

# The Moment of Continuous-Time Markov Branching Process with Immigration

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## Abstract

Let  $\{Z(t), t \geq 0\}$  be a continuous-time Markov branching process with immigration. In this paper, we mainly research the moment for  $Z(t)$ . We calculate the specific expression of the moment  $M_1(t)$  and then  $M_i(t), i > 1$  can be found subsequently. The moments of the branching process provide great help for further studying the asymptotic behavior of the branching process, so it is indispensable to calculate the moment.

## Keywords

Markov Branching Process, Moment, Immigration, Generating Function

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## 1. Introduction

The Branching process is a kind of stochastic process that describes the population evolution, and can also be used as a mathematical model to describe the random evolution of particles in the system. Galton-Watson process is the most classic and simplest model.

Early studies focused on deriving explicit solutions for mean values in classical branching processes. For instance, in single-type discrete-time branching processes, the mean follows a recursive relationship  $\mu_n = \mu^n$ , where  $\mu$  represents the expected offspring per individual [1]. However, as application scenarios grow increasingly complex—such as multi-type interactions, environmental stochasticity, and resource constraints—traditional mean computation methods face significant challenges. In epidemiological modeling, for example, individual infection rates may dynamically depend on evolving community network structures, leading to time-varying or high-dimensional Markovian transition matrices for offspring distributions. Under such conditions, closed-form solutions

for mean values become intractable. Recent advancements have expanded computational approaches through moment-generating functions, coupled stochastic differential equations, and spectral analysis of random matrices, gradually broadening the applicability of mean computation techniques.

The evolution of branching process theory has witnessed a paradigm shift from discrete-time frameworks to continuous-time models, driven by the need to capture real-time stochastic dynamics in biological, epidemiological, and physical systems. For example, significant transmission heterogeneity was observed in the COVID-19 pandemic—approximately 10% - 20% of infected individuals accounted for 80% of secondary infections. While traditional deterministic models (e.g., the SIR model) struggle to capture this phenomenon, branching processes can effectively characterize such individual-level transmission heterogeneity. Pioneering this transition, Asmussen and Hering [2] formalized the continuous-time Markov branching process (CTMBP), establishing its infinitesimal generator and exploring extinction criteria through martingale techniques. Their work laid the analytical foundation for subsequent studies on moment analysis and asymptotic behaviors in time-continuous settings.

Central to branching process theory is the characterization of moments, which quantify population dynamics such as growth rates and variability. For continuous-time models, Asmussen and Hering [2] demonstrated that the moments of CTMBP obey differential equations governed by the process's  $q$ -matrix. Specifically, the first-order moment  $M_1(t)$  follows:

$$\frac{dM_1(t)}{dt} = QM_1(t),$$

where  $Q$  encodes transition rates between states. This result generalizes the discrete-time recursive relations and enables explicit solutions under exponential offspring distributions.

Recent advancements further link moment structures to normalization constants in scaling limits. For instance, the normalization sequence  $(a_n)$  for population size scaling—critical in studying weak convergence—has been shown to depend directly on moment divergence rates [3]. Such insights unify the analysis of discrete- and continuous-time models, reinforcing moments as pivotal tools in both theoretical and applied domains.

However, current research still faces bottlenecks, as real-world models are often influenced by additional factors such as immigration and environmental stochasticity. So, in this paper, our goal is to calculate the moments of Markov branching processes with immigration MBPI.

## 2. Preparatory Work

Let  $\{Z(t); t \geq 0\}$  be a continuous-time Markov branching process with immigration (MBPI) with positive integer states and infinitesimal generator elements

$$q_{ij} := \begin{cases} ib_{j-i+1} + a_{j-i}, & i \geq 1, j \geq i, \\ -ib_0, & j = i - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where  $b_k$  denotes the branching rate

$$b_k \geq 0 (k \neq 1), \quad 0 < -b_1 = \sum_{k \neq 1} b_k < \infty. \tag{2.2}$$

$a_k$  denotes the immigration rate

$$a_k \geq 0 (k \neq 0), \quad 0 < -a_0 = \sum_{k \neq 0} a_k < \infty. \tag{2.3}$$

Define  $B(s) = \sum_{j=0}^{\infty} b_j s^j$ ,  $A(s) = \sum_{j=0}^{\infty} a_j s^j$ , and  $F_i(s, t) = \sum_{j=0}^{\infty} p_{ij}(t) s^j$ .

$$p'_{ij}(t) = \sum_k q_{ik} p_{kj}(t). \tag{2.4}$$

For the Markov branching processes with immigration MBPI, by Li [4], we know that this process has the new branching properties as follows.

$$F_i(s, t) = H(s, t) [f(s, t)]^i. \tag{2.5}$$

where  $H(s, t) = F_0(s, t)$  and  $H(s, t)$  is the generation function of the immigration section,  $f(s, t)$  is the generation function of the section without immigration.

Let  $\bar{p}_{ij}$  be the transition probabilities of the Markov branching process MBP,

$$h_1(t) = \left. \frac{\partial H(s, t)}{\partial s} \right|_{s=1}, \quad m_1(t) = \left. \frac{\partial f(s, t)}{\partial s} \right|_{s=1}, \quad M_1(t) = \left. \frac{\partial F_i(s, t)}{\partial s} \right|_{s=1},$$

$$f(s, t) = \sum_{j=0}^{\infty} \bar{p}_{1j}(t) s^j.$$

$$F_i(1, t) = H(1, t) = f(1, t) = 1.$$

From Asmussen and Hering [1],  $f(s, t)$  is the part of  $F(s, t)$  without immigration, If we regard  $f(s, t)$  as the generating function of a branching process, its corresponding q-matrix is

$$\hat{q}_{ij} := \begin{cases} ib_{j-i+1}, & i \geq 1, j \geq i, \\ -ib_0, & j = i - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $b_k$  denotes the branching rate similarly

$$b_k \geq 0 (k \neq 1), \quad 0 < -b_1 = \sum_{k \neq 1} b_k < \infty.$$

Then

$$\begin{aligned} \frac{\partial f(s, t)}{\partial t} &= \sum_{j=0}^{\infty} p'_{1j}(t) s^j \\ &= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \hat{q}_{1k} p_{kj}(t) s^j \\ &= \sum_{k=1}^{\infty} \hat{q}_{1k} f^k(s, t), \end{aligned}$$

finally,

$$\frac{\partial f(s,t)}{\partial t} = B(f(s,t)) - 2b_0. \tag{2.6}$$

According to (2.5),

$$H(s,t) = F_i(s,t) / f^i(s,t),$$

then

$$\begin{aligned} \frac{\partial H(s,t)}{\partial s} &= \left[ \frac{\partial F_i(s,t)}{\partial s} f^i(s,t) - iF_i(s,t) f^{i-1}(s,t) \frac{\partial f^i(s,t)}{\partial s} \right] / f^{2i}(s,t) \\ &= \frac{\partial F_i(s,t)}{\partial s} \frac{1}{f^i(s,t)} - iF_i(s,t) \frac{\partial f(s,t)}{\partial s}. \end{aligned}$$

Next, let  $s = 1$ , then

$$h_1(t) = \left. \frac{\partial F_i(s,t)}{f^i(s,t)} \right|_{s=1} = M_1(t) - im_1(t). \tag{2.7}$$

Through the Kolmogorov backward Equation (2.4), Athreya and Ney [1] calculated the moment of the classic MBP and obtained the exact value of the moments, i.e.,  $m_1(t) = e^{B(1)t}$ . In this paper, we still use the formula to solve the problem.

### 3. The Moment of MBPI

We begin our research by the kolmogorov backward Equation (2.4). Substitute (2.1) into (2.4),

$$p'_{ij}(t) = \sum_{k=i}^{\infty} (ib_{k-i+1} + a_{k-i}) p_{kj}(t) - ib_0 p_{i-1,j}(t). \tag{3.1}$$

Multiply both sides of the above (3.1) by  $s^j$  and then summing over  $j$  yields

$$\frac{\partial F_i(s,t)}{\partial t} = \sum_{j=0}^{\infty} \sum_{k=i}^{\infty} (ib_{k-i+1} + a_{k-i}) p_{kj}(t) s^j - ib_0 \sum_{j=0}^{\infty} p_{i-1,j}(t) s^j. \tag{3.2}$$

According to Fubini theorem together with (2.6),

$$\begin{aligned} \frac{\partial F_i(s,t)}{\partial t} &= \sum_{k=i}^{\infty} \sum_{j=0}^{\infty} (ib_{k-i+1} + a_{k-i}) p_{kj}(t) s^j - ib_0 \sum_{j=0}^{\infty} p_{(i-1)j}(t) s^j \\ &= \sum_{k=i}^{\infty} (ib_{k-i+1} + a_{k-i}) \sum_{j=0}^{\infty} p_{kj}(t) s^j - ib_0 \sum_{j=0}^{\infty} p_{i-1,j}(t) s^j \\ &= \sum_{k=i}^{\infty} (ib_{k-i+1} + a_{k-i}) F_k(s,t) - ib_0 F_{i-1}(s,t) \\ &= \sum_{k=i}^{\infty} (ib_{k-i+1} + a_{k-i}) H(s,t) f^k(s,t) - ib_0 H(s,t) f^{i-1}(s,t) \\ &= H(s,t) \left[ i \sum_{k=i}^{\infty} b_{k-i+1} f^k(s,t) + \sum_{k=i}^{\infty} a_{k-i} f^k(s,t) \right] - ib_0 H(s,t) f^{i-1}(s,t) \\ &= H(s,t) \left[ iB(f(s,t)) f^{i-1}(s,t) - 2ib_0 f^{i-1}(s,t) + A(f(s,t)) f^i(s,t) \right]. \end{aligned} \tag{3.3}$$

Differentiating both sides of Equation (2.5) with respect to  $t$  yields

$$\frac{\partial F_i(s,t)}{\partial t} = \frac{\partial H(s,t)}{\partial t} f^i(s,t) + H(s,t) \frac{\partial f(s,t)}{\partial t} f^{i-1}(s,t). \tag{3.4}$$

Combining Equation (3.3) and (3.4), and then differentiating both sides of the equation with respect to  $s$  and then letting  $s = 1$ , we have

$$\begin{aligned} \frac{dh_1(t)}{dt} + \frac{\partial H(1,t)}{\partial t} m_1(t) + i \frac{dm_1(t)}{dt} + i \frac{\partial f(1,t)}{\partial t} h_1(t) \\ = -2ib_0 h_1(t) + iB'(1)m_1(t) + A'(1)m_1(t). \end{aligned} \tag{3.5}$$

Using (2.5) again

$$\begin{aligned} \frac{\partial H(s,t)}{\partial t} &= \frac{\partial \frac{F_i(s,t)}{f^i(s,t)}}{\partial t} \\ &= \left[ \frac{\partial F_i(s,t)}{\partial t} f^i(s,t) - iF_i(s,t) f^{i-1}(s,t) \frac{\partial f(s,t)}{\partial t} \right] / f^{2i}(s,t) \\ &= \frac{\partial F_i(s,t)}{\partial t} \frac{1}{f^i(s,t)} - iF_i(s,t) \frac{\partial f(s,t)}{\partial t}. \end{aligned}$$

Let  $s = 1$ , then

$$\frac{\partial H(1,t)}{\partial t} = \frac{\partial F_i(1,t)}{\partial t} \frac{1}{f^i(1,t)} - iF_i(1,t) \frac{\partial f(1,t)}{\partial t} = 0 \tag{3.6}$$

Then Equation (3.5) is equal to

$$\frac{dM_1(t)}{dt} - [iB'(1) - A'(1)]m_1(t) = 0, \tag{3.7}$$

$$M_1(t) = \frac{iB'(1) + A'(1)}{B'(1)} e^{B'(1)t} - \frac{A'(1)}{B'(1)}, \tag{3.8}$$

with boundary condition  $M_1(0) = i$ .

### 4. Conclusions

Firstly, we give the first-order moment by finding the first derivative of the two sides of  $s$ . By the same way, other order moments can be obtained by the derivative of the corresponding order for  $s$ .

Secondly, by observing the exactly expression for the first-order moments of the process MBPI, which implies that the times limites of  $M_1(t)$  and  $m_1(t)$  are equivalent, and provides a great help to explore the limiting behavior of MBPI. For instance, Athreya and Ney [5] mentioned the normalization constant sequence of the Galton-Waston model is just the first-order moment and Seneta [6] pointed out that of G-W model with immigration is the same since the first-order moment of the two models is equivalent. Similarly, it can be seen that MBPI and MBP have the same normalization sequence.

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## Conflicts of Interest

The authors declare that they have no conflict of interest.

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