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The Extended Non-Elementary Amplitude Functions as Solutions to the Forced and Damped Pendulum Equation, Ueda's Chaotic Nonlinear Oscillator, the Shimizu-Morioka System, Lorenz System, Rössler System, Sprott-Linz F Chaotic Attractor

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Abstract

In this paper, we define some non-elementary amplitude functions that are giving solutions to some second-order nonlinear ODEs with forcing term and systems of ODEs with chaotic behavior, such as the chaotic cases of the Lorenz system. For this purpose, we will introduce a special function, that is a function of the dependent variable φ and the independent variable t, and place it into the solution-function. These solutions are equal to the amplitude, or upper limit of integration in a non-elementary integral that can be arbitrary. The first derivative to these amplitude functions contains one or two integrals that disappear at the second derivation or at the third derivation. We are giving the solutions a name, a symbol and putting them into a group of functions and into the context of other functions. Using these integral amplitude functions, we can define solutions to some well-known second-order ODEs and systems of ODEs exhibiting chaotic behavior.

Keywords

The Forced Pendulum Equation, Ueda's Chaotic Oscillator, Shimizu-Morioka System, Lorenz System, Rössler System, Sprott-Linz F Chaotic Attractor

1. Introduction

On page 1 in the book [1], we find the sentences: very few ordinary differential equations have explicit solutions expressible in finite terms. This is not because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice.

This is the main reason for this work. It should be possible to do something with this problem. If we don't have enough tools in our mathematical toolbox, we must make the tools first. For this problem we will attempt to define some new functions, or making extensions to the non-elementary amplitude functions, in order to define solutions to forced second order ODEs or to systems of ODEs exhibiting chaotic behavior.

If it is not possible to find a solution to a distinct second order nonlinear ODE, or to a system of ODEs, working in the traditional ways, why not just define a solution? How must this solution look like, and where in the landscape of functions can we place this solution? If we are thinking that every reasonable problem has a solution, even to dynamical systems, that exhibits chaotic behavior, let us try to define solutions to some well-known ODEs, or systems of ODEs, like the forced and damped pendulum equation or the chaotic Lorenz system. We will use the one-integral and two-integral non-elementary amplitude functions defined in [2], and combine them with elementary functions.

Studies of computer-generated solutions of forced second-order equations such as Duffing's equation and three-dimensional autonomous systems have revealed unexpectedly complex solution structures arising from what might appear to be relatively simple nonlinear differential equations [1].

Since a pendulum moves very slowly near the top of the swing, a small change in initial conditions can make a big change in the timing and hence cause a change in direction of the pendulum. This sort of behavior, which we now call chaos, can be observed in a system as simple as the periodically forced pendulum, it should not be surprising that it can also be found in nature [3].

If a bundle of solutions containing infinitely many unstable periodic solutions is asymptotically orbitally stable, a chaotic phenomenon appears, which results from the small uncertain factors in the real system [4].

The Lorenz equations, as they are now known, arose in a model for convective motion in the atmosphere. The solutions calculated by computer display very complex behavior for a wide range of parameter values. By complex behavior we include seemingly random or "chaotic" output from the system although the solutions remain deterministic in terms of their initial values and there is no "random" input [1].

The Rössler attractor is the attractor for the Rössler system, a system of three nonlinear ODEs originally studied by Otto Rössler in the 1970s. These differential equations define a continuous-time dynamical system that exhibits chaotic dynamics associated with the fractal properties of the attractor. Rössler interpreted it as a formalization of a taffy-pulling machine [5]. We have a chaotic attracting set for $c = 4.449$. This is known as a strange attractor [1].

2. The Amplitude Functions

2.1. The Jacobi Amplitude Function

This function is defined as equal to the Jacobi amplitude φ [6]:

$$am(u,k) = \varphi = \int_0^u dn(u',k)\,\mathrm{d}u' \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} am(u,k) = dn(u,k) \tag{2}$$

The second order nonlinear differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k^2 \sin x \cos x, \quad 0 \le k < 1 \tag{3}$$

has $am(t,k)$ as solution:

$$x(t) = am(t,k) = \varphi = \varphi(t) = \int_0^t dn(u,k)\,\mathrm{d}u \tag{4}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = dn(t) = \sqrt{1-k^2\sin^2\varphi} \tag{5}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{1}{2}\frac{\left(-k^2\right)2\sin\varphi\cos\varphi\dfrac{\mathrm{d}\varphi}{\mathrm{d}t}}{\sqrt{1-k^2\sin^2\varphi}} = -k^2\sin x\cos x \tag{6}$$

2.2. The General Amplitude Function [7]

Define a function

$$amp(u) = \varphi = \varphi(u) = \int_0^u adn(u')\,\mathrm{d}u', \; -\infty < u, \varphi < \infty \tag{7}$$

where φ is the amplitude or upper limit of integration in a non-elementary integral, that can be arbitrary.

$$\frac{\mathrm{d}}{\mathrm{d}u} amp(u) = \frac{\mathrm{d}\varphi}{\mathrm{d}u} = adn(u) = f(\varphi) \tag{8}$$

The function f can be arbitrary. It may also be a square root or a fraction. We have named the derivative to $amp(u)$ for $adn(u)$, in order to show the relationship to the Jacobi elliptic function $dn(u,k)$.

When $f(\varphi) = \sqrt{1-k^2\sin^2\varphi}$, is $adn(u) = dn(u,k)$ and $amp(u) = am(u,k)$ (9)

The integral amplitude functions are an extension to the general amplitude function [2]. The first derivative to the amplitude function contains one or two integrals. The usefulness of these functions is to define solutions when the second-order ODE is short, and when the equations in a system of ODEs are short, such as the Lorenz system.

3. The One-Integral Amplitude Function

3.1. Definition

Definition 3.1: The one-integral amplitude function.

Suppose a solution $x(t)$ is equal to the amplitude, or upper limit of integration in a non-elementary integral, and the first derivative to this solution contains an integral, then this solution is an example of the one-integral amplitude functions.

3.2. The Damped Pendulum Equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\frac{\mathrm{d}x}{\mathrm{d}t}+\omega^2\sin x=0 \tag{10}$$

$$\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t=-k\int\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t-\omega^2\int\sin x\mathrm{d}t \tag{11}$$

Summarizing the integration constants to C.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=C-kx-\omega^2\int\sin x\mathrm{d}t \tag{12}$$

Define a one-integral amplitude function as solution $x(t)=amp(t)=\varphi=\varphi(t)$.

I have consistently used φ as notation for the amplitude, or upper limit of integration, and x as the dependent variable in the ODEs. In the group of amplitude functions is the solution $x(t)=\varphi=\varphi(t)$. Therefor the change from φ to x, or from x to φ can be a bit confusing, because x and φ are the same functions, just different notation. When using the amplitude function, I change the notation of the dependent variable from x to φ, just to separate the solution x from the function φ.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-k\varphi-\omega^2\int\sin\varphi\mathrm{d}t \tag{13}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{1}{C-k\varphi-\omega^2\int\sin\varphi\mathrm{d}t} \tag{14}$$

$$t=t(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{C-k\theta-\omega^2\int\sin\varphi\mathrm{d}t} \tag{15}$$

Here we see that the solution to the damped pendulum equation is an amplitude function, the upper limit of integration in a non-elementary integral. This solution is a member in the group of the one-integral amplitude functions, where the first derivative of the function contains an integral. We can give this function the symbol amp_P, after the pendulum.

$$x(t)=amp_P(t)=\varphi=\varphi(t)=\int_0^t adn_P(u)\mathrm{d}u \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_P(t)=adn_P(t)=C-k\varphi-\omega^2\int\sin\varphi\mathrm{d}t \tag{17}$$

Now, we change the notation of the dependent variable from φ to x.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}amp_P(t)=C-kx-\omega^2\int\sin x\mathrm{d}t \tag{18}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-k\frac{\mathrm{d}x}{\mathrm{d}t}-\omega^2\sin x \tag{19}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 \sin x = 0$$

$x(t) = amp_P(t)$ is a solution to the damped pendulum equation.

3.3. The Forced and Damped Pendulum Equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 \sin x = F\cos\omega t \tag{20}$$

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = -k\int \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t - \omega^2 \int \sin x\mathrm{d}t + F\int \cos\omega t\mathrm{d}t \tag{21}$$

Summarizing the integration constants to C.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C - kx - \omega^2 \int \sin x\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{22}$$

We are doing the same as above, and defining a solution $x(t) = amp(t) = \varphi = \varphi(t)$, and changing the dependent variable from x to φ:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = C - k\varphi - \omega^2 \int \sin\varphi\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{23}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \frac{1}{C - k\varphi - \omega^2 \int \sin\varphi\mathrm{d}t + \frac{F}{\omega}\sin\omega t} \tag{24}$$

$$t = t(\varphi) = \int_0^\varphi \frac{\mathrm{d}\theta}{C - k\theta - \omega^2 \int \sin\varphi\mathrm{d}t + \frac{F}{\omega}\sin\omega t} \tag{25}$$

Here is $\varphi = x(t)$ the upper limit of integration in a non-elementary integral, and satisfy the definition of an amplitude function. We can name this amplitude function $amp_{FP}(t)$, after the forced pendulum.

$$x(t) = amp_{FP}(t) = \varphi = \varphi(t) = \int_0^t adn_{FP}(u)\mathrm{d}u \tag{26}$$

Changing the dependent variable from φ to x:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = C - kx - \omega^2 \int \sin x\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{27}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k\frac{\mathrm{d}x}{\mathrm{d}t} - \omega^2 \sin x + F\cos\omega t \tag{28}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 \sin x = F\cos\omega t$$

The amplitude function $amp_{FP}(t)$ is giving solution to the forced and damped pendulum equation.

$$\text{Notice that } \frac{\mathrm{d}x}{\mathrm{d}t} = y = y_h + y_p \tag{29}$$

$$y_h = C - kx - \omega^2 \int \sin x\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}amp_P(t) \tag{30}$$

$$y_p = \frac{F}{\omega}\sin\omega t = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{F}{\omega^2}\cos\omega t\right) \tag{31}$$

The derivative to the solution $x(t)$ is a sum of the derivative to the solution of the homogeneous equation, and the derivative to the particular solution.

$$amp_{FP}(t) \neq amp_P(t) - \frac{F}{\omega^2}\cos\omega t \tag{32}$$

3.4. Ueda's Chaotic Nonlinear Oscillator

This is a special case of the forced and damped Duffing equation.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + ax + bx^3 = F\cos\omega t \tag{33}$$

$$\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t = -k\int\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t - \int\left(ax + bx^3\right)\mathrm{d}t + F\int\cos\omega t\mathrm{d}t \tag{34}$$

Summarizing the integration constants to C.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C - kx - \int\left(ax + bx^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{35}$$

Define a solution $x(t) = amp(t) = \varphi = \varphi(t)$, and change the dependent variable from x to φ.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = C - k\varphi - \int\left(a\varphi + b\varphi^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{36}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \frac{1}{C - k\varphi - \int\left(a\varphi + b\varphi^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t} \tag{37}$$

Here is t a function of φ.

$$t = t(\varphi) = \int_0^{\varphi}\frac{\mathrm{d}\theta}{C - k\theta - \int\left(a\varphi + b\varphi^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t} \tag{38}$$

The solution is an amplitude function, denoted amp_{FD}, after forced Duffing.

$$x(t) = amp_{FD}(t) = \varphi = \varphi(t) = \int_0^t adn_{FD}(u)\mathrm{d}u \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_{FD}(t) = adn_{FD}(t) = C - k\varphi - \int\left(a\varphi + b\varphi^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{40}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}amp_{FD}(t) = C - kx - \int\left(ax + bx^3\right)\mathrm{d}t + \frac{F}{\omega}\sin\omega t \tag{41}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\frac{\mathrm{d}x}{\mathrm{d}t} - ax - bx^3 + F\cos\omega t \tag{42}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + ax + bx^3 = F\cos\omega t \tag{43}$$

$x(t) = amp_{FD}(t)$ is a solution to the forced and damped Duffing equation.

$$\text{Notice that } \frac{\mathrm{d}x}{\mathrm{d}t} = y = y_h + y_p \tag{44}$$

$$y_h = C - kx - \int \left(ax + bx^3\right) \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t} amp_D\left(t\right) \tag{45}$$

$$y_p = \frac{F}{\omega} \sin \omega t \tag{46}$$

$$amp_{FD}\left(t\right) \neq amp_D\left(t\right) - \frac{F}{\omega^2} \cos \omega t \tag{47}$$

$amp_D\left(t\right)$ is the solution to the unforced damped Duffing equation [7].

When $F = 7.5, k = 0.05, a = 0, b = 1, \omega = 1$, in the forced and damped Duffing equation, we see chaotic behavior. "Explosion of strange attractors exhibited by Duffing's equation" [8]. This is Ueda's chaotic nonlinear oscillator. The amplitude function $amp_{FD}\left(t\right)$ give solution to Ueda's chaotic oscillator.

In the same way as done above, we can define an amplitude function as solution to the forced Van der Pol equation.

4. The Two-Integral Amplitude Function

In this chapter, we will try to define amplitude functions that are giving solutions to systems of ODEs with chaotic behavior. In the previous chapter the derivative of the amplitude function was a function of φ and t, $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = f\left(\varphi, t\right)$. We will use this idea and define amplitude functions where the first and second derivative contains elementary functions of t, that disappear under the derivation, but they are absolutely necessary in order to obtain chaos behavior. But first a definition:

Definition 4.1: The two-integral amplitude function.

Suppose a solution $x\left(t\right)$ is equal to the amplitude, or upper limit of integration in a non-elementary integral, and the first derivative of this solution contain two integrals, where one integral disappears at the second derivation and the second integral disappear at the third derivation, then this solution is an example of the two-integrals amplitude functions.

4.1. The Shimizu-Morioka System

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x - \beta y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= -\alpha z + x^2 \end{aligned} \tag{48}$$

Define an integral function u:

$$u = u\left(\varphi\right) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{C - \beta\theta + \int \varphi\left(f - g\mathrm{e}^{-\alpha t} \int \varphi^2 \mathrm{e}^{\alpha t} \mathrm{d}t\right) \mathrm{d}t} \tag{49}$$

The two integrals under the fraction line are with respect to the independent variable t, in all parts of the integrals, since φ is a function of t. $\varphi = \varphi\left(t\right)$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{C-\beta\varphi+\int\varphi\left(f-g\mathrm{e}^{-\alpha t}\int\varphi^2\mathrm{e}^{\alpha t}\mathrm{d}t\right)\mathrm{d}t} \tag{50}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=C-\beta\varphi+\int\varphi\left(f-g\mathrm{e}^{-\alpha t}\int\varphi^2\mathrm{e}^{\alpha t}\mathrm{d}t\right)\mathrm{d}t \tag{51}$$

Define a two-integral amplitude function as solution:

$$x(t)=amp(t)=\varphi=\varphi(t)$$

Changing the dependent variable from φ to x.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-\beta x+\int x\left(f-\frac{g}{h}z\right)\mathrm{d}t=y \tag{52}$$

If:

$$z(t)=h\mathrm{e}^{-\alpha t}\int\varphi^2\mathrm{e}^{\alpha t}\mathrm{d}t \tag{53}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\beta y+fx-\frac{g}{h}xz \tag{54}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=h(-\alpha)\mathrm{e}^{-\alpha t}\int x^2\mathrm{e}^{\alpha t}\mathrm{d}t+h\mathrm{e}^{-\alpha t}x^2\mathrm{e}^{\alpha t} \tag{55}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-\alpha z+hx^2 \tag{56}$$

Notice that the elementary functions $\mathrm{e}^{-\alpha t}$ and $\mathrm{e}^{\alpha t}$ disappear in the second term of $\frac{\mathrm{d}z}{\mathrm{d}t}$.

We summarize the result, a system that we may name the general Shimizu-Morioka system:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=y\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=fx-\beta y-\frac{g}{h}xz\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=-\alpha z+hx^2\end{aligned} \tag{57}$$

When $f=1,g=1,h=1$, we will get the system that we started with $\alpha=0.45,\beta=0.75$ give chaos behavior, as we can see in **Figure 1**. One initial point.

The solutions to the general Shimizu-Morioka system:

$$x(t)=amp_{SM}(t)=\varphi=\varphi(t)=\int_0^t adn_{SM}(u)\mathrm{d}u, \tag{58}$$

named after Shimizu-Morioka

$$y(t)=C-\beta\varphi+\int\varphi\left(f-g\mathrm{e}^{-\alpha t}\int\varphi^2\mathrm{e}^{\alpha t}\mathrm{d}t\right)\mathrm{d}t \tag{59}$$

$$z(t)=h\mathrm{e}^{-\alpha t}\int\varphi^2\mathrm{e}^{\alpha t}\mathrm{d}t \tag{60}$$

In **Figure 2**, we can see a 3D limit cycle as 2 loops meeting at (0,0,0), and filled with spirals in **Figure 3**.

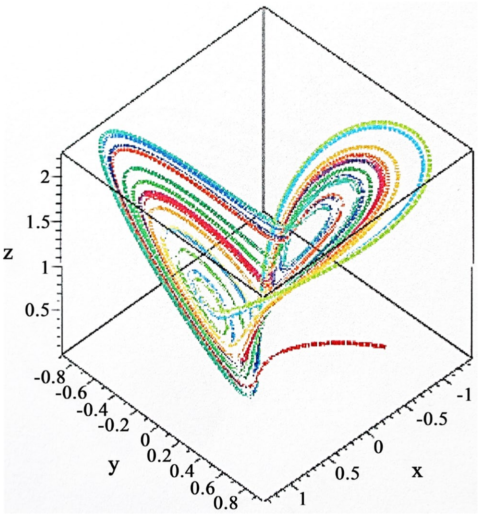

Figure 1. $\alpha = 0.45, \beta = 0.75, f = h = g = 1$.

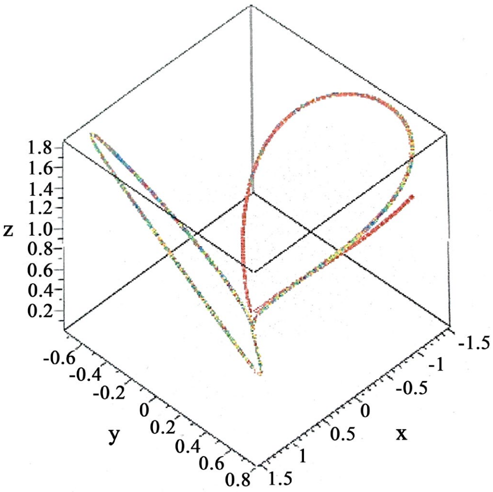

Figure 2. $\beta = 0.844, \alpha = f = g = h = 1$.

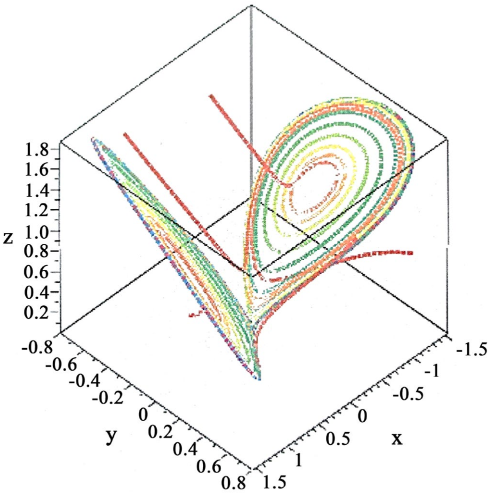

Figure 3. The LC in **Figure 2** is filled with spirals.

In **Figure 4** are the parameter values $f = 3, \beta = 0.8, \alpha = 0.5, h = 0.2, g = 0.2$, one initial point.

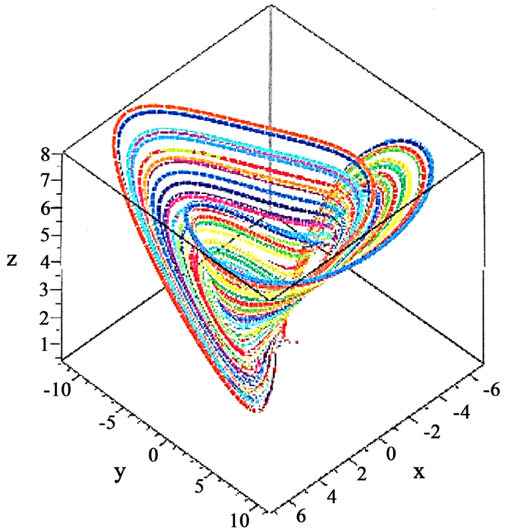

Figure 4. $f = 3, h = 0.2, g = 0.2$.

4.2. The Lorenz System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - y - xz \tag{61}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\beta z + xy$$

As we find it in some textbooks. In order to give chaos behavior, the values of the parameters are:

$$\sigma = 10, \rho = 28, \beta = \frac{8}{3}$$

We repeat the attempt we did in [2] in order to define a function giving solution to the Lorenz system. And then take a step further to define a solution to the chaos cases.

Let us try the solution that we have found to the damped and unforced Duffing equation:

$$x(t) = amp_D(t)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} amp_D(t) = C - kx - \int \left(ax + bx^3\right) \mathrm{d}t \tag{62}$$

Add and subtract cx:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C + cx - kx - cx - \int \left(ax + bx^3\right) \mathrm{d}t \tag{63}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = cx + hy \tag{64}$$

if:

$$y(t)=\frac{C}{h}-\frac{k+c}{h}x-\frac{1}{h}\int\left(ax+bx^3\right)\mathrm{d}t \tag{65}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}(k+c)(cx+hy)-\frac{1}{h}ax-\frac{b}{h}x^3 \tag{66}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(kc+c^2+a\right)x-(k+c)y-xz \tag{67}$$

if:

$$z(t)=\frac{b}{h}x^2 \tag{68}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{2b}{h}x(cx+hy)=2cz+2bxy \tag{69}$$

Then we have the system:

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=cx+hy\\ \frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(kc+c^2+a\right)x-(k+c)y-xz\\ \frac{\mathrm{d}z}{\mathrm{d}t}=2cz+2bxy\end{gathered} \tag{70}$$

This system has the solution:

$$x(t)=amp_D(t)=\varphi=\varphi(t) \tag{71}$$

Changing dependent variable from x to φ:

$$y(t)=\frac{C}{h}-\frac{k+c}{h}\varphi-\frac{1}{h}\int\left(a\varphi+b\varphi^3\right)\mathrm{d}t \tag{72}$$

$$z(t)=\frac{b}{h}\varphi^2 \tag{73}$$

Choose these values of the parameters:

$$c=-10, h=10, k=11, a=-270, b=0.5$$

This gives the system:

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=-10x+10y\\ \frac{\mathrm{d}y}{\mathrm{d}t}=28x-y-xz\\ \frac{\mathrm{d}z}{\mathrm{d}t}=-20z+xy\end{gathered} \tag{74}$$

4.2.1. The Chaotic Lorenz System

This system (74) has three equilibrium points, two spiral sink and one saddle-point. Only the parameter to the z-term in the third equation don’t have the value that is giving chaotic behavior.

This makes us thinking that the solution that is giving chaotic behavior, must have the solution $x(t)=amp_D(t)$ as a special case. The solution $z(t)=\frac{b}{h}\varphi^2$ must be a special case of the solution to the chaos-cases of the Lorenz system. Let us try to add a function to $\frac{b}{h}\varphi^2$, perhaps a function of both φ and t:

$$z(t)=\frac{b}{h}\varphi^2+g(\varphi,t) \tag{75}$$

When $g=0$, we have the special case $amp_D(t)$. Let us try the same term as for $z(t)$ in the Shimizu-Morioka system.

Define an integral function u:

$$u=u(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{C-k\theta-\int\left(a\varphi+b\varphi\left(\varphi^2+g\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t} \tag{76}$$

The two integrals under the fraction line are with respect to the independent variable t in all parts of the integrals, since φ is a function t, $\varphi=\varphi(t)$.

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{C-k\varphi-\int\left(a\varphi+b\varphi\left(\varphi^2+g\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t} \tag{77}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=C-k\varphi-\int\left(a\varphi+b\varphi\left(\varphi^2+g\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t \tag{78}$$

Define a two-integral amplitude function as solution:

$$x(t)=amp(t)=\varphi=\varphi(t)$$

Changing notation of the dependent variable from φ to x:

$$\frac{dx}{dt}=\frac{d\varphi}{dt}=C-kx-\int\left(ax+bx\left(x^2+ge^{-\beta t}\int x^2e^{\beta t}dt\right)\right)dt \tag{79}$$

Add and subtract *cx*:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=C+cx-kx-cx-\int\left(ax+bx\left(x^2+g\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t \tag{80}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=cx+hy\ , \tag{81}$$

if:

$$y(t)=\frac{C}{h}-\frac{k+c}{h}x-\frac{1}{h}\int\left(ax+bx\left(x^2+g\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t \tag{82}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}(k+c)(cx+hy)-\frac{1}{h}ax-\frac{b}{h}x\left(x^2+g\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t\right) \tag{83}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(kc+c^2+a\right)x-(k+c)y-fxz \tag{84}$$

if:

$$z(t)=\frac{b}{fh}x^2+\frac{bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t \tag{85}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{2b}{fh}x(cx+hy)-\frac{\beta bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{bg}{fh}\mathrm{e}^{-\beta t}x^2\mathrm{e}^{\beta t} \tag{86}$$

Add βz on both sides:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}+\beta z=\frac{\beta b}{fh}x^2+\frac{\beta bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{2bc}{fh}x^2+\frac{2b}{f}xy\\ -\frac{\beta bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{bg}{fh}x^2\end{aligned} \tag{87}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-\beta z+\frac{2b}{f}xy+\frac{b}{fh}x^2\left(\beta+2c+g\right) \tag{88}$$

The x^2-term is zero when $\beta+2c+g=0$.

The general Lorenz system with 8 parameters:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=cx+hy\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=-\frac{1}{h}\left(kc+c^2+a\right)x-\left(k+c\right)y-fxz\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=-\beta z+\frac{2b}{f}xy+\frac{b}{fh}x^2\left(\beta+2c+g\right)\end{aligned} \tag{89}$$

The general Lorenz system has the two integral amplitude function $amp_{Lo}\left(t\right)$ as solution.

$$x\left(t\right)=amp_{Lo}\left(t\right)=\varphi=\varphi\left(t\right)=\int_0^t adn_{Lo}\left(u\right)\mathrm{d}u \tag{90}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-k\varphi-\int\left(a\varphi+b\varphi\left(\varphi^2+g\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t$$

$$y\left(t\right)=\frac{C}{h}-\frac{k+c}{h}\varphi-\frac{1}{h}\int\left(a\varphi+b\varphi\left(\varphi^2+g\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t \tag{91}$$

$$z\left(t\right)=\frac{b}{fh}\varphi^2+\frac{bg}{fh}\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t \tag{92}$$

Test the result and change the notation of the dependent variable from φ to x:

Add and subtract cx from $\frac{\mathrm{d}x}{\mathrm{d}t}$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=C+cx-cx-kx-\int\left(ax+bx\left(x^2+g\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t\right)\right)\mathrm{d}t=cx+hy \tag{93}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(k+c\right)\left(cx+hy\right)-\frac{a}{h}x-x\left(\frac{b}{h}x^2+\frac{bg}{h}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t\right) \tag{94}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(kc+c^2+a\right)x-\left(k+c\right)y-fxz \tag{95}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{b}{fh}2x\left(cx+hy\right)-\beta\frac{bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{bg}{fh}\mathrm{e}^{-\beta t}x^2\mathrm{e}^{\beta t} \tag{96}$$

Add βz to both sides:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}+\beta z=\beta\frac{b}{fh}x^2+\beta\frac{bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{2bc}{fh}x^2+\frac{2b}{f}xy\\ -\beta\frac{bg}{fh}\mathrm{e}^{-\beta t}\int x^2\mathrm{e}^{\beta t}\mathrm{d}t+\frac{bg}{fh}x^2\end{aligned}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-\beta z+\frac{2b}{f}xy+\frac{b}{fh}\left(\beta+2c+g\right)x^2 \tag{97}$$

Choose the parameter values:

$$c=-10, h=10, k=11, a=-270, b=0.5, f=1, \beta=\frac{8}{3}, g=\frac{52}{3}$$

And the goal is achieved, as shown in **Figure 5**:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=-10x+10y\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=28x-y-xz\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=-\frac{8}{3}z+xy\end{aligned} \tag{98}$$

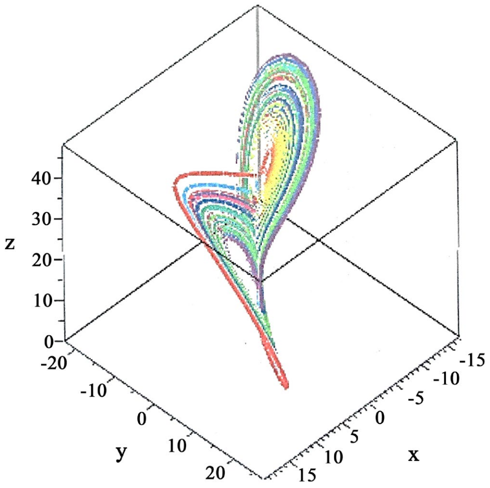

Figure 5. Phase portrait of the Lorenz system.

In **Figure 6**, are the values of the parameters:

$$c=-10, h=-16, \beta=3, f=-1.6, b=0.8, k=11.1, a=-653, g=23.4$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=-10x-16y\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=-41.5x-1.1y+1.6xz\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=-3z-xy+0.2x^2\end{aligned} \tag{99}$$

Notice the change in values of the parameters, the opposite sign in some of the parameters and the extra x^2-term in the third equation. This system has chaotic behavior. One initial point.

In **Figure 7**, the parameter values are:

$$c=-20, h=10, \beta=3, f=5, b=500, k=21.1, a=-2422, g=35$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=-20x+10y\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=240x-1.1y-5xz\end{aligned} \tag{100}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -3z + 200xy - 20x^2$$

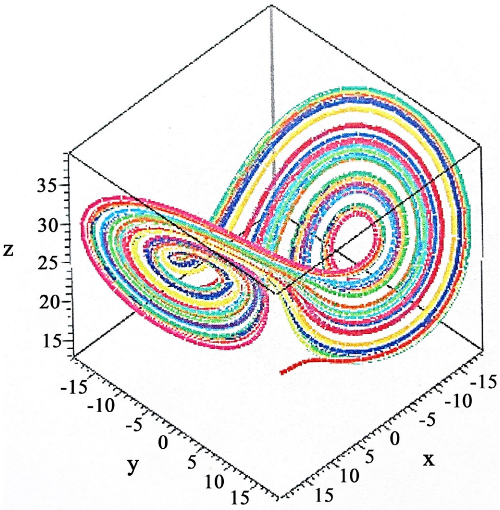

Figure 6. $\frac{\mathrm{d}z}{\mathrm{d}t}$ contain $+0.2x^2$.

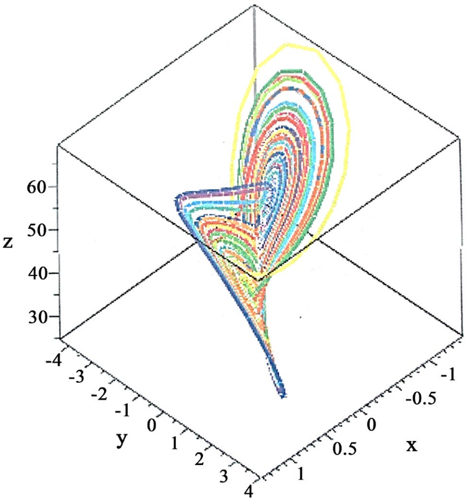

Figure 7. $\frac{\mathrm{d}z}{\mathrm{d}t}$ contain $-20x^2$.

The constant term $(\beta + 2c + g)$ don't need to be zero in order to obtain chaotic behavior. When this term is not zero, there is an addition of x^2 in the third equation. In **Figure 6**, this part is $+0.2x^2$, and in **Figure 7** is this part $-20x^2$. And we can still see chaotic behavior.

4.2.2. Chen Chaotic Attractor [9]

This system is a special case of the general Lorenz system. The two-integral amplitude function $amp_{Lo}(t)$ give solution to this system.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha(y - x)$$

$$\begin{gathered}\frac{\mathrm{d}y}{\mathrm{d}t}=(\gamma-\alpha)x-xz+\gamma y \\ \frac{\mathrm{d}z}{\mathrm{d}t}=xy-\beta z\end{gathered}\tag{101}$$

This system has chaotic behavior when $\alpha=35,\beta=3,\gamma=28$

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=35y-35x \\ \frac{\mathrm{d}y}{\mathrm{d}t}=-7x+28y-xz \\ \frac{\mathrm{d}z}{\mathrm{d}t}=xy-3z\end{gathered}\tag{102}$$

Compare with the general Lorenz system with 8 parameters.

$$c=-35,h=35,f=1,\beta=3,k=7,b=0.5,a=-735,g=67$$

Putting these parameter values into the general Lorenz system, and we become the Chen system above. See **Figure 8**.

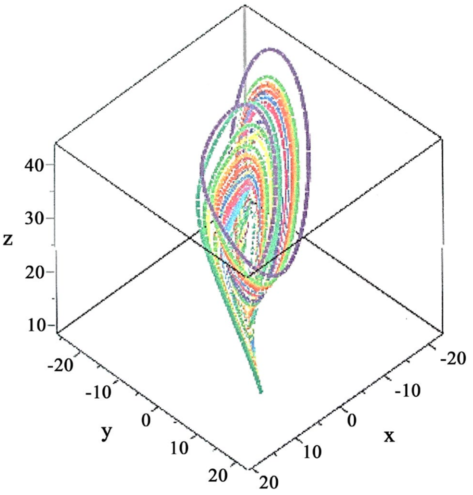

Figure 8. Phase portrait of the Chen system.

The next system has also $amp_{Lo}(t)$ as solution.

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=\alpha(y-x) \\ \frac{\mathrm{d}y}{\mathrm{d}t}=\gamma y-xz \\ \frac{\mathrm{d}z}{\mathrm{d}t}=xy-\beta z\end{gathered}\tag{103}$$

This system has chaotic behavior when $\alpha=36,\beta=3,\gamma=20$

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=36y-36x \\ \frac{\mathrm{d}y}{\mathrm{d}t}=20y-xz\end{gathered}\tag{104}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - 3z$$

Compare with the general Lorenz system with 8 parameters.

$$c = -36, h = 36, f = 1, \beta = 3, k = 16, b = 0.5, a = -720, g = 69$$

Putting these parameter values into the general Lorenz system, and we become the system above. See **Figure 9**.

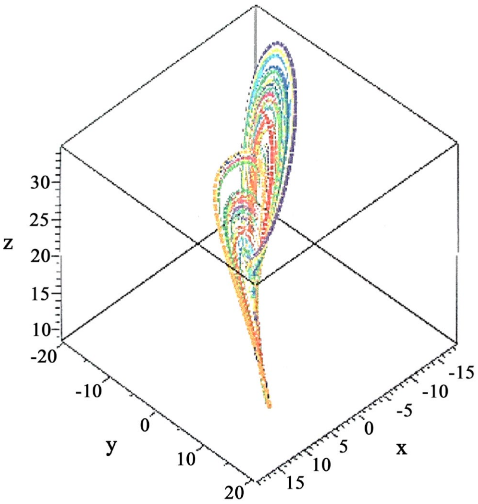

Figure 9. $x = 0$ in second equation.

4.3. The Rössler System

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + ay \tag{105}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -cz + bx + xz$$

A simple system with a complicated behavior should imply a complicated solution. My attempt to define a solution is quite complicated, with 4 integrals!

Define an integral function u:

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\varphi}{C - \mathrm{e}^{at} \int \varphi \mathrm{e}^{-at} \mathrm{d}t - b\mathrm{e}^{-ct + \int \varphi \mathrm{d}t} \int \left(\varphi \mathrm{e}^{ct - \int \varphi \mathrm{d}t} \right) \mathrm{d}t} \tag{106}$$

This is what Armitage and Eberlein says: We have permitted ourselves a familiar "abuse of notation" in using φ for the variable and for the upper limit of integration [6].

The four integrals under the fraction line are with respect to the independent variable t in all parts of the integrals, since φ is a function of t, $\varphi = \varphi(t)$.

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{C - \mathrm{e}^{at} \int \varphi \mathrm{e}^{-at} \mathrm{d}t - b\mathrm{e}^{-ct + \int \varphi \mathrm{d}t} \int \left(\varphi \mathrm{e}^{ct - \int \varphi \mathrm{d}t} \right) \mathrm{d}t} \tag{107}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=C-\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t-b\mathrm{e}^{-ct+\int\varphi\mathrm{d}t}\int\left(\varphi\mathrm{e}^{ct-\int\varphi\mathrm{d}t}\right)\mathrm{d}t \tag{108}$$

Define an amplitude function as solution $x(t)=amp(t)=\varphi=\varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=-y-z \tag{109}$$

If, the constant $C=0$, and if:

$$y(t)=\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t \tag{110}$$

$$z(t)=b\mathrm{e}^{-ct+\int\varphi\mathrm{d}t}\int\left(\varphi\mathrm{e}^{ct-\int\varphi\mathrm{d}t}\right)\mathrm{d}t \tag{111}$$

Change the notation of the dependent variable from φ to x:

$$y(t)=\mathrm{e}^{at}\int x\mathrm{e}^{-at}\mathrm{d}t \tag{112}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=a\mathrm{e}^{at}\int x\mathrm{e}^{-at}\mathrm{d}t+\mathrm{e}^{at}x\mathrm{e}^{-at}=ay+x \tag{113}$$

$$z(t)=b\mathrm{e}^{-ct+\int x\mathrm{d}t}\int\left(x\mathrm{e}^{ct-\int x\mathrm{d}t}\right)\mathrm{d}t \tag{114}$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=-cb\mathrm{e}^{-ct+\int x\mathrm{d}t}\int\left(x\mathrm{e}^{ct-\int x\mathrm{d}t}\right)\mathrm{d}t+xb\mathrm{e}^{-ct+\int x\mathrm{d}t}\int\left(x\mathrm{e}^{ct-\int x\mathrm{d}t}\right)\mathrm{d}t+b\mathrm{e}^{-ct+\int x\mathrm{d}t}x\mathrm{e}^{ct-\int x\mathrm{d}t}\\&=-cz+xz+bx\end{aligned} \tag{115}$$

We define the solution-function to the Rössler system:

$$x(t)=amp_R(t)=\varphi=\varphi(t)=\int_0^t adn_R(u)\mathrm{d}u \tag{116}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_R(t)=adn_R(t)=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t-b\mathrm{e}^{-ct+\int\varphi\mathrm{d}t}\int\left(\varphi\mathrm{e}^{ct-\int\varphi\mathrm{d}t}\right)\mathrm{d}t \tag{117}$$

See **Figure 10**, where the values of the parameters are: $a=0.4, b=0.3, c=4.449$

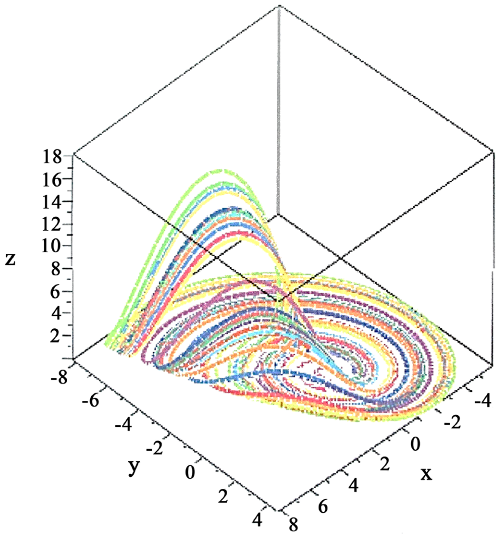

Figure 10. The Rössler chaotic attractor.

What happens if the $C\neq 0$? Then we become the system:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=C-y-z\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=x+ay\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=-cz+bx+xz\end{aligned}\tag{118}$$

Choose the constant $C=10$ and the other parameter values: $a=0.4, b=0.3, c=4.449$. See **Figure 11**. The trajectories are almost 4 times bigger than in **Figure 10**.

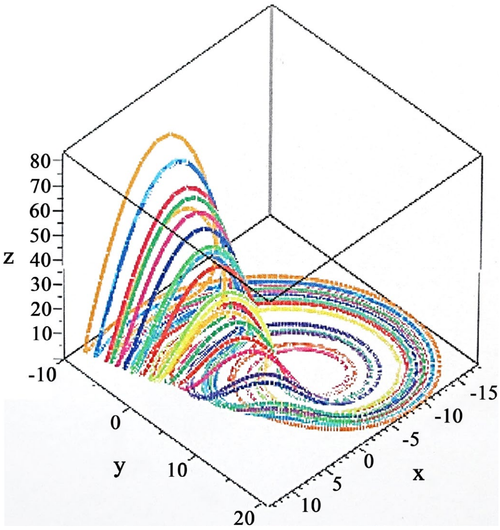

Figure 11. The constant $C=10$.

4.4. The Sprott-Linz F Chaotic Attractor [10]

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=y+z\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=ay-x\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=x^2-z\end{aligned}\tag{119}$$

Chaotic behavior when the parameter $a=0.5$.

This system reminds a bit of the Rössler system, so perhaps we can use some parts of this solution. Using more parameters, the system will become more general.

Define an integral function u:

$$u=u(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\varphi}{C+hf\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t+kb\mathrm{e}^{-ct}\int\varphi^2\mathrm{e}^{ct}\mathrm{d}t}\tag{120}$$

I have used the "abuse of notation" as for the Rössler system.

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{C+hf\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t+kb\mathrm{e}^{-ct}\int\varphi^2\mathrm{e}^{ct}\mathrm{d}t}\tag{121}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=C+hf\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t+kb\mathrm{e}^{-ct}\int\varphi^{2}\mathrm{e}^{ct}\mathrm{d}t \tag{122}$$

Define an amplitude function as solution $x(t)=amp(t)=\varphi=\varphi(t)$, and change notation of the dependent variable from φ to x.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{d\varphi}{dt}=C+hf\mathrm{e}^{at}\int x\mathrm{e}^{-at}\mathrm{d}t+kb\mathrm{e}^{-ct}\int x^{2}\mathrm{e}^{ct}\mathrm{d}t=hy+kz+C \tag{123}$$

If:

$$y(t)=f\mathrm{e}^{at}\int x\mathrm{e}^{-at}\mathrm{d}t \tag{124}$$

$$z(t)=b\mathrm{e}^{-ct}\int x^{2}\mathrm{e}^{ct}\mathrm{d}t \tag{125}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=af\mathrm{e}^{at}\int x\mathrm{e}^{-at}\mathrm{d}t+f\mathrm{e}^{at}x\mathrm{e}^{-at}=ay+fx \tag{126}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-cb\mathrm{e}^{-ct}\int x^{2}\mathrm{e}^{ct}\mathrm{d}t+b\mathrm{e}^{-ct}x^{2}\mathrm{e}^{ct}=-cz+bx^{2} \tag{127}$$

The general Sprott-Linz F system with 7 parameters:

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t}=hy+kz+C\\ \frac{\mathrm{d}y}{\mathrm{d}t}=ay+fx\\ \frac{\mathrm{d}z}{\mathrm{d}t}=-cz+bx^{2}\end{gathered} \tag{128}$$

We may name the solution to the Sprott-Linz F chaotic attractor $amp_{SL}(t)$.

$$x(t)=amp_{SL}(t)=\varphi=\varphi(t)=\int_{0}^{t}adn_{SL}(u)\mathrm{d}u \tag{129}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_{SL}(t)=adn_{SL}(t)=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C+hf\mathrm{e}^{at}\int\varphi\mathrm{e}^{-at}\mathrm{d}t+kb\mathrm{e}^{-ct}\int\varphi^{2}\mathrm{e}^{ct}\mathrm{d}t \tag{130}$$

See **Figure 12**, where $C=0,h=1,k=1,a=0.5,f=-1,b=1$

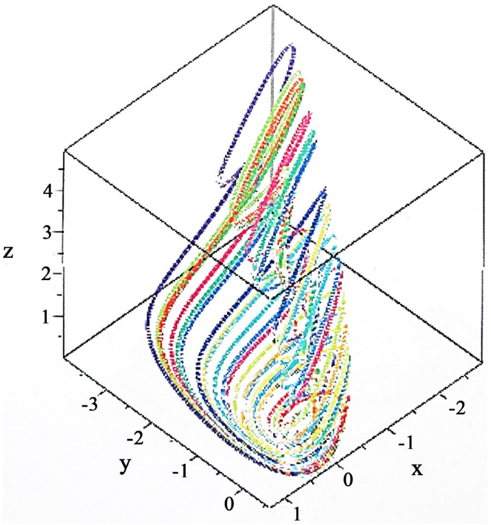

Figure 12. Sprott-Linz F chaotic attractor.

5. Conclusions

The one-integral amplitude function, that is containing a trigonometric function of the independent variable t in the first derivative of the amplitude function, is giving solution to some forced and damped second-order nonlinear ODEs.

1) The forced and damped pendulum equation
2) The forced and damped Duffing equation
3) Ueda's chaotic nonlinear oscillator

The two-integral amplitude function containing the function $g(\varphi,t)$, give solution to some 3D systems of ODEs that are exhibiting chaotic behavior.

4) The Shimizu-Morioka system with chaotic behavior
5) The chaotic cases of the Lorenz system
6) The Chen attractor
7) Rössler chaotic attractor
8) Sprott-Linz F chaotic attractor

Consider the function $g(\varphi,t)=\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t$, $\varphi=\varphi(t)$, in the solutions to the chaotic Lorenz system, in the Shimizu-Morioka system and the Sprott-Linz F system. We see a similar part in the solution to the Rössler system. This equation contains two elementary exponential functions with equal exponent, but opposite sign. It seems to me that this part of the solution is absolutely necessary in order to obtain chaotic behavior, for at least some systems of ODEs. This is the function mentioned in the Abstract.

So far, I have discovered these examples of the function $g(\varphi,t)$:

$$g_1(\varphi,t)=\mathrm{e}^{-\beta t}\int\varphi^2\mathrm{e}^{\beta t}\mathrm{d}t$$

$$g_2(\varphi,t)=\mathrm{e}^{\beta t}\int\varphi\mathrm{e}^{-\beta t}\mathrm{d}t$$

$$g_3(\varphi,t)=\mathrm{e}^{-\beta t+\int\varphi\mathrm{d}t}\int\varphi\mathrm{e}^{\beta t-\int\varphi\mathrm{d}t}\mathrm{d}t$$

A further investigation of these functions may perhaps explain the chaotic behavior of some systems of ODEs.

If the behavior of the solution curves in the phase diagrams reflects the qualities of the amplitude functions, then we can see some of these qualities in the figures in this paper.

The solutions described in this paper can be placed in the group of the amplitude functions, and in the context of the elliptic functions.

Using the two-integral amplitude function and the function $g(\varphi,t)$, it should be possible to define solutions to some other systems exhibiting chaos behavior.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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