

# Numerical Solutions of the Black-Scholes Financial Model Using the Adomian Decomposition and Power Series Collocation Methods

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**How to cite this paper:** Fatokun, J.O., Akinwale, O. and Abdulaziz, S. (2025) Numerical Solutions of the Black-Scholes Financial Model Using the Adomian Decomposition and Power Series Collocation Methods. *Journal of Applied Mathematics and Physics*, **13**, 1365-1391.

<https://doi.org/10.4236/jamp.2025.134074>

**Received:** January 16, 2025

**Accepted:** April 21, 2025

**Published:** April 24, 2025

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## Abstract

In the contractual agreement of an option, the value at which the contract is settled is one of the pertinent factors to consider and the problem is to compute and come up with a model that encapsulates the current price of an asset, strike price of the option, asset's volatility, maturity time and risk-free interest rate and the Black-Scholes model was the first model with all of these factors and this paper presents the algorithms to approximate solutions of Black Scholes financial model using the Adomian Decomposition and Power Series collocation methods, and presents comparative findings of the exact solution of Black Scholes financial model with approximate solutions from the Adomian Decomposition and Power Series collocation methods. The Adomian Decomposition method gives a better and more accurate result to the Power Series Collocation method for solving the Black Scholes financial model. Both methods are therefore efficient and agree with the exact solution of the model.

## Keywords

Black-Scholes, Power Series Collocation, Adomian Decomposition, Financial Model, Partial Differential Equation

## 1. Introduction

### 1.1. Overview

In the corporate business world, Finance is one of the fastest and consistently

growing area, and one of this area of finance is the Black-Scholes financial model which assumes that the price of asset in the financial market follows a geometric Brownian motion with recurring volatility and are efficient.

The Black-Scholes model is a mathematical model used to calculate the theoretical price of European options. This model has immensely influenced several fields including the field of quantitative finance and is an essential tool for investors and financial professionals.

The Black-Scholes model not only applies partial differential equations in option pricing and trading to determine the fair value of the option, but as the option approaches maturity it also considers the convergence of supply and demand. This model encloses a strategy that eliminates the directional risk related to the option position.

## 1.2. Problem Statement

Developed by Fischer Black and Myron Scholes in 1973 [1], the Black-Scholes model is an influential mathematical formula used to calculate the theoretical value of different types of stock. Published was the concept of the model in the paper “The Pricing of Options and Corporate Liabilities” which was later advanced by Robert Merton in the paper “Theory of Rational Options Pricing” and this model significantly changed the world of finance by providing a method to estimate the fair value of options.

In the contractual agreement of an option, the value at which the contract is settled is one of the pertinent factors to consider, because to expire the option, it signifies the price the buyer is to pay, and from the introduction of trading, the problem is to compute and come up with a model that encapsulates the current price of an asset, strike price of the option, asset’s volatility, maturity time and risk-free interest rate into consideration, and the Black-Scholes model was the first model with all of these factors.

However, different numerical methods have been used to find the numerical solution of the Black-Scholes model, and the purpose of this research is to explore mathematical methods, specifically Collocation and Decomposition method to enhance our knowledge of risk assessment in quantitative finance, option pricing and also efficiently solve the Black-Scholes financial model.

## 1.3. Paper Structure

The latter of this paper is structured thus: Section 2 is a review of relevant literatures on the Black-Scholes financial model, the Adomian Decomposition and Power Series Collocation methods, explicitly stating their applications, contributions and years of publication. Section 3 comprises of the system of methods, postulates, framework within which the research was conducted and foundation it was built, the step by step method of approach for solving the Black-Scholes financial model using the Decomposition and Power Series Collocation methods. Section 4 is basically the soul of this paper, it entails the result obtained using the

Matlab mathematical tool to solve the Black-Scholes Financial model using the Decomposition and Power Series Collocation method, while Section 5 gives the summary of the paper based on the result obtained in Section 4.

## 2. Overview of Literature

Reviews of literature with the background of the Black-Scholes financial model, other numerical solutions that have been used to find the numerical solution of the model, alongside literatures regarding the Decomposition and Power Series Collocation methods will be carried out here.

The Black-Scholes (B-S) financial model, proposed by Black and Scholes (1973) [1] has over the years contributed immensely to the attention option pricing has gotten as one of the most consistently traded financial products.

Option pricing problem was transformed by Black and Scholes (1973) into the new partial differential equation PDE with variable coefficients:

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

$$0 < S < \infty, 0 < t < T$$

And the idea behind this transformation according to Shinde and Takale (2012) [2] is the development of a safe portfolio positions in bonds, underlying stock and options. Because of the simplicity and explicit approach to the Black-Scholes model and equation, its partial differential equation PDE is applied in financial engineering to obtain the price of call options.

Amadi *et al.* (2020) [3] observed from the Crank-Nelson (CN) numerical solution for the evaluation of European call option and Black-Scholes analytic formula for the prices of European call option, that Black-Scholes analytical values are more accurate in terms of precision.

The Black-Scholes analytic formula for the prices of European call option as used by Amadi *et al.* (2020, [3]) is given below:

$$C = SN(d_1) - Ke^{-rT} N(d_2)$$

where  $d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$  and  $d_2 = d_1 - \sigma\sqrt{T}$ .

The CN discretization in tri-diagonal matrix, solvable at each steps as used by Amadi *et al.* (2020), [3] is given below:

$$a_i V_{i-1}^j + (1 + b_i) V_i^j - c_i V_{i+1}^j = a_i V_{i-1}^{j+1} + (1 - b_i) V_i^{j+1} + c_i V_{i+1}^{j+1}$$

where,  $a_i = \frac{\Delta t}{4} (\sigma^2 S_i^2 - rS_i)$ ,  $b_i = \frac{-\Delta t}{2} (\sigma^2 S_i^2 - r)$ ,  $c_i = \frac{\Delta t}{4} (\sigma^2 S_i^2 + rS_i)$ .

From **Table 1** below, BS exact values can be seen to be the best in terms of precision of the minimum values of the exact solutions of the European call options for the parameters used as shown in the 5<sup>th</sup> column for the BS and CN analytic formula respectively.

**Table 1.** Comparing the result of the Black-Scholes exact values and Crank-Nicolson method for European call option with different initial stock prices  $S_0 = 30, 35, 40, 45$  and  $50, r = 0.03, k = 25, T = 1$  and  $\sigma = 0.2$ .

Stock Prices ( $S_0$ )	Black-Scholes	Crank-Nicolson	Relative Error	Minimum Values
30	6.1368	6.1325	0.0007	6.1325:CN
35	10.8152	10.8148	0.0034	10.8148:CN
40	15.7513	15.7515	0.000095	15.7513:BS
45	20.7407	20.7409	0.0000096	20.7407:BS
50	25.7391	25.7399	0.000031	25.7391:BS

### 2.1. Adomian Decomposition and Power Series Collocation Methods

Fatokun and Gimba (2012) [4] derived and analyzed some collocation multistep methods where numerical solution of ODEs on configuration spaces formulated as homogenous manifolds evolved. Using Lie group approach, they examined the linear multi-step collocation method in solving the differential equation:

$$y' = f(y) = f_i(y)E_i, y = M$$

where  $E_i$  are vector fields on  $M$ .

At the point:

$$X = X_n + k \text{ where } k = 0, 1, 2, \dots, s$$

They collated the general linear method and obtained the discrete scheme that can be adapted to homogenous spaces.

With orders ranging from 1 to 9 and using a numerical quadrature rule, Fatokun *et al.* (2011) [5] derived steps  $k = 3, \dots, 8$  of Adams-like collocation multi-step method, established convergence of each derived step and showed the accuracy and efficiency of the proposed methods and provided option for continuous output where improved accuracy is obtainable by block hybrid methods for numerical derivatives.

### 2.2. Related Works

Iyakino and Fatokun (2015) [6] by appropriate change in variables transformed the Black-Scholes differential equation:

$$V_t + \frac{\sigma^2 S^2}{2} V_{ss} + rSV_s - rV = 0$$

Subject to  $V(0, t) = 0$ , for all  $t$

$$V(S, t) \sim S \text{ as } S \rightarrow \infty$$

$$V(S, T) = (S - K, 0)$$

Into the parabolic heat equation:

$$w_t = w_{xx}, x \in R, \tau \in \left(0, \frac{T\sigma^2}{2}\right)$$

Subject to  $w(x, 0) = \max \left\{ e^{\frac{k+1}{2}x} - e^{\frac{k-1}{2}x}, x \in R \right\}$

$$w(x, \tau) \rightarrow 0 \text{ as } x \rightarrow \pm\infty, \tau \in \left(0, \frac{T\sigma^2}{2}\right)$$

The transformed equation was then semi-discretized by the method of lines (MOL) into a system of ordinary differential equations (ODEs) numerically integrated by an L-stable trapezoidal-like integrator, the result obtained in comparison to the derived approximate theoretical solution to the transformed Black-Scholes equation:

$$w(x, \tau) = e^{(a+1)x+b\tau} - e^{ax+(b-k)\tau}$$

Using Maple 15 shows that trapezoidal-like integrator is an accurate time predictor of the solution of the Black Scholes equation.

Shinde and Takale (2012) [2] applied the Black-Scholes partial differential equation and formula in (2.1) and (1.1) using Maple software, they obtain the solution of Black-Scholes equation for valuing an option, and achieved this using the Black Scholes formula for a call price  $C$  and put price  $P$  as follows:

$$C_{call} = S\phi(d_1) - Xe^{-rT}\phi(d_2)$$

$$P_{put} = Xe^{-rT}\phi(-d_2) - S\phi(-d_1)$$

where  $d_1 = \frac{\log\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$  and  $d_2 = d_1 - \sigma\sqrt{T}$ ,

where  $C_{call}$  = the price of an option call,  $P_{put}$  = the price of an option put,  $S$  = Stock price  $X$  = Strike price of the option,  $r$  = risk free interest rate,  $t$  = Time in years,  $\sigma$  = implied volatility of underlying stock,  $\phi(\cdot)$  = the standard normal cumulative distribution function,  $\phi(d)$  denotes the standard normal cumulative distribution function, defined as follows:

$$\phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{x^2}{2}} dx$$

In application, they chose the parameters as follows:

$$E = 50, T = 0.5, r = 0.10, \sigma = 0.40$$

The result was examined using Maple by increasing the risk-free interest rate and volatility parameter,  $r = 0.11$ , and  $\sigma = 0.50$ , and find that the call option price changes by varying the parameters.

Oyakhire *et al.* (2019) [7] for computing the European call options pricing problems governed by the Black-Scholes equation, developed a proposed and efficient and fast scheme algorithm and this algorithm also works with the error analysis of the result of the European call option pricing. The proposed scheme's storage ca-

capacities and computational cost compared to Crank-Nicolson has a good performance in the sense of errors, and the accuracy of the proposed scheme compared to the Semi-implicit scheme is also better in most cases. The proposed scheme has a second order accuracy in space and time under some restrictions, the stability of the scheme was also presented in the sense of Non-Neumann analysis.

The proposed scheme is given thus:

$$A_j^{n+1} = a_1 V_{j+1}^n + b_1 V_j^n + C_1 V_{j-1}^{n+1}$$

$$B_j^{n+1} = a_2 V_{j-1}^n + b_2 V_j^n + C_2 V_{j+1}^{n+1}.$$

$$V_j^{n+1} = \frac{A_j^{n+1} + B_j^{n+1}}{2}$$

Using the proposed scheme above, numerical results were gotten using Matlab software, for the first example, European call options with the following parameters was considered for the simulation:

Set price ( $S$ ) = 100, Strike price ( $K$ ) = 100, Time ( $t$ ) = time, Volatility of asset price ( $\sigma$ ) = 0.2, risk free interest rate ( $r$ ) = 0.05, continuous dividend yield ( $q$ ) = 0.03, with a reference value of 6.029259.

**Table 2.** Below is the value obtained using the proposed scheme with fixed time steps size  $M$  and varying number of spatial grids  $N$ .

Time Step ( $M$ )	Spatial Steps ( $N$ )	Value	Error
1200	128	6.069953	0.040484
1200	256	6.047513	0.017784
1200	512	6.026426	0.003103
1200	1024	6.02884	0.000725

Using the proposed scheme, the values in **Table 2** above was gotten, and it is seen that with fixed time steps, the error of the values of the scheme decreases with decrease in spatial steps [7].

### 3. Methodology

This section discusses the Black-Scholes partial differential equation and a step by step method of solution of the PDE using the Decomposition and Power Series Collocation mathematical methods.

#### 3.1. Black-Scholes Equation

Black and Scholes (1973) transformed option pricing into the new partial differential equation with variable coefficients whose essence is the introduction of a safe portfolio positions in underlying stock, options and bonds. The Black-Scholes partial differential equation has been applied in various fields, including financial engineering to obtain the price option calls because of its simplicity and explicit approach.

As earlier established, the Black-Scholes partial differential equation is one for modelling the price of financial options and calculating the value of options considering factors like the option strike price, volatility, underlying asset price, time to expiration and risk-free interest rate, as given below:

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

$$0 < S < \infty, 0 < t < T$$

And it's important to note that the Black-Scholes formula: where the cumulative normal distribution function is denoted by  $N(\cdot)$  evaluated at  $N(d_1)$  and  $N(d_2)$  where  $d_1$  and  $d_2$  are given below:

$$d_1 = \frac{\log \frac{S_t}{K} + \left( r + \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma \sqrt{T - t}$$

Is a solution to the partial differential equation above, where  $F(s, t)$  = Option price,  $S_t/S$  = Stock price,  $K$  = Strike price of the option,  $\sigma$  = Volatility,  $T$  = Maturity time,  $r$  = Risk free interest rate,  $t$  = Time in years. One of the methods of solution we shall be using to solve the Black-Scholes partial differential equation is the Adomian Decomposition method [8].

### 3.2. Adomian Decomposition Method of Solution

Developed from early 1970s to the late 1990s by George Adomian, this method is centered on decomposing complex nonlinear equation into a series or systems of simpler sub equations. Each iteration in Adomian method involves solving simpler non-linear or linear equations and summing the solutions to obtain an approximate solution.

Below is the step by step method of solution for the power series collocation method, the approach is as follows:

**Step 1:**

Consider the Black Scholes Partial differential equation:

$$\frac{\partial F(S, t)}{\partial t} + rS \frac{\partial F(S, t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F(S, t)}{\partial S^2} - rF(S, t) = 0$$

$$0 < S < \infty, 0 < t < T$$

with the terminal condition:

$$F_T = F(S, T) = \max(S - K, 0) \text{ For a call option and,}$$

$$F_T = F(S, T) = \max(K - S, 0) \text{ For a put option.}$$

**Step 2:**

Express the Black-Scholes PDE in ADM operator form, we have:

$$L_t F(S, t) + rSL_s F(S, t) + \frac{1}{2} \sigma^2 S^2 L_{SS} F(S, t) - rF(S, t) = 0$$

$$L_t F(S, t) + rSL_s F(S, t) + \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) = rF(S, t)$$

where  $L_t = \frac{\partial}{\partial t}$ , and its inverse;  $L_t^{-1} = \int_t^T (\cdot) dx$ .

And  $L_s = \frac{\partial}{\partial S}$  while  $L_{ss} = \frac{\partial^2}{\partial S^2}$ .

**Step 3:**

Take  $L_t^{-1}$  of both side of the Black-Scholes PDE in ADM operator form, we have:

$$L_t^{-1} \left( L_t F(S, t) + rSL_s F(S, t) + \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) \right) = L_t^{-1} rF(S, t)$$

$$L_t^{-1} L_t F(S, t) + rSL_t^{-1} L_s F(S, t) + \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S, t) = rF(S, t) L_t^{-1}$$

$$L_t^{-1} L_t F(S, t) = rF(S, t) L_t^{-1} - rSL_t^{-1} L_s F(S, t) - \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S, t)$$

$$L_t^{-1} L_t F(S, t) = L_t^{-1} \left( rF(S, t) - rSL_s F(S, t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) \right)$$

But,  $L_t^{-1} L_t F(S, t) = \int_t^T \frac{\partial F(S, t)}{\partial t} dx = F(S, t) \Big|_t^T$ .

The equation becomes:

$$F(S, t) \Big|_t^T = L_t^{-1} \left( rF(S, t) - rSL_s F(S, t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) \right)$$

$$F(S, T) - F(S, t) = L_t^{-1} \left( rF(S, t) - rSL_s F(S, t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) \right)$$

$$F(S, t) = F_T - L_t^{-1} \left( rF(S, t) - rSL_s F(S, t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S, t) \right)$$

$$F(S, t) = F_T + \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S, t) + rSL_t^{-1} L_s F(S, t) - rF(S, t) L_t^{-1}$$

**Step 4:**

We assume a solution in the form of an infinite series for the unknown function  $F(S, t)$  and substitute the series:

$$F(S, t) = \sum_{n=0}^{\infty} F_n(S, t)$$

We'll have:

$$\sum_{n=0}^{\infty} F_n(S, t) = F_T + \frac{1}{2} \sigma^2 \sum_{n=0}^{\infty} L_t^{-1} S^2 L_{ss} F_n(S, t) + r \sum_{n=0}^{\infty} L_t^{-1} SL_s F_n(S, t) - r \sum_{n=0}^{\infty} L_t^{-1} F_n(S, t)$$

$$\sum_{n=0}^K F_n(S, t) = F_T + \frac{1}{2} \sigma^2 \sum_{n=0}^K L_t^{-1} S^2 L_{ss} F_n(S, t) + r \sum_{n=0}^K L_t^{-1} SL_s F_n(S, t) - r \sum_{n=0}^K L_t^{-1} F_n(S, t)$$

Each term of the approximation is then represented by:

$$F_0(S, t) = F_T$$

$$F_{n+1}(S, t) = \frac{1}{2} \sigma^2 L_t^{-1} S^2 L_{SS} F_n(S, t) + r L_t^{-1} S L_S F_n(S, t) - r L_t^{-1} F_n(S, t)$$

We'll substitute  $L_t^{-1} = \int_t^T (\cdot) dx$ ,  $L_S = \frac{\partial}{\partial S}$  and  $L_{SS} = \frac{\partial^2}{\partial S^2}$ :

$$F_{n+1}(S, t) = \frac{1}{2} \sigma^2 \int_t^T S^2 \frac{\partial^2}{\partial S^2} F_n(S, t) dx + r \int_t^T S \frac{\partial}{\partial S} F_n(S, t) dx - r \int_t^T F_n(S, t) dx$$

where  $n = 0, 1, 2, \dots, k$ .

An approximation for the solution of the PDE is given by the partial sum:

$$F(S, t) = \sum_{n=0}^K f_n(S, t)$$

### 3.3. Power Series Collocation Method of Solution

The power series collocation method is a numerical method that dates from the late 19<sup>th</sup> century to early 20<sup>th</sup> century. Carl David Tolmé Runge developed the collocation method in his effort to improving the numerical methods for ordinary and partial differential equations. This method stretches from several centuries and involves contributions of various scientists and scholars in the field of Mathematics and has been applied in Physics, Engineering among others fields with different types of differential equation.

The basic technique behind the power series collocation method is to choose a finite, say n-dimensional space called collocation points within the domain (i.e. boundary value condition and initial value condition) in consideration, and ensure the solution satisfies the ordinary or partial differential equation at these points.

Below is the step by step method of solution for the power series collocation method, the approach is as follows:

**Step 1:**

Consider the Black Scholes PDE:

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0$$

$$0 < S < \infty, 0 < t < T$$

Worthy of note is that the Black-Scholes PDE has two variables,  $t$  (time to maturity) and  $S$  (stock price), therefore, we assume the solution as:

$$F(S, t) = \sum_n \sum_m b_{nm} S^n t^m$$

**Step 2:**

Differentiate and substitute the solution into the PDE:

$$\frac{\partial F}{\partial t} = \sum_n \sum_m m b_{nm} S^n t^{m-1}$$

$$\frac{\partial F}{\partial S} = \sum_n \sum_m n b_{nm} S^{n-1} t^m$$

$$\frac{\partial^2 F}{\partial S^2} = \sum_n \sum_m n(n-1)b_{nm}S^{n-2}t^m$$

Substituting the solution in the equation, we have:

$$\sum_n \sum_m mb_{nm}S^n t^{m-1} + rS \sum_n \sum_m nb_{nm}S^{n-1}t^m + \frac{1}{2}\sigma^2 S^2 \sum_n \sum_m n(n-1)b_{nm}S^{n-2}t^m - r \sum_n \sum_m b_{nm}S^n t^m = 0$$

Simplifying:

$$\sum_n \sum_m mb_{nm}S^n t^{m-1} + r \sum_n \sum_m nb_{nm}S^n t^m + \frac{1}{2}\sigma^2 \sum_n \sum_m n(n-1)b_{nm}S^n t^m - r \sum_n \sum_m b_{nm}S^n t^m = 0$$

**Step 3:**

Using MATLAB’s symbolic algebraic solver, we’ll solve the equation to our desired degree, say  $n + m$  for each term and matching coefficients of same power degree to obtain an equal set of equations, say:

$$q_1, q_2, q_3, \dots, q_{n+m}$$

And unknown variables:

$$b_1, b_2, b_3, \dots, b_{n+m}$$

**Step 4:**

Solve equation:

$$q_1, q_2, q_3, \dots, q_{n+m}$$

We’ll solve these equations by applying the given initial value condition and boundary value condition of the Black-Scholes PDE to get the values of the unknown variables  $b_1, b_2, b_3, \dots, b_{n+m}$ .

**Step 5:**

Substitute the values of these variables  $b_1, b_2, b_3, \dots, b_{n+m}$  into the solution of the PDE, i.e.:

$$F(S, t) = \sum_n \sum_m b_{nm}S^n t^m$$

To get the result of the Black-Scholes PDE.

**4. Result**

In this section, the solution of the Black Scholes PDE using the Adomian Decomposition and Power Series Collocation Method shall be considered. The results are obtained using MATLAB.

**4.1. Solution of the Black Scholes PDE Using Adomian Decomposition Method**

**Example 1:** Consider the Black Scholes PDE:

$$\frac{\partial F(S,t)}{\partial t} + rS \frac{\partial F(S,t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F(S,t)}{\partial S^2} - rF(S,t) = 0$$

$$0 < S < \infty, 0 < t < T$$

with the initial/terminal condition:

$$F_T = F(S, T) = \max(S - K, 0)$$

where  $S = 30, T = 1, t = 0, K = 25, \sigma = 0.2, r = 0.2$  with the exact solution;  
 $BS = 6.1368$ .

Express the Black-Scholes PDE in ADM operator form:

$$L_t F(S,t) + rSL_s F(S,t) + \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) - rF(S,t) = 0$$

$$L_t F(S,t) + rSL_s F(S,t) + \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) = rF(S,t)$$

where  $L_t = \frac{\partial}{\partial t}$ , and its inverse;  $L_t^{-1} = \int_t^T (\cdot) dx$ .

And  $L_s = \frac{\partial}{\partial S}$  while  $L_{ss} = \frac{\partial^2}{\partial S^2}$ .

Take  $L_t^{-1}$  of both side of the Black-Scholes PDE in ADM operator form:

$$L_t^{-1} \left( L_t F(S,t) + rSL_s F(S,t) + \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) \right) = L_t^{-1} rF(S,t)$$

$$L_t^{-1} L_t F(S,t) + rSL_t^{-1} L_s F(S,t) + \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S,t) = rF(S,t) L_t^{-1}$$

$$L_t^{-1} L_t F(S,t) = rF(S,t) L_t^{-1} - rSL_t^{-1} L_s F(S,t) - \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S,t)$$

$$L_t^{-1} L_t F(S,t) = L_t^{-1} \left( rF(S,t) - rSL_s F(S,t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) \right)$$

But,  $L_t^{-1} L_t F(S,t) = \int_t^T \frac{\partial F(S,t)}{\partial t} dx = F(S,t) \Big|_t^T$ .

The equation becomes:

$$F(S,t) \Big|_t^T = L_t^{-1} \left( rF(S,t) - rSL_s F(S,t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) \right)$$

$$F(S,T) - F(S,t) = L_t^{-1} \left( rF(S,t) - rSL_s F(S,t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) \right)$$

$$F(S,t) = F_T - L_t^{-1} \left( rF(S,t) - rSL_s F(S,t) - \frac{1}{2} \sigma^2 S^2 L_{ss} F(S,t) \right)$$

$$F(S,t) = F_T + \frac{1}{2} \sigma^2 S^2 L_t^{-1} L_{ss} F(S,t) + rSL_t^{-1} L_s F(S,t) - rF(S,t) L_t^{-1}$$

We assume a solution in the form of an infinite series for the unknown function  $F(S,t)$  and substitute the series:

$$F(S,t) = \sum_{n=0}^{\infty} F_n(S,t)$$

We'll have:

$$\sum_{n=0}^{\infty} F_n(S, t) = F_T + \frac{1}{2} \sigma^2 \sum_{n=0}^{\infty} L_t^{-1} S^2 L_{SS} F_n(S, t) + r \sum_{n=0}^{\infty} L_t^{-1} S L_S F_n(S, t) - r \sum_{n=0}^{\infty} L_t^{-1} F_n(S, t)$$

$$\sum_{n=0}^K F_n(S, t) = F_T + \frac{1}{2} \sigma^2 \sum_{n=0}^K L_t^{-1} S^2 L_{SS} F_n(S, t) + r \sum_{n=0}^K L_t^{-1} S L_S F_n(S, t) - r \sum_{n=0}^K L_t^{-1} F_n(S, t)$$

Each term of the approximation is then represented by:

$$F_0(S, t) = F_T$$

$$F_{n+1}(S, t) = \frac{1}{2} \sigma^2 L_t^{-1} S^2 L_{SS} F_n(S, t) + r L_t^{-1} S L_S F_n(S, t) - r L_t^{-1} F_n(S, t)$$

We'll substitute  $L_t^{-1} = \int_t^T (\cdot) dx$ ,  $L_S = \frac{\partial}{\partial S}$  and  $L_{SS} = \frac{\partial^2}{\partial S^2}$

$$F_{n+1}(S, t) = \frac{1}{2} \sigma^2 \int_t^T S^2 \frac{\partial^2}{\partial S^2} F_n(S, t) dx + r \int_t^T S \frac{\partial}{\partial S} F_n(S, t) dx - r \int_t^T F_n(S, t) dx$$

where  $n = 0, 1, 2, \dots, k = 25$ .

To get the value of each term of the iteration, we'll substitute the values of  $S = 30$ ,  $T = 1$ ,  $t = 0$ ,  $K = 25$ ,  $\sigma = 0.2$ ,  $r = 0.2$ .

For  $F_0(S, t)$ :

$$F_0(S, t) = (30 - 25, 0) = 5$$

For  $F_1(S, t)$ :

$$F_{0+1}(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_0(S, t) dx$$

$$+ 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_0(S, t) dx - 0.2 \int_0^1 F_0(S, t) dx$$

$$F_1(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} 5 dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} 5 dx - 0.2 \int_0^1 5 dx = -1$$

For  $F_2(S, t)$ :

$$F_{1+1}(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_1(S, t) dx$$

$$+ 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_1(S, t) dx - 0.2 \int_0^1 F_1(S, t) dx$$

$$F_2(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-1) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-1) dx - 0.2 \int_0^1 (-1) dx$$

$$= 0.2000$$

For  $F_3(S, t)$ :

$$F_{2+1}(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_2(S, t) dx$$

$$+ 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_2(S, t) dx - 0.2 \int_0^1 F_2(S, t) dx$$

$$F_3(S, t) = \frac{1}{2} (0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (0.2000) dx$$

$$+ 0.2 \int_0^1 30 \frac{\partial}{\partial S} (0.2000) dx - 0.2 \int_0^1 (0.2000) dx$$

$$= -0.0400$$

For  $F_4(S, t)$ :

$$\begin{aligned}
 F_{3+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_3(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_3(S, t) dx - 0.2 \int_0^1 F_3(S, t) dx \\
 F_4(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-0.0400) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-0.0400) dx - 0.2 \int_0^1 (-0.0400) dx \\
 &= 0.0080
 \end{aligned}$$

For  $F_5(S, t)$ :

$$\begin{aligned}
 F_{4+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_4(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_4(S, t) dx - 0.2 \int_0^1 F_4(S, t) dx \\
 F_5(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (0.0080) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (0.0080) dx - 0.2 \int_0^1 (0.0080) dx \\
 &= -0.0016
 \end{aligned}$$

For  $F_6(S, t)$ :

$$\begin{aligned}
 F_{5+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_5(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_5(S, t) dx - 0.2 \int_0^1 F_5(S, t) dx \\
 F_6(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-0.0016) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-0.0016) dx - 0.2 \int_0^1 (-0.0016) dx \\
 &= 3.2000 \times 10^{-4}
 \end{aligned}$$

For  $F_7(S, t)$ :

$$\begin{aligned}
 F_{6+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_6(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_6(S, t) dx - 0.2 \int_0^1 F_6(S, t) dx \\
 F_7(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (3.2000 \times 10^{-4}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (3.2000 \times 10^{-4}) dx - 0.2 \int_0^1 (3.2000 \times 10^{-4}) dx \\
 &= -6.4000 \times 10^{-5}
 \end{aligned}$$

For  $F_8(S, t)$ :

$$F_{7+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_7(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_7(S, t) dx - 0.2 \int_0^1 F_7(S, t) dx$$

$$F_8(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-6.4000 \times 10^{-5}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-6.4000 \times 10^{-5}) dx - 0.2 \int_0^1 (-6.4000 \times 10^{-5}) dx = 1.2800 \times 10^{-5}$$

For  $F_9(S, t)$ :

$$F_{8+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_8(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_8(S, t) dx - 0.2 \int_0^1 F_8(S, t) dx$$

$$F_9(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (1.2800 \times 10^{-5}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (1.2800 \times 10^{-5}) dx - 0.2 \int_0^1 (1.2800 \times 10^{-5}) dx = -2.5600 \times 10^{-6}$$

For  $F_{10}(S, t)$ :

$$F_{9+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_9(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_9(S, t) dx - 0.2 \int_0^1 F_9(S, t) dx$$

$$F_{10}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-2.5600 \times 10^{-6}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-2.5600 \times 10^{-6}) dx - 0.2 \int_0^1 (-2.5600 \times 10^{-6}) dx = 5.1200 \times 10^{-7}$$

For  $F_{11}(S, t)$ :

$$F_{10+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{10}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{10}(S, t) dx - 0.2 \int_0^1 F_{10}(S, t) dx$$

$$F_{11}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (5.1200 \times 10^{-7}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (5.1200 \times 10^{-7}) dx - 0.2 \int_0^1 (5.1200 \times 10^{-7}) dx = -1.0240 \times 10^{-7}$$

For  $F_{12}(S, t)$ :

$$F_{11+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{11}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{11}(S, t) dx - 0.2 \int_0^1 F_{11}(S, t) dx$$

$$\begin{aligned}
 F_{12}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-1.0240 \times 10^{-7}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-1.0240 \times 10^{-7}) dx - 0.2 \int_0^1 (-1.0240 \times 10^{-7}) dx \\
 &= 2.0480 \times 10^{-8}
 \end{aligned}$$

For  $F_{13}(S,t)$ :

$$\begin{aligned}
 F_{12+1}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{12}(S,t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{12}(S,t) dx - 0.2 \int_0^1 F_{12}(S,t) dx \\
 F_{13}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (2.0480 \times 10^{-8}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (2.0480 \times 10^{-8}) dx - 0.2 \int_0^1 (2.0480 \times 10^{-8}) dx \\
 &= -4.0960 \times 10^{-9}
 \end{aligned}$$

For  $F_{14}(S,t)$ :

$$\begin{aligned}
 F_{13+1}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{13}(S,t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{13}(S,t) dx - 0.2 \int_0^1 F_{13}(S,t) dx \\
 F_{14}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-4.0960 \times 10^{-9}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-4.0960 \times 10^{-9}) dx - 0.2 \int_0^1 (-4.0960 \times 10^{-9}) dx \\
 &= 8.1920 \times 10^{-10}
 \end{aligned}$$

For  $F_{15}(S,t)$ :

$$\begin{aligned}
 F_{14+1}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{14}(S,t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{14}(S,t) dx - 0.2 \int_0^1 F_{14}(S,t) dx \\
 F_{15}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (8.1920 \times 10^{-10}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (8.1920 \times 10^{-10}) dx - 0.2 \int_0^1 (8.1920 \times 10^{-10}) dx \\
 &= -1.6384 \times 10^{-10}
 \end{aligned}$$

For  $F_{16}(S,t)$ :

$$\begin{aligned}
 F_{15+1}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{15}(S,t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{15}(S,t) dx - 0.2 \int_0^1 F_{15}(S,t) dx \\
 F_{16}(S,t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-1.6384 \times 10^{-10}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-1.6384 \times 10^{-10}) dx - 0.2 \int_0^1 (-1.6384 \times 10^{-10}) dx \\
 &= 3.2768 \times 10^{-11}
 \end{aligned}$$

For  $F_{17}(S, t)$ :

$$\begin{aligned}
 F_{16+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{16}(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{16}(S, t) dx - 0.2 \int_0^1 F_{16}(S, t) dx \\
 F_{17}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (3.2768 \times 10^{-11}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (3.2768 \times 10^{-11}) dx - 0.2 \int_0^1 (3.2768 \times 10^{-11}) dx \\
 &= -6.5536 \times 10^{-12}
 \end{aligned}$$

For  $F_{18}(S, t)$ :

$$\begin{aligned}
 F_{17+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{17}(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{17}(S, t) dx - 0.2 \int_0^1 F_{17}(S, t) dx \\
 F_{18}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-6.5536 \times 10^{-12}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-6.5536 \times 10^{-12}) dx - 0.2 \int_0^1 (-6.5536 \times 10^{-12}) dx \\
 &= 1.3107 \times 10^{-12}
 \end{aligned}$$

For  $F_{19}(S, t)$ :

$$\begin{aligned}
 F_{18+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{18}(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{18}(S, t) dx - 0.2 \int_0^1 F_{18}(S, t) dx \\
 F_{19}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (1.3107 \times 10^{-12}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (1.3107 \times 10^{-12}) dx - 0.2 \int_0^1 (1.3107 \times 10^{-12}) dx \\
 &= -2.6214 \times 10^{-13}
 \end{aligned}$$

For  $F_{20}(S, t)$ :

$$\begin{aligned}
 F_{19+1}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{19}(S, t) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{19}(S, t) dx - 0.2 \int_0^1 F_{19}(S, t) dx \\
 F_{20}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-2.6214 \times 10^{-13}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-2.6214 \times 10^{-13}) dx - 0.2 \int_0^1 (-2.6214 \times 10^{-13}) dx \\
 &= 5.2428 \times 10^{-14}
 \end{aligned}$$

For  $F_{21}(S, t)$ :

$$F_{20+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{20}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{20}(S, t) dx - 0.2 \int_0^1 F_{20}(S, t) dx$$

$$F_{21}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (5.2428 \times 10^{-14}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (5.2428 \times 10^{-14}) dx - 0.2 \int_0^1 (5.2428 \times 10^{-14}) dx = -1.0486 \times 10^{-14}$$

For  $F_{22}(S, t)$ :

$$F_{21+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{21}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{21}(S, t) dx - 0.2 \int_0^1 F_{21}(S, t) dx$$

$$F_{22}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-1.0486 \times 10^{-14}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-1.0486 \times 10^{-14}) dx - 0.2 \int_0^1 (-1.0486 \times 10^{-14}) dx = 2.0972 \times 10^{-15}$$

For  $F_{23}(S, t)$ :

$$F_{22+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{22}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{22}(S, t) dx - 0.2 \int_0^1 F_{22}(S, t) dx$$

$$F_{23}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (2.0972 \times 10^{-15}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (2.0972 \times 10^{-15}) dx - 0.2 \int_0^1 (2.0972 \times 10^{-15}) dx = -4.1944 \times 10^{-16}$$

For  $F_{24}(S, t)$ :

$$F_{23+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{23}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{23}(S, t) dx - 0.2 \int_0^1 F_{23}(S, t) dx$$

$$F_{24}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (-4.1944 \times 10^{-16}) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (-4.1944 \times 10^{-16}) dx - 0.2 \int_0^1 (-4.1944 \times 10^{-16}) dx = 8.3888 \times 10^{-17}$$

For  $F_{25}(S, t)$ :

$$F_{24+1}(S, t) = \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} F_{24}(S, t) dx + 0.2 \int_0^1 30 \frac{\partial}{\partial S} F_{24}(S, t) dx - 0.2 \int_0^1 F_{24}(S, t) dx$$

$$\begin{aligned}
 F_{25}(S, t) &= \frac{1}{2}(0.2)^2 \int_0^1 30^2 \frac{\partial^2}{\partial S^2} (8.3888 \times 10^{-17}) dx \\
 &\quad + 0.2 \int_0^1 30 \frac{\partial}{\partial S} (8.3888 \times 10^{-17}) dx - 0.2 \int_0^1 (8.3888 \times 10^{-17}) dx \\
 &= -1.6778 \times 10^{-17}
 \end{aligned}$$

An approximation for the solution of the PDE is given by the partial sum:

$$\begin{aligned}
 F(S, t) &= \sum_{n=0}^K f_n(S, t) \\
 &= 5 + (-1) + 0.2000 + (-0.0400) + 0.0080 + (-0.0016) + 3.2000 \times 10^{-4} \\
 &\quad + (-6.4000 \times 10^{-5}) + 1.2800 \times 10^{-5} + (-2.5600 \times 10^{-6}) + 5.1200 \times 10^{-7} \\
 &\quad + (-1.0240 \times 10^{-7}) + 2.0480 \times 10^{-8} + (-4.0960 \times 10^{-9}) + 8.1920 \times 10^{-10} \\
 &\quad + (-1.6384 \times 10^{-10}) + 3.2768 \times 10^{-11} + (-6.5536 \times 10^{-12}) + 1.3107 \times 10^{-12} \\
 &\quad + (-2.6214 \times 10^{-13}) + 5.2428 \times 10^{-14} + (-1.0486 \times 10^{-14}) + 2.0972 \times 10^{-15} \\
 &\quad + (-4.1944 \times 10^{-16}) + 8.3888 \times 10^{-17} + (-1.6778 \times 10^{-17}) \\
 &= 4.1667
 \end{aligned}$$

Using similar approach with the example above, other examples are solved for stock prices with the values: 45, 60, 70 and 75 alternating the values of the volatility from 0.2 to 0.25 for each stock prices to study the behavior of the option at maturity, while the values of other variables remain the same. The result from the solution shall be discussed in Section 5.

### 4.2. Solution of the Black Scholes PDE Using Power Series Collocation Method

**Example 1:** Consider the Black Scholes PDE:

$$\begin{aligned}
 \frac{\partial F(S, t)}{\partial t} + rS \frac{\partial F(S, t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F(S, t)}{\partial S^2} - rF(S, t) &= 0 \\
 0 < S < \infty, 0 < t < T
 \end{aligned}$$

with the initial condition:

$$F_{N,m} = (m\Delta S - K, 0) \text{ for } m = 1, 2, 3, \dots, N$$

Boundary Condition:

$$F_{n,1} = 0 \text{ for } n = 1, 2, 3, \dots, M$$

$$F_{n,M} = S_{\max} - Ke^{-r(N-n)\Delta t} \text{ for } n = 0, 1, 2, 3, \dots, M$$

where  $S = 30$ ,  $t = 1$ ,  $K = 25$ ,  $\sigma = 0.2$ ,  $r = 0.2$  with the exact solution;  $BS = 6.1368$ .

We assume the solution for  $n = 3, m = 3$ :

$$F(S, t) = \sum_{n=1}^3 \sum_{m=1}^3 b_{nm} S^n t^m$$

Expanding  $F(S, t)$ :

$$F(S, t) = \sum_{n=1}^3 (b_{n1} S^n t + b_{n2} S^n t^2 + b_{n3} S^n t^3)$$

$$F(S, t) = (b_{11}St + b_{12}St^2 + b_{13}St^3) + (b_{21}S^2t + b_{22}S^2t^2 + b_{23}S^2t^3) + (b_{31}S^3t + b_{32}S^3t^2 + b_{33}S^3t^3)$$

$$F(S, t) = b_{11}St + b_{12}St^2 + b_{13}St^3 + b_{21}S^2t + b_{22}S^2t^2 + b_{23}S^2t^3 + b_{31}S^3t + b_{32}S^3t^2 + b_{33}S^3t^3$$

Differentiating:

$$\frac{\partial F(S, t)}{\partial t} = b_{11}S + 2b_{12}St + 3b_{13}St^2 + b_{21}S^2 + 2b_{22}S^2t + 3b_{23}S^2t^2 + b_{31}S^3 + 2b_{32}S^3t + 3b_{33}S^3t^2$$

$$\frac{\partial F(S, t)}{\partial S} = b_{11}t + b_{12}t^2 + b_{13}t^3 + 2b_{21}St + 2b_{22}St^2 + 2b_{23}St^3 + 3b_{31}S^2t + 3b_{32}S^2t^2 + 3b_{33}S^2t^3$$

$$\frac{\partial^2 F(S, t)}{\partial S^2} = 2b_{21}t + 2b_{22}t^2 + 2b_{23}t^3 + 6b_{31}St + 6b_{32}St^2 + 6b_{33}St^3$$

Substitute  $\frac{\partial F(S, t)}{\partial t}$ ,  $\frac{\partial F(S, t)}{\partial S}$  and  $\frac{\partial^2 F(S, t)}{\partial S^2}$  into the Black-Scholes PDE:

$$b_{11}S + 2b_{12}St + 3b_{13}St^2 + b_{21}S^2 + 2b_{22}S^2t + 3b_{23}S^2t^2 + b_{31}S^3 + 2b_{32}S^3t + 3b_{33}S^3t^2 + rS(b_{11}t + b_{12}t^2 + b_{13}t^3 + 2b_{21}St + 2b_{22}St^2 + 2b_{23}St^3 + 3b_{31}S^2t + 3b_{32}S^2t^2 + 3b_{33}S^2t^3) + \frac{1}{2}\sigma^2S^2(2b_{21}t + 2b_{22}t^2 + 2b_{23}t^3 + 6b_{31}St + 6b_{32}St^2 + 6b_{33}St^3) - r(b_{11}St + b_{12}St^2 + b_{13}St^3 + b_{21}S^2t + b_{22}S^2t^2 + b_{23}S^2t^3 + b_{31}S^3t + b_{32}S^3t^2 + b_{33}S^3t^3) = 0$$

Simplifying:

$$b_{11}S + 2b_{12}St + 3b_{13}St^2 + b_{21}S^2 + 2b_{22}S^2t + 3b_{23}S^2t^2 + b_{31}S^3 + 2b_{32}S^3t + 3b_{33}S^3t^2 + rStb_{11} + rSt^2b_{12} + rSt^3b_{13} + 2rS^2tb_{21} + 2rS^2t^2b_{22} + 2rS^2t^3b_{23} + 3rS^3tb_{31} + 3rS^3t^2b_{32} + 3rS^3t^3b_{33} + \sigma^2S^2tb_{21} + \sigma^2S^2t^2b_{22} + \sigma^2S^2t^3b_{23} + 3\sigma^2S^3tb_{31} + 3\sigma^2S^3t^2b_{32} + 3\sigma^2S^3t^3b_{33} - rStb_{11} - rSt^2b_{12} - rSt^3b_{13} - rS^2tb_{21} - rS^2t^2b_{22} - rS^2t^3b_{23} - rS^3tb_{31} - rS^3t^2b_{32} - rS^3t^3b_{33} = 0$$

Simplifying in order of like terms:

$$b_{11}S + rStb_{11} - rStb_{11} + 2b_{12}St + rSt^2b_{12} - rSt^2b_{12} + 3b_{13}St^2 + rSt^3b_{13} - rSt^3b_{13} + b_{21}S^2 + 2rS^2tb_{21} + \sigma^2S^2tb_{21} - rS^2tb_{21} + 2b_{22}S^2t + 2rS^2t^2b_{22} + \sigma^2S^2t^2b_{22} - rS^2t^2b_{22} + 3b_{23}S^2t^2 + 2rS^2t^3b_{23} + \sigma^2S^2t^3b_{23} - rS^2t^3b_{23} + b_{31}S^3 + 3rS^3tb_{31} + 3\sigma^2S^3tb_{31} - rS^3tb_{31} + 2b_{32}S^3t + 3rS^3t^2b_{32} + 3\sigma^2S^3t^2b_{32} - rS^3t^2b_{32} + 3b_{33}S^3t^2 + 3rS^3t^3b_{33} + 3\sigma^2S^3t^3b_{33} - rS^3t^3b_{33} = 0$$

Simplifying further:

$$Sb_{11} + 2Stb_{12} + 3St^2b_{13} + (S^2 + \sigma^2S^2t + rS^2t)b_{21} + (2S^2t + \sigma^2S^2t^2 + rS^2t^2)b_{22} + (3S^2t^2 + \sigma^2S^2t^3 + rS^2t^3)b_{23} + (S^3 + 3\sigma^2S^3t + 2rS^3t)b_{31} + (2S^3t + 3\sigma^2S^3t^2 + 2rS^3t^2)b_{32} + (3S^3t^2 + 3\sigma^2S^3t^3 + 2rS^3t^3)b_{33} = 0$$

Recall:

$$S = 30, t = 1, K = 25, \sigma = 0.2, r = 0.2$$

We'll discretize the values of  $S$  and  $t$ :

$$S_1 = 10, S_2 = 20, S_3 = 30$$

$$t_1 = 0.333, t_2 = 0.667, t_3 = 1$$

We'll apply the initial and boundary conditions:

$$F_{N,m} = (m\Delta S - K, 0) \text{ for } m = 1, 2, 3, \dots, N$$

$$F_{n,1} = 0 \text{ for } n = 1, 2, 3, \dots, M$$

$$F_{n,M} = S_{\max} - Ke^{-r(N-n)\Delta t} \text{ for } n = 0, 1, 2, 3, \dots, M$$

And approximate at the mesh points:

$$S_1 t_1 = 0 \quad S_1 t_2 = 0 \quad S_1 t_3 = 8.1177$$

$$S_2 t_1 = 0 \quad S_2 t_2 = 0 \quad S_2 t_3 = 6.6108$$

$$S_3 t_1 = 0 \quad S_3 t_2 = 0 \quad S_3 t_3 = 5$$

Substitute the values of  $\sigma = 0.2, r = 0.2$  and the values of  $S$  and  $t$  at each mesh points into the simplified equation to get a system of 9 equations:

$$10b_{11} + 6.66b_{12} + 3.3267b_{13} + 107.9920b_{21} + 69.2613b_{22} + 34.1529b_{23} \\ + 1173.2b_{31} + 723.6623b_{32} + 351.8685b_{33} = 0$$

$$10b_{11} + 13.34b_{12} + 13.3467b_{13} + 116.0080b_{21} + 144.0773b_{22} + 140.5885b_{23} \\ + 1346.8b_{31} + 1565.3b_{32} + 1489b_{33} = 0$$

$$10b_{11} + 20b_{12} + 30b_{13} + 124b_{21} + 224b_{22} + 324b_{23} + 1520b_{31} + 2520b_{32} \\ + 3520b_{33} = 8.1177$$

$$20b_{11} + 13.32b_{12} + 6.6533b_{13} + 431.9680b_{21} + 277.0453b_{22} + 136.6117b_{23} \\ + 9385.3b_{31} + 5789.3b_{32} + 2814.9b_{33} = 0$$

$$20b_{11} + 26.68b_{12} + 26.6933b_{13} + 464.0320b_{21} + 576.3093b_{22} + 562.3539b_{23} \\ + 10775b_{31} + 12523b_{32} + 11912b_{33} = 0$$

$$20b_{11} + 40b_{12} + 60b_{13} + 496b_{21} + 896b_{22} + 1296b_{23} + 12160b_{31} + 20160b_{32} \\ + 28160b_{33} = 6.6108$$

$$30b_{11} + 19.98b_{12} + 9.98b_{13} + 971.9280b_{21} + 623.3520b_{22} + 307.3763b_{23} \\ + 31675b_{31} + 19539b_{32} + 9500.5b_{33} = 0$$

$$30b_{11} + 40.02b_{12} + 40.04b_{13} + 1044.1b_{21} + 1296.7b_{22} + 1265.3b_{23} \\ + 36365b_{31} + 42264b_{32} + 40202b_{33} = 0$$

$$30b_{11} + 60b_{12} + 90b_{13} + 1116b_{21} + 2016b_{22} + 2916b_{23} + 41040b_{31} \\ + 68040b_{32} + 95040b_{33} = 5$$

Solving the system of equations using Matlab to get values of the unknown:

$$b_{11} = 1.6103$$

$$b_{12} = -3.6251$$

$$\begin{aligned}
 b_{13} &= 2.4168 \\
 b_{21} &= -0.088356 \\
 b_{22} &= 0.20584 \\
 b_{23} &= -0.13804 \\
 b_{31} &= 0.0013342 \\
 b_{32} &= -0.0032245 \\
 b_{33} &= 0.0021831
 \end{aligned}$$

Substitute the values of the unknowns and  $S = 30, t = 1$  into the assumed solution to get an approximation for the solution of the PDE:

$$\begin{aligned}
 F(S, t) &= \sum_{n=1}^3 \sum_{m=1}^3 b_{nm} S^n t^m \\
 &= b_{11}St + b_{12}St^2 + b_{13}St^3 + b_{21}S^2t + b_{22}S^2t^2 + b_{23}S^2t^3 + b_{31}S^3t + b_{32}S^3t^2 + b_{33}S^3t^3 \\
 &= (1.6103 \times 30 \times 1) + ((-3.6251) \times 30 \times 1^2) + (2.4168 \times 30 \times 1^3) \\
 &\quad + ((-0.088356) \times 30^2 \times 1) + (0.20584 \times 30^2 \times 1^2) + ((-0.13804) \times 30^2 \times 1^3) \\
 &\quad + (0.0013342 \times 30^3 \times 1) + ((-0.0032245) \times 30^3 \times 1^2) + (0.0021831 \times 30^3 \times 1^3) \\
 &= 1.4652
 \end{aligned}$$

Using similar approach with the examples above, other examples are solved for stock prices with the values: 45, 60, 70 and 75 alternating the values of the volatility from 0.2 to 0.25 for each stock prices to study the behavior of the option at maturity while the values of other variables remain the same. The result from the solution shall be discussed in Section 5.

## 5. Discussion

In this section, we shall discuss results of the solutions from Section 4.

### 5.1. Solution of the Black Scholes PDE Using Adomian Decomposition Method

Ten examples of Black Scholes PDE were solved using the Adomian Decomposition method as presented in the table below:

**Table 3.** Exact and adomian decomposition method solution of black scholes PDE.

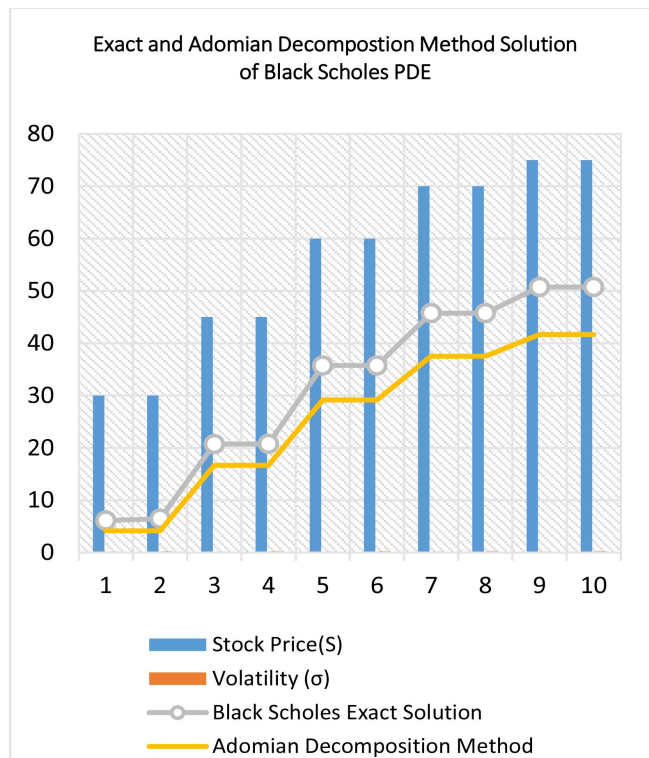
Stock Price ( $S$ )	Strike Price ( $K$ )	Volatility ( $\sigma$ )	Interest rate ( $r$ )	Time ( $t$ )	Black Scholes Exact Solution	Adomian Decomposition Method
30	25	0.2	0.2	1	6.1368	4.1667
30	25	0.25	0.2	1	6.478	4.1667
45	25	0.2	0.2	1	20.7407	16.667
45	25	0.25	0.2	1	20.7568	16.667
60	25	0.2	0.2	1	35.7389	29.1666
60	25	0.25	0.2	1	35.7392	29.1666

**Continued**

70	25	0.2	0.2	1	45.7389	37.5
70	25	0.25	0.2	1	45.7389	37.5
75	25	0.2	0.2	1	50.7389	41.6667
75	25	0.25	0.2	1	50.7389	41.6667

Worthy of note from the ADM method is that the volatility of the *asset alone* does not significantly determine the value of the call option at maturity. In example two, the volatility of same asset in example 1 was increased from 0.2 to 0.25 to study the behavior of the asset at maturity, and both gives exactly the same approximate values. Same experiment was carried out in example 3 and 4, 5 and 6, 7 and 8, 9 and 10, and they all give the same approximate values.

The approximate solution of the ADM in the ten examples as seen in **Table 3** and **Figure 1** shows that they are not exactly the same as the exact solution, but are within the same digit value, which indicates that the method gives a good approximation of the solution of the Black Scholes PDE.



**Figure 1.** Exact and adomian decomposition method solution of black scholes PDE.

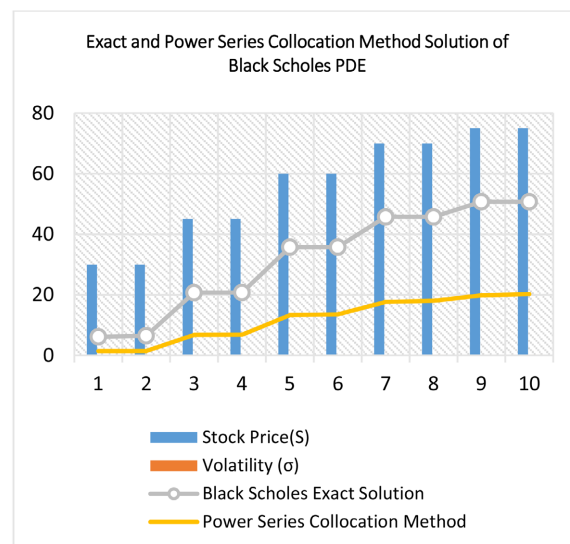
**5.2. Solution of the Black Scholes PDE Using Power Series Collocation Method**

Ten examples of Black Scholes PDE were solved using the Power Series Collocation method as presented in the table below:

**Table 4.** Exact and power series collocation method solution of black scholes PDE.

Stock Price ( $S$ )	Strike Price ( $K$ )	Volatility ( $\sigma$ )	Interest rate ( $r$ )	Time ( $t$ )	Black Scholes Exact Solution	Adomian Decomposition Method
30	25	0.2	0.2	1	6.1368	1.4652
30	25	0.25	0.2	1	6.478	1.4499
45	25	0.2	0.2	1	20.7407	6.7498
45	25	0.25	0.2	1	20.7568	6.827
60	25	0.2	0.2	1	35.7389	13.2881
60	25	0.25	0.2	1	35.7392	13.5538
70	25	0.2	0.2	1	45.7389	17.6479
70	25	0.25	0.2	1	45.7389	18.0175
75	25	0.2	0.2	1	50.7389	19.8178
75	25	0.25	0.2	1	50.7389	20.2679

Worthy of note from the PSCM method is that the volatility of an asset unlike the ADM affects and significantly determine the value of the call option at maturity. In example two, the volatility of same asset in example 1 was increased from 0.2 to 0.25 to study the behavior of the asset at maturity, and both gives different approximate values. Same experiment was carried out in example 3 and 4, 5 and 6, 7 and 8, 9 and 10, and they all give different approximate values. Not only are the approximate values of the call option different, the PSCM method also shows that the volatility of an asset can result to either an increase or decrease in the value of the call option at maturity.



**Figure 2.** Exact and Power Series Collocation Method Solution of Black Scholes PDE.

The approximate solution of the PSCM in the ten examples as seen in **Table**

4 and Figure 2 above shows that they are not exactly the same as the exact solution, and that there are examples where the approximate values of the PSCM and the exact solution are within the same digit value, as seen in example one and two, five and six, seven and eight, nine and ten and there are examples where they are not in the same digit value, as shown in example three and four, which indicates that the method gives a good approximation of the solution of the Black Scholes PDE but with room for improvement in accuracy and precision.

### 5.3. Comparative Analysis of the Approximate Solutions of Adomian Decomposition Method and Power Series Collocation Method

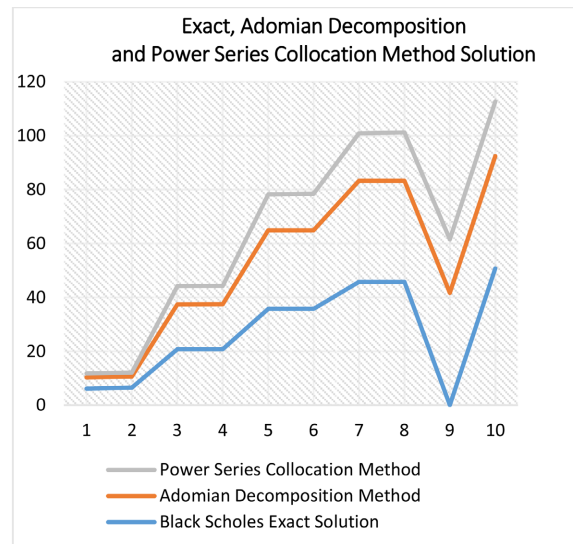
Ten same examples of Black Scholes PDE were solved using the Adomian Decomposition and Power Series Collocation method. Presented in the table below is the exact solution and approximate solution of the Adomian Decomposition and Power Series Collocation method, and the relative error of the methods.

**Table 5.** Relative errors of the adomian decomposition and power series collocation method.

Black Scholes Exact Solution	Adomian Decomposition Method	Relative Error-ADM	Power Series Collocation Method	Relative Error-PSCM
6.1368	4.1667	0.321	1.4652	0.7612
6.478	4.1667	0.3568	1.4499	0.7762
20.7407	16.667	0.1964	6.7498	0.6746
20.7568	16.667	0.197	6.827	0.6711
35.7389	29.1666	0.1839	13.2881	0.6282
35.7392	29.1666	0.1839	13.5538	0.6208
45.7389	37.5	0.1801	17.6479	0.6142
45.7389	37.5	0.1801	18.0175	0.6061
50.7389	41.6667	0.1788	19.8178	0.6094
50.7389	41.6667	0.1788	20.2679	0.6005

The approximate values of the PSCM and ADM method shows that both methods give good approximation of the solution of Black Scholes PDE, and are within the same digit value with the exact solution. However, all the examples in the ADM are within the same digit value with the exact solution, while in PSCM some examples are within the same digit value with the exact solution while some are not.

As seen in Table 5 and Figure 3, ADM and PSCM gives a good approximation of the Black Scholes PDE. However, the Adomian Decomposition method gives more accurate result and precision of the exact solution of the Black Scholes PDE.



**Figure 3.** Exact, adomian decomposition and power series collocation method solution.

### 5.4. Error Bound Analysis of the Adomian Decomposition Method and Power Series Collocation Method

**Table 6.** Error bound analysis of the ADM and PSCM.

Black Scholes Exact Solution	Adomian Decomposition Method	ADM Error Bounds/ Absolute Error	Power Series Collocation Method	PSCM Error Bounds/ Absolute Error
6.1368	4.1667	1.9701	1.4652	4.6716
6.478	4.1667	2.3113	1.4499	5.0281
20.7407	16.667	4.0737	6.7498	13.9909
20.7568	16.667	4.0898	6.827	13.9298
35.7389	29.1666	6.5723	13.2881	22.4508
35.7392	29.1666	6.5726	13.5538	22.1854
45.7389	37.5	8.2389	17.6479	28.091
45.7389	37.5	8.2389	18.0175	27.7214
50.7389	41.6667	9.0722	19.8178	30.9211
50.7389	41.6667	9.0722	20.2679	30.471

For the ADM, increase in the values of the volatility from 0.2 to 0.25 has no significant impact on the value of the absolute error from the exact solution of the Black-Scholes PDE to the approximate solution, as seen in the analysis above, however, there was increase in value of error bounds as the volatility increases from 0.2 to 0.25 for stock prices 30, 45 and 60, which shows that absolute error increase for this method with increase in volatility.

For the PSCM on the other hand, increase in the values of volatility from 0.2 to 0.25 has a very significant impact on the value of the absolute error from the exact solution of the Black-Scholes PDE to the approximate solution has also seen in

the error bound analysis in **Table 6**, the absolute error decreases as the volatility increases from 0.2 to 0.25 across the stock prices from 45, 60, 70 and 75, which shows that the absolute error decreases with increase in volatility for the PSCM.

### 5.5. Sensitivity Analysis of the Range of Stock Prices and Volatility

The rationale behind the chosen parameters for the range of stock prices and volatility values is to study the behavior of the European call option for each stock prices at the intermittent increase in the volatility from 0.2 to 0.25, because the volatility is the measure of risk associated to a financial instrument or the degree of fluctuation in stock prices over a period of time, and we have been able to show using the PSCM that the volatility of an asset can result in either an increase or decrease in the value of the call option at maturity, while for the ADM method, the volatility of the *asset alone* does not significantly determine the value of the call option at maturity.

### 5.6. Financial Implications of the Approximate Solutions of the Adomian Decomposition Method and Power Series Collocation Methods

The approximate solution of the ADM and PSCM as shown in this research are not exactly the same value as the exact solution, but agree with the exact solution in digit value, however there are financial implications for approximate solution with such outcome, and they include:

**1) Model Assumption:** Black Scholes financial model relies on assumptions such as no transaction costs and dividend payments, and availability of risk-free instruments amongst others, which may not perfectly reflect a particular market condition, therefore, approximations that are not exactly same with the exact solution are financially accepted if they align with these assumptions.

**2) Market Conditions and Liquidity:** The exact solution of less commonly traded assets are often difficult to derive and approximate solutions that agree with the exact solution in digit value like the PSCM may be used when market conditions make precision unachievable.

**3) Real Life Application:** As long as the approximate solutions are in sync with the exact solution in pattern and digit value, they are considered in real life asset trading environment to estimate prices or risks due to the need for urgent decision-making.

## 6. Conclusion

In this paper, Black Scholes PDE were successfully solved using the Adomian Decomposition and Power Series Collocation methods. The Adomian Decomposition method requires more computation efforts to the exact solution of the Black Scholes PDE because this method involves continuous iteration from 0 to the value of  $K$  (i.e. the strike price of the option) and a summation of these iterations to get the approximate solution of the PDE. The solutions of each example in the

Adomian Decomposition method agree with the exact solutions of the Black Scholes PDE in the sense that the solutions of all the examples are the same digit value with the exact solution. The Power Series Collocation method, on the other hand, requires lesser computational efforts to the Adomian Decomposition method, in this method, we solve to get a system of consistent equation which is then solved using Matlab to get values of the unknowns before substituting these unknowns in the assumed solution to get an approximate solution of the PDE. The solution of some examples of the Power Series Collocation Method agrees with the exact solution by being in the same digit value with the exact solution while some are not and this is due to numerical instabilities during computation and the method's sensitivity to changes in initial conditions and parameters.

### Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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