \quad \forall \varphi \in V_m, \tag{4.19}$$

$$\left( \bar{u}_m, \varphi \right) = \left( \bar{v}_m, \varphi \right) + \varepsilon (\nabla \bar{v}_m, \nabla \varphi) \text{ in } \mathcal{D}'(0, T), \quad \forall \varphi \in V_m, \tag{4.20}$$

$$\frac{d}{dt} \left( \bar{\beta}_m, \varphi \right) + (\nabla \bar{\beta}_m, \nabla \varphi) + (\nabla \bar{\alpha}_m, \nabla \varphi) = -\frac{d}{dt} \left( \bar{u}_m, \varphi \right) \text{ in } \mathcal{D}'(0, T), \quad \forall \varphi \in V_m, \tag{4.21}$$

$$\bar{u}_m|_{t=0} = \bar{u}_{0,m}, \quad \bar{\alpha}_m|_{t=0} = \bar{\alpha}_{0,m}, \quad \bar{\beta}_m|_{t=0} = \bar{\alpha}_{1,m}, \tag{4.22}$$

where

$$\bar{u}_{0,m} = P_m \bar{u}_0, \quad \bar{\alpha}_{0,m} = P_m \bar{\alpha}_0, \quad \bar{\alpha}_{1,m} = P_m \bar{\alpha}_1, \tag{4.23}$$

$P_m$  is the orthogonal projector from  $L^2(\Omega)$  onto  $E_m$  (for the  $L^2$ -metric). This means that

$$\bar{u}_{0,m} = \sum_{i=1}^m (u_0, e_i) e_i, \quad \bar{\alpha}_{0,m} = \sum_{i=1}^m (\alpha_0, e_i) e_i, \quad \bar{\alpha}_{1,m} = \sum_{i=1}^m (\alpha_1, e_i) e_i. \tag{4.24}$$

Note that (4.19)-(4.22) is equivalent to the following problem:

$$A^{-1} \frac{d\bar{u}_m}{dt} + A\bar{v}_m + P_m \overline{f(u_m)} = \frac{d\bar{\alpha}_m}{dt} \text{ in } \mathcal{D}'(0, T), \tag{4.25}$$

$$\bar{u}_m = \bar{v}_m + \varepsilon A\bar{v}_m \text{ in } \mathcal{D}'(0, T), \tag{4.26}$$

$$\frac{d^2 \bar{\alpha}_m}{dt^2} + A \frac{d\bar{\alpha}_m}{dt} + A\bar{\alpha}_m = -\frac{d\bar{u}_m}{dt} \text{ in } \mathcal{D}'(0, T). \tag{4.27}$$

**Secondly: Existence of a local in time solution**

The existence of a local, and then maximal, (in time) solution to the approximate problem (4.19)-(4.23) is standard. Indeed, we have to solve a Lipschitz finite-dimensional system of ODEs to find  $(u_m, v_m, \alpha_m, \beta_m)$ , defined on  $[0, T_*)$ .

**Thirdly: Energy decay**

All constants below are independent of  $m$ .

We note that (3.20) can be rewritten, equivalently, as  $(u_m = u, v_m = v \text{ and } \alpha_m = \alpha)$

$$\frac{dE_{1,m}}{dt} + 2 \left\| \frac{\partial u_m}{\partial t} \right\|_{-1}^2 + 2 \left\| \nabla \frac{\partial \alpha_m}{\partial t} \right\|^2 = 0, \tag{4.28}$$

where

$$E_{1,m} = \|\nabla v_m\|^2 + \varepsilon \|\Delta v_m\|^2 + 2 \int_{\Omega} F(u_m) dx + \|\nabla \alpha_m\|^2 + \left\| \frac{\partial \alpha_m}{\partial t} \right\|^2,$$

i.e., the energy decay also holds for the approximated problems. This also yields that the maximal solution is global in time, i.e.,  $T_* = T$ .

**Fourth: Further a priori estimates**

Similarly, we rewrite (3.24) as

$$\frac{dE_{2,m}}{dt} + c \left( E_{2,m} + \left\| \frac{\partial u_m}{\partial t} \right\|_{-1}^2 + \left\| \nabla \frac{\partial \alpha_m}{\partial t} \right\|^2 \right) \leq c_M, \tag{4.29}$$

where

$$E_{2,m} = E_{1,m} + \delta_1 \|\bar{u}_m\|_{-1}^2 + \delta_2 \|\nabla \alpha_m\|^2 + 2\delta_2 \left( \bar{\alpha}_m, \frac{\partial \bar{\alpha}_m}{\partial t} \right)$$

satisfies

$$E_{2,m} \geq c \left( \|u_m\|_{L^4(\Omega)}^2 + \varepsilon \|v_m\|_{H^2(\Omega)}^2 + \|\alpha_m\|_{H^1(\Omega)}^2 + \left\| \frac{\partial \alpha_m}{\partial t} \right\|^2 \right) - c'_M, \quad c > 0. \tag{4.30}$$

**Fifth: Passage to the limit**

It follows from the above and standard Aubin-Lions compactness results, and  $\varphi$  such that

$$u_m \rightarrow u \text{ in } L^4((\Omega) \times (0, T)) \text{ weakly and a.e.,}$$

$$\frac{\partial u_m}{\partial t} \rightarrow \frac{\partial u}{\partial t} \text{ in } L^2(0, T; H^{-1}(\Omega)) \text{ weakly,}$$

$$v_m \rightarrow v \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak star,}$$

$$\varepsilon^{\frac{1}{2}} v_m \rightarrow \varphi \text{ in } L^\infty(0, T; H^2(\Omega)) \text{ weak star,}$$

$$\alpha_m \rightarrow \alpha \text{ in } L^\infty(0, T; H^1(\Omega)) \text{ weak star}$$

and

$$\frac{\partial \alpha_m}{\partial t} \rightarrow \frac{\partial \alpha}{\partial t} \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak star and in } L^2(0, T; H^1(\Omega)) \text{ weakly}$$

as  $m \rightarrow +\infty$ , for all  $T > 0$ .

Next, it follows from the above almost everywhere convergence of  $f(u_m)$ , we can note that it follows from on classical (Aubin-Lions type) compactness results that, at least for a subsequence that we do not relabel,

$$u_m \rightarrow u \text{ in } L^4(\Omega \times (0, T)) \text{ weakly and a.e.,}$$

for a proper  $u$ ,  $f(u_m) = u_m^3 - u_m$ , which implies that

$$f(u_m) \rightarrow f(u) \text{ a.e. and } f(u_m) \text{ is bounded in } L^{\frac{4}{3}}(\Omega \times (0, T)).$$

Therefore,  $f(u_m) \rightarrow f(u)$  in  $L^{\frac{4}{3}}(\Omega \times (0, T))$  weakly (see, e.g., [18]), which is sufficient to pass to the limit in the weak formulation.

**Sixth: Continuity with respect to time**

We first note that it follows from standard results that, since, e.g.,  $u \in C([0, T]; L^4(\Omega)_w)$ ,  $v \in C([0, T]; H^2(\Omega))$ ,  $\alpha \in C([0, T]; H^1(\Omega))$  and  $\frac{\partial \alpha}{\partial t} \in C([0, T]; L^2(\Omega))$ , where the index  $w$  denotes the weak topology and the weak continuity follows from the Strauss lemma (see, e.g., [19]).

It follows from Theorem 4.1 that we can define the family of solving operators

$$\bar{S}_\varepsilon(t) : \bar{\Phi}_{MN} \rightarrow \bar{\Phi}_{MN}, (u_0, \bar{\alpha}_0, \alpha_1) \mapsto \left( u(t), \bar{\alpha}(t), \frac{\partial \alpha}{\partial t}(t) \right), \forall t \geq 0$$

where

$$\bar{\Phi}_{MN} = \{(\varphi, \psi, \chi) \in \Phi_{MN}, \langle \varphi \rangle = 0\}$$

with

$$\Phi_{MN} = \{(\varphi, \psi, \chi) \in \Phi, |\langle \varphi \rangle| \leq M, |\langle \chi \rangle| \leq N\}$$

and

$$\Phi = H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega).$$

Furthermore, this family of solving operators forms a semigroup, i.e.,  $\bar{S}_\varepsilon(0) = I$  and  $\bar{S}_\varepsilon(t + \tau) = \bar{S}_\varepsilon(t) \circ \bar{S}_\varepsilon(\tau)$ ,  $t, \tau \geq 0$ , which is continuous with respect to the  $H^{-1} \times L^2 \times H^{-1}$ -topology.

Finally, it follows from (3.29) that we have the

**Theorem 4.2.** The semigroup  $\bar{S}_\varepsilon(t)$  is dissipative in  $\bar{\Phi}_{MN}$ , in the sense that it possesses a bounded absorbing set  $\mathcal{B}_0 \subset \bar{\Phi}_{MN}$  (i.e.,  $\forall \mathcal{B} \subset \Phi \exists t_0 = t_0(\mathcal{B})$  such that  $t \geq t_0$  implies  $\bar{S}_\varepsilon(t)\mathcal{B} \subset \mathcal{B}_0$ ).

**Remark 4.1.** The dissipativity is a first step in view of the study of the (temporal) asymptotic behavior of the associated dynamical system. In particular, an important issue is to prove the existence of finite-dimensional attractors: such objects describe all possible dynamics of the system; furthermore, the finite-dimensionality means, very roughly speaking, that, even though the initial phase space  $\bar{\Phi}_{MN}$  has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [20] for discussions on this subject). This will be studied elsewhere.

## 5. Conclusion

We proposed in this article a phase-field model based on Type III Heat Conduction. In particular, we proved the existence and uniqueness of solutions, as well as the dissipativity of the associated solution operators.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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