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Relaxation Oscillations in a Slow-Fast Modified Rosenzweig-MacArthur Model

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Abstract

In this paper, the relaxation oscillation of a slow-fast Rosenzweig-MacArthur model is investigated. The existence of the relaxation oscillation cycle is obtained by means of geometric singular perturbation theory and entry-exit function. The asymptotical stability of the cycle which follows from the negativity of the Floquet multiplier further guarantees the uniqueness of the relaxation oscillation. The theoretical results are verified by numerical simulations.

Keywords

Rosenzweig-Macarthur Model, Slow-Fast Dynamics, Relaxation Oscillation, Geometric Singular Perturbation, Entry-Exit Function

1. Introduction

Predator-prey models, which describe the interaction of two or more species with respect to predation, have been researched to a large extent for their abundant dynamic behaviors and phenomena. Since introduced by Lotka [1] in 1925, the complexity of models has been increasing and the forms of research have been enriched. In these aspects, we only give a few examples: The competition of predators (see [2]), growth patterns of prey (see [3]) or various predation functions (see [4]). And predator-prey models which have two or more time scales, especially slow-fast systems, have been studied in various fields. In recent years, the discussion on the slow-fast predator-prey models has dominated the research (see [5]-[8]).

The Rosenzweig-MacArthur predator-prey model introduced by Rosenzweig & MacArthur [9] in 1963 then by Berryman [10] in 1992 is one of the most classic systems in bioresearch. It describes the interaction of two species between prey and predator using the linear mortality of the predator, logistic growth of the prey, and Holling II function response:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau}&=au\left(1-\frac{u}{K}\right)-\frac{\alpha uv}{1+\beta u},\\ \frac{\mathrm{d}v}{\mathrm{d}\tau}&=\gamma\frac{\alpha uv}{1+\beta u}-cv,\end{aligned}\tag{1}$$

where u and v represent the density of prey and predator respectively. Logistic growth is related to the intrinsic growth rate a and environmental capacity K that determines the maximum population density of the prey. The predation response function is in connection with half-saturation constant $1/\beta$ and maximum predation rate α/β. γ is the conversion rate to specify the ratio between the increase in biomass and food intake since not all prey taken up are converted into biomass of the predator. Predator mortality is represented by c. Another is that the feeding rate α must be high to compensate for the low conversion rate γ [11]. To satisfy the biological significance, we know that they're all positive.

System (1) has been studied by many scholars in many different ways. Vanselow *et al.* [3] studied the collapse of the system when very slow is too fast. Using the bifurcation theory approach, Seo and Wolkowicz [12] studied the sensitivity of the system and obtained that the model gives rise to subcritical Hopf bifurcations. Moustafa *et al.* [13] got the existence, uniqueness, non-negativity and boundedness of the solutions as well as the local and global asymptotic stability of the equilibrium points by studying a Rosenzweig-MacArthur model incorporating a prey refuge.

Consider the following coordinate transformations

$$x=\alpha\gamma u/c,\ \ y=\alpha v/a,\ \ t=\tau a.$$

Then one can get the following modified Rosenzweig-MacArthur predator-prey model:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=x(1-\phi x)-\frac{xy}{1+\eta x}:=xf(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=\varepsilon\left(\frac{xy}{1+\eta x}-y\right):=\varepsilon g(x,y),\end{aligned}\tag{2}$$

where $\eta=c\beta/(\alpha\gamma)$, $1/\phi=\alpha\gamma K/c$ and $\varepsilon=c/a$. The parameter $\varepsilon=c/a$ can be considered as the ratio of time constants of two different time scales. In nature, predator and prey generally correspond to different levels of nutrition and biosphere status, so they have different longevities, reproductive rates, and evolutionary rates. For example, mice can conceive 3 - 4 times a year and give birth to 12 - 15 cubs at a time, while cats can conceive 1 - 2 times a year and give birth to 3 - 5 cubs at a time; a mouse lives only 2 - 3 years while a cat can live 13 - 20 years. In these situations, the predator variable could be slow in comparison to the prey variable. Therefore we can assume that the growth rate of the prey a is much bigger than the death rate of the predator c. In other words, $0<\varepsilon\ll 1$ is a small parameter. Then, system (2) can be viewed as a slow-fast system with a slow variable y and a fast variable x. This kind of system can be investigated by geometric singular perturbation theory [14]-[19].

In the past few decades, geometric singular perturbation technology has become a powerful mathematical analysis tool in studying the dynamic behaviors of predator-prey system with slow-fast structures. Utilizing Floquet theory and geometric singular perturbation theory, Hsu [20] derived characteristic functions that determine the location and the stability of relaxation oscillations as $\varepsilon \to 0$. Using geometric singular perturbation theory, Wang and Zhang [21] achieved much richer dynamical phenomena than the existing ones, such as the existence of canard cycles and canard explosion. Some other theoretical applications of singular perturbation theory, such as slow-fast normal form theory and blow-up technique, please see [22]-[26]. In this paper, using the singular perturbation technique including Fenichel's theory [18] and the entry-exit function [19] [27], we get the existence and uniqueness of relaxation oscillation cycle of the modified Rosenzweig-MacArthur predator-prey model (2).

The rest of this paper is formed as follows. We first introduce the entry-exit function and the limit systems of system (2) in Section 2, some definitions are also given. The existence and uniqueness of the relaxation oscillation of the system is proved in Section 3. A numerical example is presented to verify the theoretical results in Section 4. Conclusions are given in the final section.

2. Preliminaries

2.1. Equilibria and Positive Invariance

By simple proofs we can have the following results.

Theorem 2.1. The set $\Omega = \left\{ (x, y) \mid 0 \le x \le 1/\phi; y \ge 0 \right\}$ is positively invariant of system (2).

Proof. Note that the y-axis and x-axis are both invariant for the flow of system (2). Integrating both sides of the second equation for system (2), we obtain

$$y(t) = \mathrm{e}^{\varepsilon\left(\frac{x}{1+\eta x}-1\right)t} > 0$$

for any t. Clearly, the y value is non-negative. Since the y-axis is invariant, restricting the first equation of system (2) on $x = 1/\phi$ we get

$$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=1/\phi} = -\frac{xy}{1+\eta x} < 0$$

for $y > 0$. Obviously, for any t, we have $0 \le x \le 1/\phi$. Therefore the set Ω is positive invariance of the system (2).

Theorem 2.2. System (2) satisfies the following statements.

a) System (2) has two trivial equilibria $E_0(0,0)$ and $E_x(1/\phi, 0)$. E_0 is a hyperbolic saddle. If $\phi + \eta > 1$, E_x is a stable node, $\phi + \eta = 1$, E_x is a saddle-node, $0 < \phi + \eta < 1$, E_x is a saddle.

b) If $1 - \eta > 0$ and $\frac{1}{1-\eta} < \frac{1}{\phi}$ hold, system (2) has a unique equilibrium

$E_*(x_*, y_*)$ with $x_* = \frac{1}{1-\eta}$ and $y_* = (1-\phi x_*)(1+\eta x_*)$. And E_* is unstable if $\frac{1}{1-\eta} < \frac{\eta-\phi}{2\eta\phi} < \frac{1}{\phi}$ ($\operatorname{tr}(J(E_*)) < 0$), E_* is stable if $\frac{\eta-\phi}{2\eta\phi} < \frac{1}{1-\eta} < \frac{1}{\phi}$.

Proof. a) By simple calculation, we know that system (2) has two trivial equilibria $E_0(0,0)$ and $E_x\left(\frac{1}{\phi}, 0\right)$. Linearizing system (2) at equilibrium E_0 and E_x and studying the Jacobian matrixes $J(E_0)$ and $J(E_x)$. We examine the signs of $\operatorname{tr}(J(E_0))$ and $\det(J(E_0))$, and find that $\operatorname{tr}(J(E_0)) = 0$ and $\det(J(E_0)) < 0$, thus E_0 is a hyperbolic saddle point. Similarly, we calculate the $\operatorname{tr}(J(E_x))$ and $\det(J(E_x))$ and obtain $\operatorname{tr}(J(E_x)) = -\left(2 - \frac{1}{\phi+\eta}\right)$, $\det(J(E_x)) = -\left(\frac{1}{\phi+\eta} - 1\right)$. Thus we have E_x is a saddle when $0 < \phi+\eta < 1$, and E_x is a saddle-node when $\phi+\eta = 1$. Furthermore, we can get that

$$\operatorname{tr}(J(E_x))^2 - 4\det(J(E_x)) = \left(2 - \frac{1}{\phi+\eta}\right)^2 - 4\left(\frac{1}{\phi+\eta}\right) > 0,$$

when $\phi+\eta > 1$. Therefore E_x is a stable node.

b) By simple calculation, we know that system (2) have a unique equilibrium $E_*(x_*, y_*)$ with $x_* = \frac{1}{1-\eta}$ and $y_* = (1-\phi x_*)(1+\eta x_*)$ when $1-\eta > 0$ and $\frac{1}{1-\eta} < \frac{1}{\phi}$. If this unique positive equilibrium point exists, we have $\operatorname{tr}(J(E_*)) = \eta + \phi - \frac{2\phi}{1-\eta}$ and $\det(J(E_*)) = 1-\eta-\phi > 0$. Therefore, we know that E_* is unstable if $\frac{1}{1-\eta} < \frac{\eta-\phi}{2\eta\phi} < \frac{1}{\phi}$ ($\operatorname{tr}(J(E_*)) < 0$), and E_* is stable if $\frac{\eta-\phi}{2\eta\phi} < \frac{1}{1-\eta} < \frac{1}{\phi}$ ($\operatorname{tr}(J(E_*)) > 0$).

2.2. Definitions

In this subsection, we will use a more general formulation to introduce some definitions.

Utilize the time scale transformation $\bar{t} = \varepsilon t$ instead of the fast time scale t for system (2), we can get

$$\begin{aligned} \varepsilon \frac{\mathrm{d}x}{\mathrm{d}\bar{t}} &= \varepsilon \dot{x} = x f(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\bar{t}} &= \dot{y} = g(x,y). \end{aligned}$$

Let $\varepsilon \to 0$, we get slow subsystem

$$\begin{aligned} 0 &= x f(x,y), \\ \dot{y} &= g(x,y), \end{aligned} \tag{3}$$

which is restricted to the critical manifold

$$C_0 = \{(x,y) \in \Omega \mid f(x,y)x = 0\}.$$

Next, we use slow subsystem above to derive the definitions.

Definition 2.1. For all $p \in S$, if the matrix $(D_x(xf))(p,0)$ has no eigenvalues with zero real part, then we call subset $S \subset C_0$ normally hyperbolic.

Definition 2.2. A normally hyperbolic subset $S \subset C_0$ is called attracting if all eigenvalues of $(D_x(xf))(p,0)$ have negative real part for $p \in S$; inversely, S is called repelling if all eigenvalues have positive real part.

Next, we introduce the definition of the generic fold point.

The critical manifold C_0 has a nondegenerate fold point at $P = (x_0, y_0)$ if

$$(xf)_x(x_0,y_0) = 0,\ (xf)_{xx}(x_0,y_0) \neq 0,\ (xf)_y(x_0,y_0) \neq 0.$$

In addition, it is natural to assume a transversality condition given by

$$g(x_0,y_0) > 0 \text{ or } g(x_0,y_0) < 0.$$

Definition 2.3. A fold point satisfying the above two formulas is called a generic fold point.

In the following we consider system (2) in the set Ω.

2.3. Entry-Exit Function

From the expressions of functions f and g in system (2) that

$$f(x,y) = (1-\phi x) - \frac{y}{1+\eta x},\quad g(x,y) = \frac{xy}{1+\eta x} - y,$$

it can be easily verified that f and g are continuous derivable functions and satisfy

$$g(0,y) < 0 \text{ for } y > 0;\ f(0,y) < 0 \text{ for } y > 1 \text{ and } f(0,y) > 0 \text{ for } 0 < y < 1.$$

Hence, when $\varepsilon = 0$, $Y^a := \{(x,y) \mid x = 0, y > 1\}$ is attracting, while $Y^r := \{(x,y) \mid x = 0, 0 < y < 1\}$ is repelling, as can be seen in **Figure 1(a)**. For $\varepsilon > 0$, according to Definition 2.1 and Definition 2.2, we know that the part of the positive y-axis with respect to $y > 1$ is normally hyperbolic attracting while the rest of the positive y-axis is normally hyperbolic repelling by calculation. The orbit of system (2) which starts at (x_0, y_0) with $x_0 > 0$ small and $y_0 > 1$ is attracted toward the y-axis, then it stays close to the y-axis and drifts slowly downward, see **Figure 1(b)**. When the orbit crosses the line $y = 1$, it will leave the vicinity of the repelling part of the positive y-axis at a point $(0, p_0(y_0))$, and then it will re-intersect the line $x = x_0$ at the point whose y-coordinate is $p_\varepsilon(y_0)$ satisfying $\lim_{\varepsilon \to 0} p_\varepsilon(y_0) = p_0(y_0)$, where $p_0(y_0)$ is determined by

$$\int_{y_0}^{p_0(y_0)} \frac{f(0,y)}{g(0,y)} \mathrm{d}y = 0. \tag{4}$$

The function $y_0 \to p_0(y_0)$ defined by Eq. (4) implicitly is called the entry-exit function.

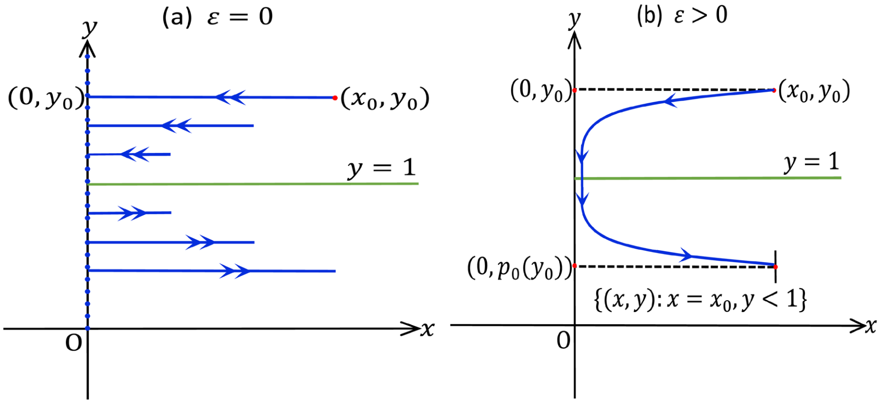

Figure 1. (a) The orbits of system (2) when $\varepsilon = 0$. (b) A typical orbit of system (2) for $0 < \varepsilon \ll 1$.

2.4. The Slow and Fast Subsystems

From (3), we obtain the slow subsystem of system (2)

$$\begin{aligned} 0 = \varepsilon \frac{\mathrm{d}x}{\mathrm{d}\bar{t}} &= x(1-\phi x) - \frac{xy}{1+\eta x}, \\ \frac{\mathrm{d}y}{\mathrm{d}\bar{t}} &= \frac{xy}{1+\eta x} - y, \end{aligned} \tag{5}$$

which is a differential-algebraic equation on the critical manifold

$$C_0 := \{(x, y) \in \Omega \mid x = 0 \text{ or } y = (1-\phi x)(1+\eta x) := F(x)\}.$$

Letting $\varepsilon = 0$ in system (2) we can obtain the fast subsystem

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(1-\phi x) - \frac{xy}{1+\eta x}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 0, \end{aligned} \tag{6}$$

where y can be treated as a constant and it is called a layer problem.

From the fast subsystem (6), we know that the positive y-axis which constitutes equilibria of the fast subsystem is repelling for $0 < y < 1$ and attracting for $y > 1$, seen in **Figure 1(a)**. Additionally, from the slow subsystem (5), we can see that the slow flow on positive y-axis directs downward. On the other hand, one branch of C_0 depicted by $\{(x, y) \in \Omega \mid y = F(x)\}$ has a unique generic fold point $D(x_M, y_M) = \left(\frac{\eta - \phi}{2\eta\phi}, \frac{(\eta+\phi)^2}{4\eta\phi} \right)$ according to Definition 2.3. The intersection $E(x, y) = (0,1)$ of the two branches of C_0 is called transcritical point, seen in **Figure 2**. Then the critical manifold C_0 can be split into four parts by the points $D(x_M, y_M)$ and $E(0,1)$:

$$C_0^a = \left\{ (x, y) \in \Omega \mid \frac{\eta - \phi}{2\eta\phi} < x < \frac{1}{\phi}, y = F(x) \right\}, \ C_0^+ = \{x = 0, y > 1\},$$

$$C_0^r = \left\{ (x, y) \in \Omega \mid 0 < x < \frac{\eta - \phi}{2\eta\phi}, y = F(x) \right\}, \ C_0^- = \{x = 0, 0 < y < 1\}.$$

According Definition 2.1, Definition 2.2 and calculation, we know that C_0^a and C_0^+ are normally hyperbolic attracting while C_0^r and C_0^- are normally hyperbolic repelling. The dynamics of slow subsystem (5) and fast subsystem (6) are shown in Figure 2.

We restrict the parameters ϕ and η on the U parameter space, where

$$U=\left\{(\phi,\eta)\in R_+^2 \mid 0<\frac{1}{1-\eta}<\frac{\eta-\phi}{2\eta\phi}<\frac{1}{\phi} \text{ and } \eta\neq\phi\right\}.$$

Obviously, slow subsystem (5) has the same unique positive equilibrium as the system (2). Since $\frac{1}{1-\eta}<\frac{\eta-\phi}{2\eta\phi}$, the equilibrium point $E_*(x_*,y_*)\in C_0^r$. For C_0^r is normally hyperbolic repelling, then the Fenichel's theory indicated that $E_*(x_*,y_*)$ is hyperbolic and is an unstable node, as shown in Figure 2 (the black dot point).

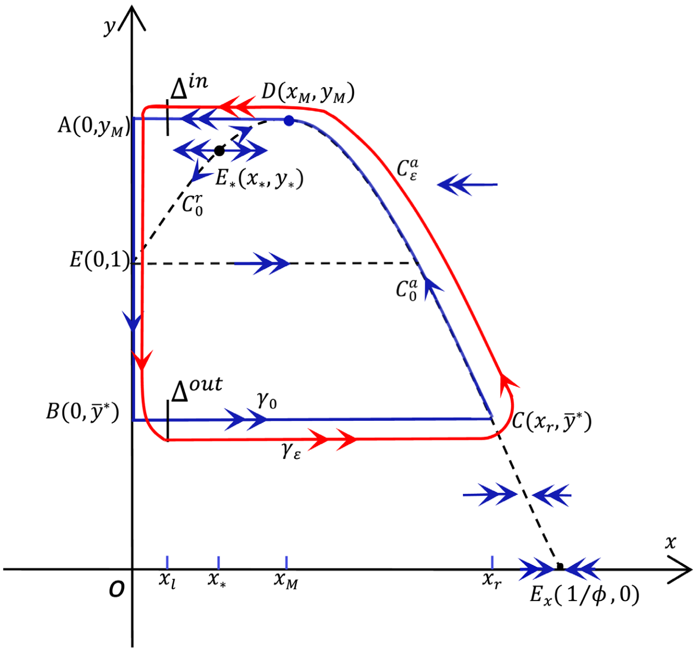

Figure 2. Relaxation oscillation of system (2). The critical manifold C_0 (dashed black) consisting of normally hyperbolic repelling part C_0^r and attracting part C_0^a, the fold point $D(x_M,y_M)$ (blue dot), the unstable node (black dot) on C_0^r, a slow-fast cycle γ_0 (solid blue), fast and slow flows (double and single arrows) and the relaxation oscillation γ_ε (red) are shown.

3. Relaxation oscillation of system (2)

Based on the above analysis, we have the following results about the entry-exit function for system (2).

Lemma 3.1. For system (2), there exists a unique $\bar{y}^*$ $(0<\bar{y}^*<1)$ such that

$$\int_{\bar{y}^*}^{y_M}\frac{f(0,y)}{g(0,y)}\mathrm{d}y=0.$$

Proof. We can obtain from system (2) that for $\bar{y} \in (0,1)$,

$$I(\bar{y}) := \int_{\bar{y}}^{y_M} \frac{f(0,y)}{g(0,y)} \mathrm{d}y = \int_{\bar{y}}^{y_M} \left(1 - \frac{1}{y}\right) \mathrm{d}y = (y - \ln y)\Big|_{\bar{y}}^{y_M} \to -\infty, \text{ as } \bar{y} \to 0^+.$$

$$I(1) = \int_{1}^{y_M} \left(1 - \frac{1}{y}\right) \mathrm{d}y > 0, \text{ since } y_M > 1.$$

On the other hand, $\frac{\mathrm{d}I}{\mathrm{d}\bar{y}} = \frac{1-\bar{y}}{\bar{y}} > 0$, hence, there exists a unique $\bar{y}^* \left(0 < \bar{y}^* < 1\right)$ such that

$$I(\bar{y}^*) = \int_{\bar{y}^*}^{y_M} \frac{f(0,y)}{g(0,y)} \mathrm{d}y = 0.$$

This finished the proof.

Let x_r denote the x-coordinate of the intersection point of C_0^a and $y = \bar{y}^*$. **Define a singular slow-fast cycle** γ_0: γ_0 is constituted by two fast segments $\overrightarrow{D(x_M, y_M)A(0, y_M)}$ and $\overrightarrow{B(0,\bar{y}^*)C(x_r,\bar{y}^*)}$ and two slow segments $\overrightarrow{A(0,y_M)B(0,\bar{y}^*)}$ and C_0^a from $C(x_r,\bar{y}^*)$ to $D(x_M, y_M)$, seen in Figure 2. The following theorem displays the existence of the relaxation oscillation for system (2).

Theorem 3.1. Assume that V represents a tubular δ-neighborhood of the slow-fast cycle γ_0, where $\delta > 0$ is sufficiently small. Suppose parameters $(\phi,\eta) \in U$, then for $0 < \varepsilon \ll 1$, system (2) possesses a limit cycle $\gamma_\varepsilon \subset V$ converging to γ_0 with the Hausdorff distance when $\varepsilon \to 0$.

Proof. Choose a small positive number as x_l. Define two vertical sections

$$\Delta^{in} := \left\{(x_l, y) \in V \mid y \in I_{in}\right\} \text{ and } \Delta^{out} := \left\{(x_l, y) \in V \mid y \in I_{out}\right\},$$

where $I_{in} = [y_M - \delta_1, y_M + \delta_1]$ and $I_{out} = \left[p_0(y_M) - \delta_2, p_0(y_M) + \delta_2\right]$, respectively, δ_1 and δ_2 are sufficiently small positive constants not larger than δ. Define a transition map $\Pi : \Delta^{in} \to \Delta^{in}$, which is induced by the flow of system (2) and is constituted by the following two maps

$$\Pi_1 : \Delta^{in} \to \Delta^{out} \text{ and } \Pi_2 : \Delta^{out} \to \Delta^{in}.$$

Next we focus on the properties of the two maps.

a) *Analysis about map* Π_1. By Lemma 3.1, with respect to each $y_0 \in I_{in}$ there exists a unique $p_0(y_0)$ with $0 < p_0(y_0) < 1$ defined by

$$\int_{p_0(y_0)}^{y_0} \frac{f(0,y)}{g(0,y)} \mathrm{d}y = 0.$$

According to Lemma 3.1, $p_0(y_M) = \bar{y}^*$. As what displays in Figure 1(b), the orbit of system (2) initiating from $(x_0, y_0) \in \Delta^{in}$ will go through the part Δ^{out} at a point $(x_0, p_\varepsilon(y_0))$ with $\lim_{\varepsilon \to 0} p_\varepsilon(y_0) = p_0(y_0)$. Therefore, $\Pi_1 : \Delta^{in} \to \Delta^{out}$ can be defined by

$$\Pi_1\left(x_0, y_0\right)=\left(x_0, p_\varepsilon\left(y_0\right)\right).$$

b) *Analysis about map* Π_2. Due to the Fenichel's theory, there exists a slow manifold C_ε^a in the neighborhood of $C_0^a=\left\{(x, y) \in \Omega \mid \frac{\eta-\phi}{2\eta\phi}<x<\frac{1}{\phi}, y=F(x)\right\}$, which is a perturbation of C_0^a with a higher order term $O(\varepsilon)$ for C_0^a is normally hyperbolic attracting. According to Theorem 2.1 in [23] about the analysis of the dynamics near a generic fold point, the slow manifold C_ε^a moves upward lying close to C_0^a until it arrives at the neighbourhood of the fold point $D\left(x_M, y_M\right)$, and then it jumps to the neighbourhood of C_0^+. Two orbits γ_ε^1 and γ_ε^2 which start at Δ^{out} will be attracted to C_ε^a at an exponential rate $O\left(\mathrm{e}^{-1/\varepsilon}\right)$ according to Fenichel's theory. And then they moves upward lying close to C_0^a until arriving at the neighbourhood of the fold point $D\left(x_M, y_M\right)$, where they contract toward each other exponentially. Thereafter, they jump to Δ^{in}.

According to the above analysis, we know that the transition map $\Pi: \Delta^{in} \rightarrow \Delta^{in}$ is a contraction map whose exponential rate is $O\left(\mathrm{e}^{-1/\varepsilon}\right)$. The contraction mapping theorem then indicates that Π possesses a fixed point in Δ^{in}, which provides a relaxation oscillation cycle $\gamma_\varepsilon \subset V$ of system (2) passing through Δ^{in} for $0<\varepsilon \ll 1$. Further, γ_ε converges to the singular slow-fast cycle γ_0 as $\varepsilon \rightarrow 0$ in the Hausdorff distance by using the Fenichel's theory and Theorem 2.1 in [23].

Next, we show the uniqueness of the relaxation oscillation cycle γ_ε of system (2) by proving γ_ε is asymptotically stable.

Theorem 3.2. Suppose $x=G(y)$ is the inverse function of $y=F(x)$ on C_0^a. If the minimal period of γ_ε is $T>0$, then γ_ε is a locally asymptotically stable orbit since its nontrivial Floquet multiplier

$$\lambda:=\int_0^T\left(\frac{\partial(xf)}{\partial x}+\varepsilon \frac{\partial g}{\partial y}\right) \mathrm{d} t=\frac{1}{\varepsilon}\left(O(\delta)+\int_{\bar{y}^*}^{y_M} \frac{G(y) f_x(G(y), y)}{g(G(y), y)} \mathrm{d} y+\varepsilon \tilde{M}\right)<0,$$

where $\tilde{M}>0$ is independent of positive ε and δ. Then the uniqueness of the relaxation oscillation cycle follows.

Proof. Assume that γ_ε starts in the right δ-neighborhood of the point $A\left(0, y_M\right)$; $t_1>0$ is the time that γ_ε reaches the right δ-neighborhood of the point $B\left(0, \bar{y}^*\right)$; $t_2>t_1$ is the time γ_ε reaches the δ-neighborhood of the point $C\left(x_r, \bar{y}^*\right)$; $t_3>t_2$ is the time that γ_ε reaches the right δ-neighborhood of the point $D\left(x_M, y_M\right)$. Note that the functions xf and g as well as their first and second order partial derivatives are bounded in V, and $|xf(x, y)| \geq m_0$, $|g(x, y)| \geq m_0$ in V for some $m_0>0$.

Next, we compute λ by the following five steps.

Step 1. From 0 to t_1 on γ_ε, $x(t) \leq \delta$ follows that

$$\frac{\partial(xf)}{\partial x}=f(x, y)+x f_x(x, y)=f(x, y)+O(\delta).$$

Due to $y'=\varepsilon g(x, y) \leq -m_0 \varepsilon$ and $x'=x f(x, y)=O(\delta)$, regard $x=x(t)$ as a function of y and there holds

$$\begin{aligned}\int_0^{t_1}\frac{\partial(xf)}{\partial x}\mathrm{d}t&=\frac{1}{\varepsilon}\int_{y_M}^{\bar{y}^*}\frac{f(x(y),y)+O(\delta)}{g(x(y),y)}\mathrm{d}y\\&=\frac{1}{\varepsilon}\left[O(\delta)-\int_{\bar{y}^*}^{y_M}\frac{f(0,y)}{g(0,y)}\mathrm{d}y\right]\\&=\frac{1}{\varepsilon}O(\delta).\end{aligned}$$

Step 2. From t_1 to t_2 on γ_ε, we have $x'(t)>m_0$ and $y'=\varepsilon g(x,y)$, regard $y=y(t)$ as a function of x and get

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\partial(xf)}{\partial x}\mathrm{d}t&=\int_{\delta}^{x_r+\delta}\frac{f(x,y(x))+xf_x(x,y(x))}{xf(x,y(x))}\mathrm{d}x\\&=\ln\frac{x_r+\delta}{\delta}+\int_{\delta}^{x_r+\delta}\frac{f_x(x,y)}{f(x,y)}\mathrm{d}x\\&\le M'\end{aligned}$$

for some $M'>0$.

Step 3. From t_2 to t_3 on γ_ε, due to $y'(t)=\varepsilon g(x,y)>\varepsilon m_0$, reconsider x as a function of y. Note that $x(y)-G(y)=O(\delta)$ for $t\in(t_2,t_3)$ and $f(G(y),y)=0$ on C_0^a, then

$$\begin{aligned}f(x(y),y)&=f(x(y),y)-f(G(y),y)=f_x(x(y),y)(x(y)-G(y))\\&=O(\delta)\\f_x(x(y),y)&=f_x(G(y),y)+f_x(x(y),y)-f_x(G(y),y)\\&=f_x(G(y),y)+f_{xx}(x(y),y)(x(y)-G(y))\\&=f_x(G(y),y)+O(\delta),\end{aligned}$$

and $g(x(y),y)=g(G(y),y)+O(\delta)$. Thus,

$$\begin{aligned}\int_{t_2}^{t_3}\frac{\partial(xf)}{\partial x}\mathrm{d}t&=\frac{1}{\varepsilon}\int_{\bar{y}^*}^{y_M}\frac{f(x(y),y)+x(y)f_x(x(y),y)}{g(x(y),y)}\mathrm{d}y\\&=\frac{1}{\varepsilon}\left[O(\delta)+\int_{\bar{y}^*}^{y_M}\frac{G(y)f_x(G(y),y)}{g(G(y),y)}\mathrm{d}y\right].\end{aligned}$$

Step 4. From t_3 to T on γ_ε, since $x'(t)<-m_0<0$ and $y'=\varepsilon g(x,y)$, consider y as a function of x, and apply similar procedures in step 2, there holds

$$\begin{aligned}\int_{t_3}^{T}\frac{\partial(xf)}{\partial x}\mathrm{d}t&=-\int_{\delta}^{x_M+\delta}\frac{f(x,y(x))+xf_x(x,y(x))}{xf(x,y(x))}\mathrm{d}x\\&=-\ln\frac{x_M+\delta}{\delta}-\int_{\delta}^{x_M+\delta}\frac{f_x(x,y)}{f(x,y)}\mathrm{d}x\\&\le M''\end{aligned}$$

for some $M''>0$.

Step 5. Using similar methods as those in Steps 1 - 4, we can obtain

$$\int_0^T \frac{\partial g}{\partial y}\mathrm{d}t \le \frac{M'''}{\varepsilon},\quad \text{for some } M'''>0.$$

Summarizing the formulaes in Steps 1 - 5, one can get

$$\lambda = \frac{1}{\varepsilon}\left(O(\delta) + \int_{\bar{y}^*}^{y_M} \frac{G(y) f_x\left(G(y),y\right)}{g\left(G(y),y\right)}\mathrm{d}y + \varepsilon\tilde{M}\right),$$

where $\tilde{M} = M' + M'' + M'''$ is independent of $\delta > 0$ and $\varepsilon > 0$ from the above analysis.

Since C_0^a is normally hyperbolic attracting, then $f_x\left(G(y),y\right) < 0$. Hence, we have

$$\int_{\bar{y}^*}^{y_M} \frac{G(y) f_x\left(G(y),y\right)}{g\left(G(y),y\right)}\mathrm{d}y < 0,$$

which indicates that $\lambda < 0$. Therefore, γ_ε is a locally asymptotically stable orbit, which follows the uniqueness of the relaxation oscillation cycle γ_ε.

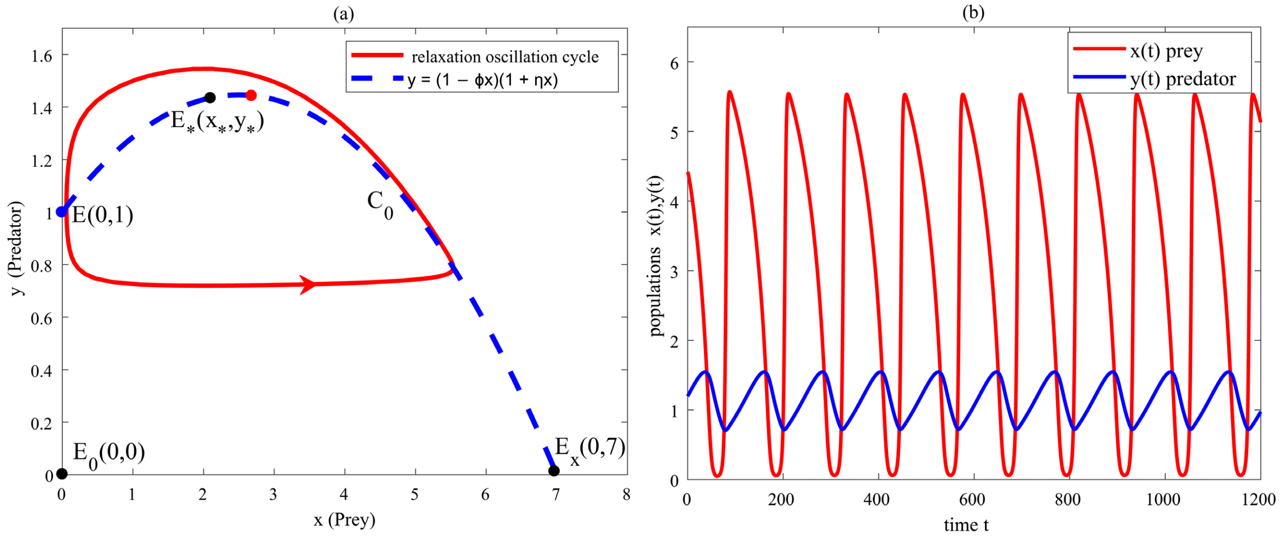

Figure 3. Numerical simulations of system (2) with initial point $(x_0, y_0) = (4.391, 1.223)$. (a) The relaxation oscillation cycle in red and the manifold C_0 in dashed blue. The three equilibria (black square), transcritical point (blue square) and fold point (red square) are also shown. (b) Time series of the relaxation oscillation.

4. Numerical Simulation

An example is given to verify the theoretical results.

Example 4.1. For system (2), choose $\varepsilon = 0.028, \phi = 1/7, \eta = 1/2$, then system (2) has a unique positive equilibrium point $E_*(2, 10/7)$, which is an unstable node. After calculation, we know that these parameters satisfy the conditions of Theorem 3.1 and Theorem 3.2, and our numerical simulation also shows that system (2) has a unique limit cycle, as seen in Figure 3, which displays the time series of limit cycle and the phasigram. Although the system admits a unique positive equilibrium, due to the slow-fast nature, the prey and predator finally varies periodically other than converging to the positive equilibrium.

5. Conclusion

In our work, we put forward and analyzed a classical Rosenzweig-MacArthur predator-prey model, which is dominated by logistic-based growth of prey, linear mortality rate of predator and Holling II function response. Predators and prey are located at different positions in the food chain, so they have different nutrition levels, evolution rates, birth rates and death rates. Generally, predators have higher nutrition levels, but prey has much higher evolution rates, birth rates and death rates than predators. Then we obtain the singularly perturbed (slow-fast) system with a small parameter ε by coordinate transformation and assuming that the mortality rate of prey c is far greater than the intrinsic growth rate of predator a. Next, we can investigate this modified Rosenzweig-MacArthur predator-prey model by geometric singular perturbation theory. We got the existence of the relaxation oscillation cycle by geometric singular perturbation theory and entry-exit function, and the uniqueness of the relaxation oscillation by the negativity of the Floquet multiplier. Finally, a numerical example is presented to verify the theoretical results.

6. Discussion

Wang and Zhang [15] got the existence and uniqueness of relaxation oscillation cycle of a slow-fast modified Leslie-Gower model. However, their work has too many restrictions on parameters and the proof of the unique is too brief. In their paper, the slow flow on positive y-axis directs downward for $y > e_2$ and directs upward for $0 < y < e_2$, while positive y-axis is normally hyperbolic attracting for $y > \frac{e_1}{a}$ and normally hyperbolic repelling for $0 < y < \frac{e_1}{a}$. Therefore, for existing a relaxation oscillation cycle, we must limit the parameters so that $e_2 < y_{out} < \frac{e_1}{a}$ (please refer to [15] for details). And The form of the unique positive equilibrium is also complex, which makes the parameters satisfying the parameter set U too limited. In our work, the slow flow on positive y-axis directs downward and positive y-axis is attracting for $y > 1$ and repelling for $0 < y < 1$, which makes many parameter restrictions less. Moreover, our work gives the concrete proof of uniqueness by proving the asymptotical stability of the cycle which follows from the negativity of the Floquet multiplier, which makes the existence and uniqueness of the relaxation oscillation cycle are more complete and rigorous.

We know that the Filippov system (please see [28]-[30]) where appears pervasively in real life models can better describe the predation relationship between species in nature, what are the effects of the Filippov slow-fast modified Rosenzweig-MacArthur model on the existence and uniqueness of relaxation oscillation cycle? We leave these to future investigations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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