

Casimir Operators on the \mathfrak{sl}_2 -Modules of the Linear Differential Operators and the Associated Symbols on the Real Line

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Abstract

The research paper investigates the computation of Casimir operators on \mathfrak{sl}_2 -modules associated with linear differential operators and symbols on the real line. The study considers spaces of differential operators and principal symbols related to tensor densities, and derives explicit formulas for the Casimir operators. It extends previously known results about \mathfrak{sl}_2 -modules and Schur's Lemma, enriching the understanding of their role in differential geometry and representation theory.

Keywords

Module Structures, Adjointe Action of Lie Algebra $\text{Vect}(\mathbb{R})$, Schur Lemma, Equivalence of Representations

1. Introduction

Consider $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ the space of linear differential operators of order k on tensor densities $\mathcal{F}_\mu(\mathbb{R})$ with values in the space of tensor densities $\mathcal{F}_\lambda(\mathbb{R})$. The space $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ has a $\text{Vect}(\mathbb{R})$ -module structure. It is isomorphic to a direct sum of tensor densities spaces as follow [1] [2]:

$$\mathcal{D}_{\lambda\mu}^k(\mathbb{R}) \cong \mathcal{F}_{\delta+k}(\mathbb{R}) \oplus \mathcal{F}_{\delta+k-1}(\mathbb{R}) \oplus \cdots \oplus \mathcal{F}_\delta(\mathbb{R}).$$

We then consider an isomorphism of vector spaces called symbol map

$$\sigma_{\lambda\mu} : \mathcal{D}_{\lambda\mu}^k(\mathbb{R}) \rightarrow \text{Pol}(T^*\mathbb{R}) : a_k(x) \frac{d^k}{dx^k} + \cdots + a_1(x) \frac{d}{dx} + a_0(x) \mapsto \xi^\delta \sum_{p=0}^k \xi^k \bar{a}_p(x)$$

where (x, ξ) denotes the standard coordinates on $T^*(\mathbb{R})$ and

$\bar{a}_p(x) \in \mathcal{F}_{\delta+p}(\mathbb{R}) \cong C^\infty(\mathbb{R})$. We use the general formula given in [2] and make it explicit in the case of the space of second linear differential operators.

The vector space $\text{Pol}(T^*\mathbb{R})$ of polynomials of degree $\leq k$ is also called the space of symbols denoted $\mathcal{S}_\delta^k(\mathbb{R})$. We can see that the vector space $\mathcal{S}_\delta^k(\mathbb{R})$ is also isomorphic to the vector space $\mathcal{F}_{\delta+k}(\mathbb{R}) \oplus \mathcal{F}_{\delta+k-1}(\mathbb{R}) \oplus \dots \oplus \mathcal{F}_\delta(\mathbb{R})$.

The goal of this paper is to compute, on the one hand, the Casimir operator on the \mathfrak{sl}_2 -module $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ and on the other hand, the Casimir operators on the \mathfrak{sl}_2 -module $\mathcal{S}_\delta^k(\mathbb{R})$ of principal symbols and on the \mathfrak{sl}_2 -module $\mathcal{F}_{\delta+k}(\mathbb{R})$ of tensor densities. We finally establish the Schur lemma.

2. Notation and Problem Setting

2.1. Tensor Densities and Differential Operators on \mathbb{R}

Firstly, we begin by the definition of a tensor-density on the real line [1]-[3]. Consider the determinant bundle $\wedge^1 T\mathbb{R} \rightarrow \mathbb{R}$.

Definition 2.1. A homogeneous function of degree λ on the complement of the zero section $\wedge^1 T\mathbb{R} \setminus \mathbb{R}$ of the determinant bundle

$$F : \wedge^1 T\mathbb{R} \setminus \mathbb{R} \rightarrow \mathbb{R} : k\omega \mapsto k^\lambda F(\omega)$$

is called tensor-density of degree λ on \mathbb{R} .

By $\mathcal{F}_\lambda(\mathbb{R})$, we denote the space of tensor-densities of degree $-\lambda$. It is clear that on the real line a tensor-density ϕ takes the form $\phi = \phi(x)(dx)^{-\lambda} \in \mathcal{F}_\lambda(\mathbb{R})$.

Consider a 1-parameter family of $\text{Vect}(\mathbb{R})$ -actions on $C^\infty(\mathbb{R})$ defined by

$$L_{f(x)\frac{d}{dx}}^\lambda(\phi) = f(x)\phi'(x) - \lambda f'(x)\phi(x) \tag{2.1}$$

where $\phi(x) \in C^\infty(\mathbb{R})$ and $\lambda \in \mathbb{R}$. By $\mathcal{F}_\lambda(\mathbb{R})$, we denote the corresponding $\text{Vect}(\mathbb{R})$ -module structure on $C^\infty(\mathbb{R})$. It is clear that $L_{f(x)\frac{d}{dx}}^\lambda$ is the operator of Lie derivative on $\mathcal{F}_\lambda(\mathbb{R})$.

Now, we define a differential operator A as follow.

Definition 2.2. A differential operator A is a linear operator

$$A : \mathcal{F}_\lambda(\mathbb{R}) \rightarrow \mathcal{F}_\mu(\mathbb{R}), \lambda, \mu \in \mathbb{R}.$$

By $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$, we denote the space of the all k -order differential operators on \mathbb{R} .

Thus a k -order differential operator on \mathbb{R} is defined by

$$A(\phi) = a_k(x)\frac{d^k\phi}{dx^k} + a_{k-1}(x)\frac{d^{k-1}\phi}{dx^{k-1}} + \dots + a_1(x)\frac{d\phi}{dx} + a_0(x)\phi, \tag{2.2}$$

where $a_i(x), \phi(x) \in C^\infty(\mathbb{R})$ and $\phi \in \mathcal{F}_\lambda(\mathbb{R})$.

The action of $\text{Vect}(\mathbb{R})$ on the spaces $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ is defined by

$$L_X^{\lambda\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda$$

where L_X^λ is given by (2.1). Thus, we obtain a 1-parameter family of $\text{Vect}(\mathbb{R})$ -modules on $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$. The spaces $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ are endowed with their structure of $\text{Vect}(\mathbb{R})$ -modules. In other words, the module $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ means the representation $(\mathcal{D}^k(\mathbb{R}), L_X^{\lambda\mu})$.

2.2. The Lie Subalgebra \mathfrak{sl}_2 of $\text{Vect}(\mathbb{R})$

Consider [4] the Lie algebra \mathfrak{sl}_2 of the special linear group $\text{SL}(2)$ which consists of all 2×2 matrices with trace zero where we use the basis

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Then we obtain the commutator table

[...]	A_1	A_2	A_3
A_1	0	A_1	$-2A_2$
A_2	$-A_1$	0	A_3
A_3	$2A_2$	$-A_3$	0

such that the structure constants are

$$C_{12}^1 = C_{23}^3 = 1, C_{32}^3 = C_{21}^1 = -1, C_{13}^2 = -2, C_{31}^2 = 2.$$

Now consider the three-dimensional Lie algebra spanned by the following vector fields

$$v_1 = \partial_x, v_2 = x\partial_x, v_3 = x^2\partial_x. \tag{2.4}$$

The commutator table for this Lie algebra is as follows:

[...]	v_1	v_2	v_3
v_1	0	v_1	$2v_2$
v_2	$-v_1$	0	v_3
v_3	$-2v_2$	$-v_3$	0

If we replace v_3 by $-v_3 = -x^2\partial_x$, then we find the same commutator table as that of \mathfrak{sl}_2 with basis given by (2.3). This shows that there is a local action of the special linear group $\text{SL}(2)$ on the real line with $\partial_x, x\partial_x, -x^2\partial_x$ serving as the infinitesimal generators. We can see that this group action is just the projective group

$$x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2)$$

being the real analogue of the complex group of linear fractional transformations. This shows that the Lie algebra \mathfrak{sl}_2 with the basis (2.3) can be embedded as a Lie subalgebra of $\text{Vect}(\mathbb{R})$ generated by the basis $\{X = \partial_x, H = x\partial_x, Y = -x^2\partial_x\}$ of vector fields.

2.3. Casimir Operator of a Representation

Definition 2.3. The Killing form of a Lie algebra \mathfrak{g} on the field \mathbb{K} is a

symmetric bilinear application defined by

$$\beta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K} : (A, B) \mapsto \text{tr}(ad(A)ad(B))$$

where ad denotes the adjoint representation of \mathfrak{g} .

Definition 2.4. A Lie algebra is semi-simple if and only if its Killing form is nondegenerate.

In this case, the Killing form defines a duality in the Lie algebra and obtain the following definition.

Definition 2.5. If \mathfrak{g} is an n -dimensional semi-simple Lie algebra, then for every basis (u_1, \dots, u_n) of \mathfrak{g} , there exists the dual basis (u_1^*, \dots, u_n^*) such that $\beta(u_i, u_j^*) = \delta_{ij}$ for all $i, j \leq n$. Such basis is dual-Killing.

We can define the Casimir operator associated to any representation (V, ρ) of a semi-simple Lie algebra as follow.

Definition 2.6. If (V, ρ) is a representation of a semi-simple Lie algebra \mathfrak{g} , then the casimir operator of this representation is defined by

$$C_\rho : V \rightarrow V : x \mapsto \sum_{i=1}^n \rho(u_i)\rho(u_i^*)x$$

where $(u_i : i \leq n)$ and $(u_i^* : i \leq n)$ are given by the definition 2.5.

Proposition 2.7 For a representation (E, ρ) of a semi simple Lie algebra \mathfrak{g} , the following hold:

i) The Casimir operator C_ρ is an intertwining operator of the representation (E, ρ) and

$$\rho(x) \circ C_\rho = C_\rho \circ \rho(x), \forall x \in \mathfrak{g}. \tag{2.5}$$

ii) If (E', ρ') is another representation of \mathfrak{g} and if

$$T : (E, \rho) \rightarrow (E', \rho') \tag{2.6}$$

is an intertwining operator, i.e.

$$T \circ \rho(x) = \rho'(x) \circ T, \forall x \in \mathfrak{g}$$

then

$$C_{\rho'} \circ T = T \circ C_\rho \tag{2.7}$$

iii) If $\gamma : (E, \rho) \rightarrow (E', \rho')$ is an isomorphism of vector spaces E and E' and v is an eigenvector of C_ρ associated to the eigenvalue α , then $\gamma(v)$ is an eigenvector of $C_{\rho'}$ with the same eigenvalue α .

3. Main Results

3.1. Explicit Formula of Casimir Operator

We compute explicitly this result in this theorem.

Theorem 3.1. Let's consider $\lambda \neq \mu \neq 0$. Then the Sturm-Liouville operator

$A = \frac{d^2}{dx^2} + u(x)$ at $\mathcal{F}_{\frac{1}{2}}(\mathbb{R})$ of the $-\frac{1}{2}$ -densities having values in the space

$\mathcal{F}_{-\frac{3}{2}}(\mathbb{R})$ of $\frac{3}{2}$ -densities on the real line.

Proof. Let's consider the Sturm-Liouville operator $A = \frac{d^2}{dx^2} + u(x)$. This operator acts as an differential operator

$$A: \mathcal{F}_\lambda(\mathbb{R}) \rightarrow \mathcal{F}_\mu(\mathbb{R}).$$

We must compute λ and μ by using the following formula $A(L_X^\lambda(\phi)) = L_X^\mu(A(\phi))$, where $\phi \in \mathcal{F}_\lambda(\mathbb{R})$.

Computing the expression $A(L_X^\lambda(\phi))$, we have

$$\begin{aligned} A(L_X^\lambda(\phi)) &= \left(\frac{d^2}{dx^2} + u(x) \right) (X(x)\phi' - \lambda X'(x)\phi) \\ &= \frac{d^2}{dx^2} (X(x)\phi' - \lambda X'(x)\phi) + u(x)X(x)\phi' - \lambda u(x)X'(x)\phi. \end{aligned}$$

Further computations provide that $A(L_X^\lambda(\phi))$ is equal to

$$\begin{aligned} &(1 - 2\lambda)X''(x)\phi' + (2 - \lambda)X'(x)\phi'' + X(x)\phi''' - \lambda X'''(x)\phi \\ &+ u(x)X(x)\phi' - \lambda u(x)X'(x)\phi. \end{aligned}$$

Now, we compute the expression $L_X^\mu(A(\phi))$, and have

$$\begin{aligned} L_X^\mu(A(\phi)) &= (X(x) - \mu X'(x))(\phi'' + u(x)\phi) \\ &= X(x)\phi'' + X(x)u'(x)\phi + X(x)u(x)\phi' \\ &\quad - \mu X'(x)\phi'' - \phi X'(x)u(x)\phi. \end{aligned}$$

Equating the corresponding monoms, we obtain the following:

$$\begin{cases} 1 - 2\lambda = 0 \\ 2 - \lambda = -\mu \end{cases}$$

and hence $\lambda = \frac{1}{2}$ and $\mu = -\frac{3}{2}$. □

Theorem 3.2. Consider the spaces $\mathcal{D}_{\lambda,\mu}^2(\mathbb{R})$ and the representation $(\mathcal{D}_{\lambda,\mu}^2(\mathbb{R}), \rho)$ of the semi-simple Lie algebra \mathfrak{sl}_2 . The Casimir operator C_ρ is an intertwining operator and it is a multiple of the identity.

Proof. Consider the differential operator

$A = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x) \in \mathcal{D}_{\lambda,\mu}^2(\mathbb{R})$. With respect to the basis

$$X = \frac{d}{dx}, H = x\frac{d}{dx}, Y = -x^2\frac{d}{dx}$$

of the Lie algebra \mathfrak{sl}_2 it corresponds a matrix basis

$$\left\{ A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Due to the subsection 2.3, we compute its dual basis and find the vector fields $X^* = Y, H^* = 2H, Y^* = X$ corresponding to the matrix of the dual basis. Therefore, we can compute the Casimir operator by using the formula

$$C_\rho(A) = L_X^\lambda \circ L_Y^\lambda(A) + L_Y^\lambda \circ L_X^\lambda(A) + L_H^\lambda \circ L_{2H}^\lambda(A).$$

Firstly, to see that C_ρ (i.e., C_rho) is an intertwining operator, it suffices to compute the commutators and verify that

$$[C_\rho, L_X^\lambda] = [C_\rho, L_H^\lambda] = [C_\rho, L_Y^\lambda] = 0.$$

Secondly, the contribution of the first term of $C_{L_X}(A)$ is

$$L_X^\lambda \circ L_Y^\lambda(A) = L_X^\lambda \left(-x^2 \frac{d}{dx} + 2\lambda x \right) \left(a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \right)$$

and we find

$$\begin{aligned} L_X^\lambda \circ L_Y^\lambda(A) = & -2xa_2'(x) \frac{d^2}{dx^2} - x^2 a_2''(x) \frac{d^2}{dx^2} - x^2 a_2'(x) \frac{d^3}{dx^3} - 2xa_2(x) \frac{d^3}{dx^3} \\ & - x^2 a_2'(x) \frac{d^3}{dx^3} - x^2 a_2(x) \frac{d^4}{dx^4} - 2xa_1'(x) \frac{d}{dx} - x^2 a_1''(x) \frac{d}{dx} \\ & - x^2 a_1'(x) \frac{d}{dx} - 2xa_1(x) \frac{d^2}{dx^2} - x^2 a_1'(x) \frac{d^2}{dx^2} - x^2 a_1(x) \frac{d^3}{dx^3} \\ & - 2xa_0'(x) - x^2 a_0''(x) + 2\lambda a_2(x) \frac{d^2}{dx^2} + 2\lambda xa_2'(x) \frac{d^2}{dx^2} \\ & + 2\lambda xa_2'(x) \frac{d^3}{dx^3} + 2\lambda xa_1(x) \frac{d}{dx} + 2\lambda xa_1'(x) \frac{d}{dx} \\ & + 2\lambda xa_1(x) \frac{d^2}{dx^2} + 2\lambda a_0(x) + 2\lambda xa_0'(x). \end{aligned}$$

For the second term of $C_\rho(A)$, we find

$$\begin{aligned} L_Y^\lambda \circ L_X^\lambda(A) = & -x^2 a_2''(x) \frac{d^2}{dx^2} - x^2 a_2'(x) \frac{d^3}{dx^3} + 2\lambda xa_2'(x) \frac{d^2}{dx^2} - x^2 a_2'(x) \frac{d^3}{dx^3} \\ & - x^2 a_2(x) \frac{d^4}{dx^4} + 2\lambda xa_2(x) \frac{d^3}{dx^3} - x^2 a_1''(x) \frac{d}{dx} - x^2 a_1'(x) \frac{d^2}{dx^2} \\ & + 2\lambda xa_1'(x) \frac{d}{dx} - x^2 a_1(x) \frac{d^2}{dx^2} - x^2 a_1(x) \frac{d^2}{dx^3} + 2\lambda xa_1(x) \frac{d^2}{dx^2} \\ & - x^2 a_0''(x) + 2\lambda xa_0'(x) \end{aligned}$$

and for the third term,

$$\begin{aligned} L_H^\lambda \circ L_{2H}^\lambda(A) = & 2xa_2'(x) \frac{d^2}{dx^2} + 2x^2 a_2''(x) \frac{d^2}{dx^2} + 2x^2 a_2'(x) \frac{d^3}{dx^3} - 2\lambda xa_2'(x) \frac{d^2}{dx^2} \\ & + 2xa_2(x) \frac{d^3}{dx^3} + 2x^2 a_2'(x) \frac{d^3}{dx^3} + 2xa_2(x) \frac{d^4}{dx^4} - 2\lambda xa_2(x) \frac{d^3}{dx^3} \\ & + 2xa_1'(x) \frac{d}{dx} + 2x^2 a_1''(x) \frac{d}{dx} + 2x^2 a_1'(x) \frac{d^2}{dx^2} - 2\lambda xa_1'(x) \frac{d}{dx} \\ & + 2xa_1(x) \frac{d^2}{dx^2} + 2x^2 a_1'(x) \frac{d^2}{dx^2} + 2x^2 a_1(x) \frac{d^3}{dx^3} - 2\lambda xa_1(x) \frac{d^2}{dx^2} \\ & + 2xa_0'(x) + 2xa_0''(x) - 2\lambda xa_2' \frac{d^2}{dx^2} - 2\lambda xa_2(x) \frac{d^3}{dx^3} \\ & + 2\lambda^2 a_2(x) \frac{d^2}{dx^2} - 2\lambda xa_1'(x) \frac{d}{dx} - 2\lambda xa_1(x) \frac{d^2}{dx^2} + 2\lambda^2 a_1(x) \frac{d}{dx} \\ & - 2\lambda xa_0'(x) + 2\lambda^2 a_2(x). \end{aligned}$$

The sum of the all contributions of $C_\rho(A)$ gives

$$C_\rho\left(a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right) = 2\lambda(\lambda + 1)\left(a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right).$$

Thus, we conclude that $C_\rho = 2\lambda(\lambda + 1)\text{Id}$.

□

We can do the similar computations on the space $\mathcal{D}_{\lambda\mu}^3(\mathbb{R})$ and generalize these computations on the space $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ of differential operators of an arbitrary order on the real line.

Theorem 3.3. Consider the space $\mathcal{D}_{\lambda\mu}^k(\mathbb{R})$ with k an arbitrary integer and the representation $(\mathcal{D}_{\lambda\mu}^k(\mathbb{R}), \rho)$ of the semi-simple Lie algebra \mathfrak{sl}_2 . Then the Casimir operator C_ρ is a intertwining operator and it is a multiple of the identity.

Proof. The proof is similar to that of the previous Theorem 3.2. Considering a k -order differential operator on \mathbb{R} as follows

$$A = a_k(x)\frac{d^k}{dx^k} + a_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \dots + a_1(x)\frac{d}{dx} + a_0(x),$$

we obtain

$$C_\rho = 2\lambda(\lambda + 1)\text{Id}.$$

□

3.2. The \mathfrak{sl}_2 -Equivariant Symbol Map

We begin with the computations for $k = 2$.

Proposition 3.4. The Lie algebra $\text{Vect}(\mathbb{R})$ acts on $\mathcal{D}_{\lambda\mu}^2(\mathbb{R})$ by

$$L_X^{\lambda\mu}(A) = a_2^X(x)\frac{d^2}{dx^2} + a_1^X(x)\frac{d}{dx} + a_0^X(x) \tag{3.1}$$

where

$$a_2^X(x) = L_X^{\delta+2}(a_2)$$

$$a_1^X(x) = L_X^{\delta+1}(a_1) + (2\lambda - 1)X'(x)a_2(x)$$

$$a_0^X(x) = L_X^\delta(a_0) + \lambda(a_1(x)X''(x) + a_2(x)X'''(x))$$

Proof. We use the action $L_X^{\lambda\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda$ and compute

$$\begin{aligned} L_X^{\lambda\mu}(A) &= \left(X(x)\frac{d}{dx} - \mu X'(x)\right)\left(a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right) \\ &\quad - \left(a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right)\left(X(x)\frac{d}{dx} - \lambda X'(x)\right). \end{aligned}$$

□

Consider the cotangent bundle $T^*\mathbb{R} \cong \mathbb{R}^2$ with the coordinate ξ on the fiber. The space $\text{Pol}(T^*\mathbb{R})$ of functions on the cotangent bundle $T^*\mathbb{R}$ of polynomials on the fibers is also denoted by $\mathcal{S}_\delta(\mathbb{R})$ and $\mathcal{S}_\delta^k(\mathbb{R})$ is its subspace. The space of symbols is isomorphic to sum of subspaces of $-(\delta + p)$ -tensor densities as follows:

$$\mathcal{S}_\delta^k(\mathbb{R}) \cong \bigoplus_{p=0}^k \mathcal{F}_{\delta+p}$$

In a local coordinate system (x, ξ) ,

$$P(x, \xi) = \xi^\delta \sum_{p=0}^k \xi^p a_p(x), \quad a_p(x) \in C^\infty(\mathbb{R}).$$

As in [2], for an element $A \in \mathcal{D}_{\lambda\mu}^k(\mathbb{R})$, we define the symbol map $\sigma_{\lambda\mu}$ as a map from $\mathcal{D}_{\lambda}^k(\mathbb{R})$ to $\mathcal{S}_\delta^k(\mathbb{R})$

$$\sigma_{\lambda\mu} : \mathcal{D}_{\lambda\mu}^k(\mathbb{R}) \rightarrow \mathcal{S}_\delta^k(\mathbb{R}) : A \mapsto \sigma_{\lambda\mu}(A) = \xi^\delta \sum_{p=0}^k \xi^p \bar{a}_p(x), \quad \delta = \mu - \lambda \quad (3.2)$$

where $\bar{a}_p(x)$ are defined by

$$\bar{a}_p(x) := \sum_{j=p}^k \alpha_p^j a_p^{(j-p)}(x) \quad (3.3)$$

and the coefficients α_p^j are given by

$$\alpha_p^j = \frac{\binom{j}{p} \binom{2\lambda - p}{2\lambda - j}}{\binom{2\delta + j + p + 1}{2\delta + 2p + 1}} \quad (3.4)$$

Due to the formula (3.4), for the particular case of $k = 2$, we obtain that

$$\sigma_{\lambda\mu}(A) = \xi^{\delta+2} \bar{a}_2(x) + \xi^{\delta+1} \bar{a}_1(x) + \xi^\delta \bar{a}_0(x) \quad (3.5)$$

where

$$\begin{aligned} \bar{a}_2(x) &= a_2(x) \\ \bar{a}_1(x) &= a_1(x) + \frac{2\lambda - 1}{\delta + 2} a_2'(x) \\ \bar{a}_0(x) &= a_0(x) + \frac{\lambda}{\delta + 1} a_1'(x) + \frac{2\lambda(2\lambda - 1)}{(2\delta + 1)(2\delta + 3)} a_2''(x). \end{aligned}$$

The Lie algebra $\text{Vect}(\mathbb{R})$ acts on an element $\sigma_{\lambda\mu}(A) = P(x, \xi) = \xi^\delta \sum_{p=0}^2 \xi^p \bar{a}_p$ of $\mathcal{S}_\delta^2(\mathbb{R})$ as follows:

$$L_X^\delta(P(x, \xi)) = \xi^{\delta+2} L_X^{\delta+2}(\bar{a}_2) + \xi^{\delta+1} L_X^{\delta+1}(\bar{a}_1) + \xi^\delta L_X^\delta(\bar{a}_0) \quad (3.6)$$

or equivalently, if we use the identification

$$\mathcal{S}_\delta^2(\mathbb{R}) \leftrightarrow \mathcal{F}_{\delta+2}(\mathbb{R}) \oplus \mathcal{F}_{\delta+1}(\mathbb{R}) \oplus \mathcal{F}_\delta(\mathbb{R}),$$

such that

$$\xi^\delta \sum_{p=0}^2 \xi^p \bar{a}_p(x) \leftrightarrow F = (\bar{a}_2(x), \bar{a}_1(x), \bar{a}_0(x)),$$

this action is written as follows:

$$L_X^\delta(\bar{a}_2(x), \bar{a}_1(x), \bar{a}_0(x)) = (L_X^{\delta+2}(\bar{a}_2), L_X^{\delta+1}(\bar{a}_1), L_X^\delta(\bar{a}_0)). \quad (3.7)$$

Theorem 3.5. A second differential operator $L_X^{\lambda\mu}(A)$ has a symbol \mathfrak{sl}_2 -equivariant $\sigma_{\lambda\mu}(L_X^{\lambda\mu}(A)) = \xi^\delta \sum_{p=0}^2 \xi^p \bar{a}_p^X(x)$ where $\bar{a}_p^X(x) = L_X^{\delta+2}(\bar{a}_p)$.

Proof. The formula defined by (3.3) implies that $\bar{a}_p^X = \sum_{j=p}^2 \alpha_p^j (a_j^X)^{(j-p)}$. Replacing $a_j^X(x)$ by its values in (3.1) and using (3.5) which show the relation between $\bar{a}_j(x)$ and $a_j(x)$, we obtain

$$\begin{aligned} \bar{a}_2^X &= a_2^X(x) = L_X^{\delta+2}(a_2) = L_X^{\delta+2}(\bar{a}_2) \\ \bar{a}_1^X &= a_1^X(x) + \frac{2\lambda-1}{\delta+1} \{X(x)a_2'(x) - (\delta+2)a_2(x)\}' \\ &= L_X^{\delta+1} \left(a_1(x) + \frac{2\lambda-1}{\delta+1} a_2'(x) \right) \\ &= L_X^{\delta+1}(\bar{a}_1) \\ \bar{a}_0^X &= a_0^X + \frac{\lambda}{\delta+1} \{L_X^{1+\delta}(a_1) + (2\lambda-1)X''(x)a_2(x)\} \\ &\quad + \frac{2\lambda(2\lambda-1)}{(2\delta+3)(2\delta+2)} (L_X^{\delta+2}(a_2))'' \\ &= L_X^\delta(\bar{a}_0) + \lambda \left(\frac{2\lambda+\delta}{\delta+1} - \frac{(2\lambda-1)(\delta+2)}{(2\delta+3)(\delta+1)} \right) X'''(x)a_2(x). \end{aligned}$$

For $X \in \mathfrak{sl}_2$ which are at most of degree two, these last formulae become

$$\bar{a}_p^X = L_X^{p+\delta}(\bar{a}_p), \quad p = 0, 1, 2$$

and $\sigma_{\lambda,\mu}$ defines an \mathfrak{sl}_2 -equivariant symbol map. □

Remark 3.6. The computations were made in the case of $k = 2$, but these computations can be generalized to arbitrary k (see [2]).

Definition 3.7. A symbol $P(x, \xi) = \xi^\delta \sum_{p=0}^k \xi^p \bar{a}_p(x)$ of the higher term $\xi^{\delta+k} \bar{a}_k(x)$ of $P(x, \xi)$, is called a principal symbol on the space $\mathbb{S}_\delta^k(\mathbb{R})$. Denote the space of principal symbols on \mathbb{R} by $\mathbb{S}_\delta^k(\mathbb{R})$.

Note that

$$\mathbb{S}_\delta^k(\mathbb{R}) = \{S = \xi^{\delta+k} \bar{a}_k(x), \bar{a}_k(x) \in \mathcal{F}_{\delta+k}(\mathbb{R})\}.$$

We use the isomorphism $\gamma: \mathbb{S}_\delta^k(\mathbb{R}) \rightarrow \mathcal{F}_{\delta+k}(\mathbb{R}): \xi^{\delta+k} \bar{a}_k(x) \mapsto \bar{a}_k(x)$ of vector spaces and to transport the structure of \mathfrak{sl}_2 -modules. Thus, the Lie algebra \mathfrak{sl}_2 acts on $\mathbb{S}_\delta^k(\mathbb{R})$ by the formula

$$L_X^\delta(S) := \xi^{\delta+k} L_X^{\delta+k}(\bar{a}_k) \tag{3.8}$$

while it acts on the space $\mathcal{F}_{\delta+k}(\mathbb{R})$ by

$$\mathcal{L}_X^\delta(\bar{a}_k) = L_X^{\delta+k}(\bar{a}_k). \tag{3.9}$$

The spaces $\mathbb{S}_\delta^k(\mathbb{R})$ and $\mathcal{F}_{\delta+k}(\mathbb{R})$ become the \mathfrak{sl}_2 -modules.

Now, we compute the Casimir operators on the \mathfrak{sl}_2 -modules $\mathbb{S}_\delta^k(\mathbb{R})$ and the $\mathcal{F}_{\delta+k}(\mathbb{R})$.

Theorem 3.8. Consider $\gamma: \mathbb{S}_\delta^k(\mathbb{R}) \rightarrow \mathcal{F}_{\delta+k}(\mathbb{R}): \xi^{\delta+k} \bar{a}_k(x) \mapsto \bar{a}_k(x)$. If S is an eigenvector of the Casimir operator $C_{\rho'}$ related to the eigenvalue α on

$\mathbb{S}_\delta^k(\mathbb{R})$, then $\gamma(S)$ is the eigenvector of C_π related to the same eigenvalue α .

Proof. The computation of the Casimir operator $C_{\rho'}$ on $\mathbb{S}_\delta^k(\mathbb{R})$ is obtained by using the formula

$$C_{\rho'}(S) = L_Y^\delta L_X^\delta(S) + 2L_H^\delta L_H^\delta(S) + L_X^\delta L_Y^\delta(S)$$

for the principal symbol S , where the vector fields are $X = \frac{d}{dx}$, $Y = -x^2 \frac{d}{dx}$ and $H = x \frac{d}{dx}$ and the formula (3.8). We obtain that $C_{\rho'}$ is the multiple of the identity on $\mathcal{F}_{\delta+p}(\mathbb{R})$ and

$$C_{\rho'}(S) = 2(\delta+k)(\delta+k+1)\xi^{\delta+k}\bar{a}_k(x).$$

Similar computation is done on the space $\mathcal{F}_{\delta+k}(\mathbb{R})$ using

$$C_\pi(\bar{a}_k) = \mathcal{L}_Y^\delta \mathcal{L}_X^\delta(\bar{a}_k) + 2\mathcal{L}_H^\delta \mathcal{L}_H^\delta(\bar{a}_k) + \mathcal{L}_X^\delta \mathcal{L}_Y^\delta(\bar{a}_k),$$

and the formula (3.9). The Casimir operator C_π is related to the projection

$$\pi: \bigoplus_{p=0}^k \mathcal{F}_{\delta+p}(\mathbb{R}) \rightarrow \mathcal{F}_{\delta+k}(\mathbb{R}): (\bar{a}_k(x), \bar{a}_{k-1}(x), \dots, \bar{a}_0(x)) \mapsto \bar{a}_k(x).$$

Thus the common eigenvalue is $\alpha = 2(\delta+k)(\delta+k+1)$. □

We note that these results can be used to compute the explicit formulas of the \mathfrak{sl}_2 -equivariant quantization on the real line. In other words, it is to compute the isomorphism of the representation of the Lie algebra \mathfrak{sl}_2 between the space of symbols on the real line and the spaces of differential operators on the real line satisfying the normalisation condition [5] [6].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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