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Invariance of Plurigenera in Smooth Projective Families: An Algebraic Approach

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Abstract

This article presents an algebraic proof of the invariance of plurigenera for families of smooth projective varieties under deformations. While Siu's original proof relied on analytic tools such as multiplier ideal sheaves and L^2-extension theorems, our approach reformulates these techniques within the framework of algebraic geometry, emphasizing multiplier ideals, Castelnuovo-Mumford regularity, and Nadel vanishing theorem. Key steps include establishing the surjectivity of restriction maps for pluricanonical sections via careful analysis of base ideals and asymptotic multiplier ideals. This work aligns with recent efforts to translate Siu's results into algebraic settings and provides a foundation for extending the invariance theorem to singular varieties.

Keywords

Algebraic Proof, Invariance of Plurigenera, Multiplier Ideal Sheaves, Nadel Vanishing Theorem

1. Introduction

The plurigenera is fundamental invariant in classification theory, encoding information about the birational geometry. Their deformation invariance implies that certain properties of canonical models persist in families, a cornerstone of the minimal model program. In 1998, Yum-Tong Siu proved the invariance of plurigenera for families of smooth projective varieties under deformations [1]. His proof relied on three key tools: Nadel multiplier ideal sheaves, Skoda's L^2 estimates, and the Ohsawa-Takegoshi-Manivel extension theorem. These tools collectively ensured that the plurigenera of algebraic varieties remained invariant during the deformation process. Although Siu's proof was analytic in nature, in recent years, researchers have also attempted to describe the invariance of plurigenera under deformations using algebraic methods. For example, by translating

Siu's analytic approach into a more algebraic framework, Kawamata has successfully extended Siu's results [2]. The core of this method lies in constructing singular Hermitian metrics and utilizing their semipositivity to apply the L^2 extension theorem. While Siu's analytic methods are powerful, algebraic techniques offer advantages in settings where analytic tools are limited—for instance, in positive characteristic or for singular varieties. By rephrasing the proof using multiplier ideals and vanishing theorems, we unify the argument with broader principles in birational geometry.

2. Preliminaries

We work throughout with a smooth algebraic variety X of dimension n defined over $\mathbb{C}$.

Multiplier Ideals. Let $\mu: X' \to X$ be a log resolution of divisor D or of $\mathfrak{a}\left(\subseteq \mathcal{O}_X\right)$. $c > 0$ is a rational number, the multiplier ideals associated to D and to $\mathfrak{a}$ are defined to be:

$$\mathcal{J}(D) = \mu_* \mathcal{O}_{X'}\left(K_{X'/X} - \left[\mu^*\right]\right)$$

$$\mathcal{J}\left(\mathfrak{a}^c\right) = \mu_* \mathcal{O}_{X'}\left(K_{X'/X} - [cF]\right).$$

If L is a divisor on X such that the complete linear series $|L|$ is non-trivial, $\mu: X' \to X$ be a log resolution of $|L|$, $\mu^*|L| = |M| + F$, where $|M|$ is a basepoint-free linear series. Given $c > 0$, we can define

$$\mathcal{J}(c \cdot |L|) = \mu_* \mathcal{O}_{X'}\left(K_{X'/X} - [cF]\right).$$

Assume that L is big, then for $p \gg 0$ the multiplier ideals $\mathcal{J}\left(\frac{c}{p} \cdot |pL|\right)$ all coincide. The resulting ideal, written $\mathcal{J}(c \cdot \|L\|)$, is the asymptotic multiplier of $|L|$ with coefficient c.

Plurigenera. Let X be a smooth projective variety, for $m > 0$, the m^{th} plurigenus $P_m(X)$ of X is the dimension of the space of m-canonical forms on X:

$$P_m(X) = dim H^0\left(X, \mathcal{O}_X\left(mK_X\right)\right).$$

Nadel vanishing theorem. Let X be a smooth complex projective variety, let D be any Q-divisor on X, and let L be any integer divisor such that $L - D$ is nef and big. Then for $i > 0$,

$$H^i\left(X, \mathcal{O}_X\left(K_X + L\right) \otimes \mathcal{J}(D)\right) = 0.$$

The core value of Nadel's vanishing theorem lies in: Within the framework of multiplier ideal sheaves, even when line bundles exhibit weak positivity or singularities, the theorem still enables critical support for geometric problems—such as section lifting and surjectivity of restriction maps—through adjusted vanishing cohomology. We may consult [3] for details.

Subadditivity theorem. For ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ and $c > 0$,

$$\mathcal{J}\left(X, \mathfrak{a}^c \cdot \mathfrak{b}^c\right) \subseteq \mathcal{J}\left(X, \mathfrak{a}^c\right) \cdot \mathcal{J}\left(X, \mathfrak{b}^c\right).$$

For a detailed treatment, see [3].

3. Siu's Theorem on Plurigenera

Lemma 3.1. Let X be a smooth projective variety, and $S \subseteq X$ a smooth irreducible divisor. Define $L = K_X + S + B$, where B is a nef divisor. Assume that L is big and $S \not\subseteq \boldsymbol{B}_+(L)$. If $\mathfrak{a} \subseteq \mathcal{J}\left(S, \|(m-1)L\|_s\right)$ and $\mathcal{O}_S(mL_S) \otimes \mathfrak{a}$ is globally generated, then

$$\mathfrak{a} \subseteq \mathcal{J}\left(S, \|mL\|_s\right).$$

Proof. We make the following remark: Let M be a Cartier divisor on a projective variety V, and let $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_V$ be ideal sheaves. If $\mathcal{O}_V(M) \otimes \mathfrak{a}$ is globally generated, then $\mathfrak{a} \subseteq \mathfrak{b}$ if and only if

$$H^0\left(V, \mathcal{O}_V(M) \otimes \mathfrak{a}\right) \subseteq H^0\left(V, \mathcal{O}_V(M) \otimes \mathfrak{b}\right),$$

where both sides are viewed as subspaces of $H^0\left(V, \mathcal{O}_V(M)\right)$. In our situation, it therefore suffices to show that

$$H^0\left(S, \mathcal{O}_S(mL_S) \otimes \mathfrak{a}\right) \subseteq H^0\left(S, \mathcal{O}_S(mL_S) \otimes \mathcal{J}\left(S, \|mL\|_s\right)\right).$$

Suppose there exists a section $t \in H^0\left(S, \mathcal{O}_S(mL_S) \otimes \mathfrak{a}\right)$, i.e., t is a section of $\mathcal{O}_S(mL_S)$ vanishing along the ideal sheaf $\mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathcal{J}\left(S, \|(m-1)L\|_s\right)$, it follows from the adjoint sequence

$$0 \to \mathcal{J}\left(X, \|(m-1)L\|\right) \otimes \mathcal{O}_X(-S) \to Adj_S\left(X, \|(m-1)L\|\right) \to \mathcal{J}\left(S, \|(m-1)L\|_S\right) \to 0$$

that t lifts to a section

$$\tilde{t} \in H^0\left(X, \mathcal{O}_X(mL)\right).$$

By definition, $\tilde{t}$ vanishes along the base ideal $\mathfrak{b}\left(|mL|\right) \subseteq \mathcal{O}_X$, and hence its restriction $t \in H^0\left(S, \mathcal{O}_S(mL_S)\right)$ must vanish along $\mathfrak{b}\left(|mL|\right) \cdot \mathcal{O}_S \subseteq \mathcal{J}\left(S, \|mL\|_S\right)$, which establishes the required inclusion.

Lemma 3.2. Let X be a smooth projective variety, and $S \subseteq X$ a smooth irreducible divisor. Define $L = K_X + S + B$, where B is a nef divisor. Assume that L is big and $S \not\subseteq \boldsymbol{B}_+(L)$. There exist a very ample divisor A on X, a positive integer $k_0 > 0$, and a divisor $D \in |k_0 L - A|$ meeting S properly, such that for every $p \geq 0$, the inclusion

$$\mathcal{J}\left(S, \|pL_S\|\right) \otimes \mathcal{O}_s(-D_S) \subseteq \mathcal{J}\left(S, \|(p + k_0 - 1)L\|_S\right) \qquad (*)$$

holds.

Proof. By Nadel's vanishing theorem and Castelnuovo-Mumford regularity, we may choose a very ample divisor A on X such that for every $q \geq 0$, the sheaf $\mathcal{O}_s(qL_S + A_S) \otimes \mathcal{J}\left(S, \|qL\|_S\right)$ is globally generated. Furthermore, since $S \not\subseteq \boldsymbol{B}_+(L)$, for $k_0 \gg 0$, we may select a divisor $D \in |k_0 L - A|$ that does not contain S. We prove the inclusion (*) for these choices of data by induction on p.

For $p = 0$, the required inclusion holds because $D + A \sim k_0 L$, which implies

$$\mathcal{O}_s(-D_s) \subseteq \mathcal{J}\left(S, \|k_0 L\|_S\right) \subseteq \mathcal{J}\left(S, \|(k_0 - 1)L\|_S\right).$$

Assuming (*) holds for a given p, we show it also holds for $p+1$. To this end, first observe that by the choice of A, the sheaf

$\mathcal{O}_s((p+k_0)L_S - D_S) \otimes \mathcal{J}(S, \|pL_S\|)$ is globally generated. Applying lemma 3.1. with $m = p + k_0$ and the ideal sheaf $\mathfrak{a} = \mathcal{J}(S, \|pL_S\|) \otimes \mathcal{O}_s(-D_s)$, and invoking the inductive hypothesis $\mathcal{J}(S, \|pL_S\|) \otimes \mathcal{O}_s(-D_s) \subseteq \mathcal{J}(S, \|(p+k_0-1)L\|_S)$, that in fact $\mathcal{J}(S, \|pL_S\|) \otimes \mathcal{O}_s(-D_s) \subseteq \mathcal{J}(S, \|(p+k_0)L\|_S)$.

Therefore also $\mathcal{J}(S, \|(p+1)L_S\|) \otimes \mathcal{O}_s(-D_s) \subseteq \mathcal{J}(S, \|(p+k_0)L\|_S)$, which completes the induction.

Theorem 3.3. Let X be a smooth projective variety, and $S \subseteq X$ a smooth irreducible divisor. Define $L = K_X + S + B$, where B is a nef divisor. Assume that L is big and $S \not\subseteq \boldsymbol{B}_+(L)$. Then for every $m \geq 2$, the restriction map

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(S, \mathcal{O}_S(mL_S))$$

is surjective.

Proof. Fix an integer m, and apply inclusion in lemma 3.2. with $p = qm$ for all $q \gg 0$. We obtain the following chain of inclusions:

$$\begin{aligned} \mathfrak{b}(|mL_S|)^q \otimes \mathcal{O}_S(-D_S) &\subseteq \mathfrak{b}(|mqL_S|) \otimes \mathcal{O}_S(-D_S) \\ &\subseteq \mathcal{J}(S, \|mqL_S\|) \otimes \mathcal{O}_s(-D_s) \\ &\subseteq \mathcal{J}(S, \|(mq+k_0-1)L\|_S) \\ &\subseteq \mathcal{J}(S, \|mqL\|_S) \\ &\subseteq \mathcal{J}(S, \|mL\|_S)^q \end{aligned}$$

where the final inclusion follows from the subadditivity theorem. However, we assert that if the inclusion

$$\mathfrak{b}(|mL_S|)^q \otimes \mathcal{O}_S(-D_S) \subseteq \mathcal{J}(S, \|mL\|_S)^q$$

holds for all $q \gg 0$, then it necessarily implies

$$\mathfrak{b}(|mL_S|) \subseteq \mathcal{J}(S, \|mL\|_S),$$

and consequently, the inclusion $\mathfrak{b}(|mL_S|) \subseteq \mathcal{J}(S, \|(m-1)L\|_S)$ holds.

The inclusion $\mathfrak{b}(|mL_S|) \subseteq \mathcal{J}(S, \|(m-1)L\|_S)$ holds, we complete the proof. In fact, we twist by $\mathcal{O}_X(mL)$ using the exact sequence
$0 \to \mathcal{J}(X, \|(m-1)L\|) \otimes \mathcal{O}_X(-S) \to Adj_S(X, \|(m-1)L\|) \to \mathcal{J}(S, \|(m-1)L\|_S) \to 0$
(*). Noting the linear equivalence

$$mL - S \sim (m-1)L + K_X + B,$$

the Nadel vanishing theorem and the assumptions imply that for $m \geq 2$,

$$H^1(X, \mathcal{O}_X(mL - S) \otimes \mathcal{J}(X, \|(m-1)L\|)) = 0.$$

Consequently, the exact sequence (*) ensures the surjectivity of the map

$$H^0(X, \mathcal{O}_X(mL) \otimes Adj_S(\|(m-1)L\|)) \to H^0(S, \mathcal{O}_S(mL_S) \otimes \mathcal{J}(\|(m-1)L\|_S)).$$

Since the left-hand group is a subspace of $H^0(X, \mathcal{O}_X(mL))$, this means any setion of $\mathcal{O}_S(mL_S)$ vanishing along $\mathcal{J}(S, \|(m-1)L\|_S)$ can be lifted to a section of $\mathcal{O}_X(mL)$ .Therefore, if the inclusion $\mathfrak{b}(|mL_S|) \subseteq \mathcal{J}(S, \|(m-1)L\|_S)$ is known, it follows that all sections of $\mathcal{O}_S(mL_S)$ can be lifted to X.

Theorem 3.4.(Siu's Theorem on Plurigenra)

Let

$$\pi : Y \to T$$

be a smooth projective family of varieties of general type. Then for each $m \geq 0$, the plurigenera

$$P_m(Y_t) = h^0\left(Y_t, \mathcal{O}_{Y_t}\left(mK_{Y_t}\right)\right)$$

are independent of t.

Proof. Without loss of generality, assume $m \geq 2$ and that T is a smooth affine curve. Let $K_t = K_{Y_t}$. Fix $0 \in T$. By semicontinuity, $P_m(Y_0) \geq P_m(Y_t)$ for generic t. Thus, it suffices to prove the reverse inequality:

$$h^0\left(Y_0, \mathcal{O}_{Y_0}(mK_0)\right) \leq h^0\left(Y_t, \mathcal{O}_{Y_t}(mK_t)\right) \qquad (*)$$

For $t \in T$ near 0.

Consider the sheaf $\pi_* \mathcal{O}_Y(mK_{Y/T})$ on T. This is torsion-free (hence locally free) sheaf whose rank computes the generic value of the m-th plurigenus $P_m(Y_t)$. The fiber of $\pi_* \mathcal{O}_Y(mK_{Y/T})$ at 0 consists of pluricanonical forms on Y_0 that extend to forms on Y over a neighborhood of 0. To prove(*), it suffices to show that any $\eta \in H^0\left(Y_0, \mathcal{O}_{Y_0}(mK_0)\right)$ extends (after possibly shrinking T) to $\tilde{\eta} \in H^0\left(Y, \mathcal{O}_Y(mK_{Y/T})\right)$.This follows from Theorem 3.3.

Specifically, complete π to a morphism $\bar{\pi} : \bar{Y} \to \bar{T}$ of smooth projective varieties, where $\bar{T} \supseteq T$ and $Y = \bar{\pi}^{-1}(T)$. Regard $Y_0 \subseteq \bar{Y}$ as a smooth divisor on $\bar{Y}$. Let A be a very ample divisor on $\bar{T}$, set $B = \pi^*\left(A - K_{\bar{T}}\right)$, and define

$$L = K_{\bar{Y}} + Y_0 + B \sim K_{\bar{Y}/\bar{T}} + Y_0 + \pi^* A.$$

By taking A sufficiently positive, we may assume B is nef. Adjust A further to ensure L is big and $Y_0 \not\subseteq \boldsymbol{B}_+(L)$.

Bigness of L: Let D be an ample divisor on $\bar{Y}$. For A sufficiently positive, there exists $k \gg 0$, such that $kL - D$ is effective. Since Y_t is of general type for general t, choose k large enough so that $kK_{\bar{Y}/\bar{T}} - D$ is effective on a very general fiber of $\bar{\pi}$. When A is sufficiently positive, $\bar{\pi}_* \mathcal{O}_{\bar{Y}}(kL - D)$ is non-zero and globally generated, imply $h^0\left(\bar{Y}, \mathcal{O}_{\bar{Y}}(kL - D)\right) \neq 0$.

Avoidance of $\boldsymbol{B}_+(L)$: Since $|rY_0| = \bar{\pi}^* |r \cdot 0|$ is free for $r \gg 0$, Y_0 does not lie in the base locus of $|kL - D + qY_0|$ for $q \gg 0$. By further increasing A, replace L with $L + pY_0$ to ensure $Y_0 \not\subseteq \boldsymbol{B}_+(L)$.

Therefore, we can apply Theorem 3.3. with $X = \bar{Y}$ and $S = Y_0$, concluding that for every $m \geq 2$, the restriction map

$$H^0\left(\bar{Y}, \mathcal{O}_{\bar{Y}}(mL)\right) \to H^0\left(Y_0, \mathcal{O}_{Y_0}(mL)\right) \qquad (*)$$

is surjective. By taking T sufficiently small, then

$$\mathcal{O}_Y(mL) \simeq \mathcal{O}_Y(mK_{Y/T}).$$

The surjective of (*) then implies the required surjectivity of

$$H^0\left(Y, \mathcal{O}_Y\left(mK_{Y/T}\right)\right) \to H^0\left(Y_0, \mathcal{O}_{Y_0}\left(mK_0\right)\right),$$

Which completes the proof.

Remark 3.5. Siu established that the same statement holds even when Y_t is not of general type [4]. Kawamata and Nakayama establish a number of striking results from [1]. Kawamata shows in [5] that canonical singularities are preserved under deformation, Nakayama [6] proves that if X_0 is a variety of general type having only canonical singularities, then any deformation of X_0 is again of general type.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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