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Hopf Bifurcation in a Diffusive Predator-Prey Model with Predation-Driven Allee Effect and Gestation Time Delay

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Abstract

In this paper, a diffusive predator-prey system with predation-driven Allee effect and delay is considered. The effect of time delay on the model, including stability of the positive equilibrium and Hopf bifurcation is studied. To validate our theoretical analysis results, some numerical simulations are realized. The research results indicate that time delay can affect the stability of coexisting equilibrium points and cause periodic oscillations in predator and prey density.

Keywords

Prey-Predator, Allee Effect, Stability, Hopf Bifurcation

1. Introduction

The direct predation relationship between prey and predators is one of the most important relationships in population dynamics in nature. Currently, many scholars have studied this relationship by establishing predator-prey models [1] [2]. Many scholars have also considered the impact of the Allee effect on predator-prey models [3] [4]. In [5] authors proposed a predator-prey model with predation-driven Allee effect with the following form.

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = ru\left(1-\dfrac{u}{K}\right)\left(1-\dfrac{f+\theta v}{f+u}\right)-\dfrac{muv}{a+u}, \\ \dfrac{\mathrm{d}v(t)}{\mathrm{d}t} = sv\left(1-\dfrac{v}{\beta+\gamma u}\right), \end{cases} \tag{1}$$

where $u(t)$ and $v(t)$ respectively represent the populations of the prey and the predator. The significance of specific parameters can be referred to in reference

[5]. They studied the local stability, Bogdanov-Tankens bifurcation, and Hopf bifurcation near the coexisting equilibrium point.

In nature, the spatial arrangement of populations is uneven, and diffusion phenomena often occur. Therefore, it is wise to introduce the reaction-diffusion to predator-prey model [6] [7]. In [8], Mi Y Y, Song C and Wang Z C investigated the steady-state patterns and dynamical behaviors of a modified Leslie-Gower predator-prey model with diffusion, which incorporates density-dependent movement in the predators. They validated the existence of regular solutions with the uniform-in time bound and analyzed the global and local stability of the spatially homogeneous co-existence steady state under certain parameter conditions. In [9], Yang W S examined a diffusive predator-prey model with no-flux boundary condition and modified Holling-Tanner functional response. He confirmed persistence of the system by obtaining a sufficient condition. Moreover, he used a comparison method to confirm sufficient conditions for the global asymptotical stability of the system's unique positive equilibrium.

In addition, time delay phenomenon is also widely present [11] [12]. In [10], Guo S J applied the S-1-equivariant degree method to a Hopf bifurcation problem for functional differential equations with a state-dependent delay. He used the homotopy invariance of S-1-equivariant degree, then the linearization of the system at a stationary state is extracted and translated into a bifurcation invariant. In [13], authors considered the fractional-order Leslie-Gower model with a single time delay and Holling type II functional response, they determined the stability range and bifurcation points by analytic extrapolation with regarding time delay as a bifurcation parameter. In [14], authors studied a predator-prey model with a Holling type II functional response and gestation delay. Focusing on the effect of predation-induced fear, they proved positivity boundedness and permanence under certain parametric conditions. We consider the following model with diffusion and the gestation time delay of predator.

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + ru\left(1-\dfrac{u}{K}\right)\left(1-\dfrac{f+\theta v}{f+u}\right)-\dfrac{muv}{a+u}, & x\in(0,l\pi), t>0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + sv\left(1-\dfrac{v(t-\tau)}{\beta+\gamma u(t-\tau)}\right), & x\in(0,l\pi), t>0, \\ u_x(0,t)=v_x(0,t)=0, u_x(l\pi,t)=v_x(l\pi,t)=0, & t>0, \\ u(x,\theta)=u_0(x,\theta)\ge 0, v(x,\theta)=v_0(x,\theta)\ge 0, & x\in[0,l\pi], \theta\in[-\tau,0], \end{cases} \tag{2}$$

where d_1 and d_2 are self diffusion coefficients of prey and predators. τ is the gestation time delay, K is the carrying capacity of prey, the term $1-\dfrac{f+\theta v}{f+u}$ describes the predator-driven Allee effect in prey. s is the conversion co-effcient.

The structure of this article is as follows. In Section 2, the stability of the positive equilibrium and time delay inducing Hopf bifurcation are analyzed. In Section 3, some numerical simulations are given.

2. Stability Analysis

The existence of equilibrium points has been discussed in [5], then we present the relevant results as follow. The model (2) has three boundary equilibrium $(0,0)$, $(K,0)$ and $(0,\beta)$. If $\gamma\theta > 1$ and $a < K$, then the model (2) obtain an unique positive equilibrium (u_*, v_*), where u_* is the positive root of the following equation and $v_* = \beta + \gamma u_*$.

$$\begin{aligned}
&\sigma_0 u^3 + \sigma_1 u^2 + \sigma_2 u + \sigma_3 = 0, \\
&\sigma_0 = r(1-\gamma\theta),\ \sigma_1 = -r(K-a)(1-\gamma\theta) + \gamma K m - \beta\theta r, \\
&\sigma_2 = \beta\theta r(K-a) - aKr(1-\gamma\theta) + Km(\beta + \gamma f), \\
&\sigma_3 = \beta K(a\theta r + fm).
\end{aligned} \tag{3}$$

If $\gamma\theta < 1$, then the model (2) may have two positive equilibria.

Denote a positive equilibrium of (2) by $E_*(u_*, v_*)$. Linearize system (2) at $E_*(u_*, v_*)$ has the following form

$$\frac{\partial u}{\partial t}\begin{pmatrix} u(x,t) \\ u(x,t) \end{pmatrix} = J_1 \begin{pmatrix} \Delta u(t) \\ \Delta v(t) \end{pmatrix} + J_2 \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + J_3 \begin{pmatrix} u(x,t-\tau) \\ v(x,t-\tau) \end{pmatrix}, \tag{4}$$

where

$$J_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\quad J_2 = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix},\quad J_3 = \begin{pmatrix} 0 & 0 \\ s\gamma & -s \end{pmatrix},$$

and

$$\begin{aligned}
a_1 &= u_* \left(\frac{m v_*}{(a+u_*)^2} + \frac{r\left(1-\frac{u_*}{K}\right)(f+\theta v_*)}{(f+u_*)^2} + \frac{r(\theta v_* - u_*)}{K(f+u_*)} \right), \\
a_2 &= -u_* \left(\frac{m}{a+u_*} + \frac{\theta r(1-u_*/K)}{f+u_*} \right) < 0.
\end{aligned} \tag{5}$$

2.1. The Non-Delay Model

When the time delay $\tau = 0$ and diffusion $d_1 = d_2 = 0$. The characteristic equation are

$$\lambda^2 - \kappa_n \lambda + \nu_n = 0,\ n \in \mathbb{N}_0, \tag{6}$$

where

$$\kappa_n = a_1 - s,\ \nu_n = -s(a_1 + a_2\gamma),\ \mu_n = \frac{n^2}{l^2}.$$

The roots of (6) come from

$$\lambda_{1,2} = \frac{1}{2}\left[(a_1 - s) \pm \sqrt{(a_1-s)^2 + 4s(a_1 + a_2\gamma)} \right]. \tag{7}$$

It easy to know that the roots of (6) have negative real parts if and only if $a_1 - s < 0$ and

$$(\mathbf{H_2})\quad a_1 + a_2\gamma < 0,$$

hold. When s near a_1, Equation (6) has a pair of complex eigenvalues $\alpha(s)\pm i\omega(s)$ with

$$\alpha(s)=\frac{1}{2}(a_1-s),\omega(s)=\frac{1}{2}\sqrt{(a_1-s)^2+4s(a_1+\gamma a_2)},$$

We have

$$\alpha(a_1)=0,\ \alpha'(a_1)=-\frac{1}{2},\ \omega(a_1)>0.$$

Obviously, ODE system of (2) without delay undergoes a Hopf bifurcation at $E_*=(u_*,v_*)$ when $s=a_1$.

Theorem 1 When ($\mathbf{H_2}$) holds, for ODE system of (2) without delay the following statements are true.

i) If $a_1\le 0$ and $s>0$, the equilibrium $E_*=(u_*,v_*)$ is local asymptotically stable.

ii) If $a_1>0$ and $s>a_1$, the equilibrium $E_*=(u_*,v_*)$ is local asymptotically stable.

iii) If $a_1>0$ and $s=a_1$, the ODE system of (2) without delay undergoes Hopf bifurcation at $E_*=(u_*,v_*)$.

When $d_1\neq 0,d_2\neq 0$, we define the real-valued Sobolev space

$$\mathbf{X}:=\left\{(u,v)\in\left[H^2(0,l\pi)\right]^2:(u_x,v_x)\big|_{x=0.l\pi}=0\right\},$$

and the complexification of $\mathbf{X}$:

$$\mathbf{X}_{\mathbb{C}}:=\mathbf{X}\oplus \mathrm{i}\mathbf{X}=\{x_1+\mathrm{i}x_2:x_1,x_2\in\mathbf{X}\}.$$

The linearized system of (2) without delay at $(0,0)$ has the following form

$$\begin{pmatrix}u_t\\ v_t\end{pmatrix}=L(\rho)\begin{pmatrix}u\\ v\end{pmatrix}:=\begin{pmatrix}d_1 & 0\\ 0 & d_2\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}+\begin{pmatrix}a_1 & a_2\\ s\gamma & -s\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}.$$

Then the linearized operator of the steady state evaluated at $(s,0,0)$ is

$$L(s)=\begin{pmatrix}d_1\frac{\partial^2}{\partial x^2}+a_1 & a_2\\ s\gamma & d_2\frac{\partial^2}{\partial x^2}-s\end{pmatrix},$$

with the domain $D_{L(c)}=\mathbf{X}_{\mathbb{C}}$.

By the [15], we konw that the eigenvalues of $L(s)$ are given by the eigenvalues of $L_n(s)$ for $n=0,1,2,\cdots$. where

$$L_n(s):=\begin{pmatrix}a_1-d_1\mu_n & a_2\\ s\gamma & -s-d_2\mu_n\end{pmatrix}.$$

The characteristic equation of $L_n(s)$ is

$$\lambda^2-\lambda T_n(s)+D_n(s)=0,\ n=0,1,2,\cdots, \tag{8}$$

with

$$\begin{cases}T_n(s)=-\mu_n(d_1+d_2)+a_1-s,\\ D_n(s)=\mu_n d_1 d_2-\mu_n(d_2 a_1-d_1 s)-s(a_1+\gamma a_2),\end{cases} \tag{9}$$

the eigenvalues of Equation (8) are given by

$$\lambda_{1,2}^{(n)}(s)=\frac{\pm\sqrt{T_n^2(s)-4D_n(s)}+T_n(s)}{2},\quad n=0,1,2,\cdots. \tag{10}$$

It is obvious that when

$$\textbf{(H}_3\textbf{)}\quad s>a_1 \text{ and } s\geq\frac{d_2a_1}{d_1}$$

holds, all the roots of (8) have negative real parts. We make a hypothesis

$$\textbf{(H}_4\textbf{)}\quad a_1<s<\frac{d_2a_1}{d_1},$$

and denote

$$\mu_{n\pm}=\frac{(d_2a_1-d_1s)\pm\sqrt{(d_2a_1-d_1s)^2+4sd_1d_2(a_1+\gamma a_2)}}{2d_1d_2},$$

$$s_\pm=\frac{d_2}{d_1}\left[-(a_1+2\gamma a_2)\pm\sqrt{\gamma a_2(\gamma a_2+2a_1)}\right].$$

Then the following statements are true.

Theorem 2. Suppose (**H**$_2$) holds, for system (2) with $\tau=0$,

i) If (**H**$_3$) holds, then the equilibrium $E_*(u_*,v_*)$ is asymptotically stable;

ii) If (**H**$_4$) and $s\in(s_-,s_+)$ hold, then the equilibrium $E_*(u_*,v_*)$ is asymptotically stable;

iii) If (**H**$_4$) hold, $s\in(0,s_-)\cup(s_+,+\infty)$ and $\mu_n\in(0,\mu_{n-})\cup(\mu_{n+},+\infty)$ hold, then the equilibrium $E_*(u_*,v_*)$ is asymptotically stable;

iv) If (**H**$_4$) hold, $s\in(0,s_-)\cup(s_+,+\infty)$ and $\mu_n\in(\mu_{n-},\mu_{n+})$ hold, then the equilibrium $E_*(u_*,v_*)$ is Turing unstable;

v) If $a_1>0$, when $s=s_n:=a_1-\mu_n(d_1+d_2)$, for $0\leq n\leq n^*$, the system (2) undergoes Hopf bifurcation at $E_*(u_*,v_*)$.

Proof. From [15], (i), (ii), (iii), (iv) are easy to argue, we only prove (v).

We assume $\alpha_n(s)\pm\mathrm{i}\omega_n(s)$ are eigenvalues of (6), then $\alpha_n(s)=\frac{\kappa_n(s)}{2}$, $\omega_n(s)=\sqrt{\nu_n(s)-\alpha^2(s)}$. By straightforward computation, we obtain $\alpha'(s_n)=-\frac{1}{2}<0$. If $\pm i\omega_n$ are eigenvalues of (6), then $T_n(s_n)=0$, we could obtain $s=s_n$. Obviously, s_n is monotonically decreasing with regard to n, then there is a $n_1^*\in\mathbb{N}_0$, we have $s_n\leq0$ for $n=n_1^*+1,n_1^*+2,\cdots$ and $s_n>0$ for $n=0,1,2,\cdots,n_1^*$.

Substitution s_n into $\nu_n(c)$ yields

$$\nu_n(s_n)=-a_1(a_1+a_2\gamma)+\mu_n(2a_1d_1+a_2\gamma(d_1+d_2))-d_1^2\mu_n^2$$

By $\nu_0(s_0)=-a_1(a_1+a_2\gamma)>0$, there yields an integer $n_2^*\geq1$ then $\nu_n(s_n)>0$ when $n=0,1,\cdots,n_2^*$. Define $n^*=\min\{n_1^*,n_2^*\}$, such that the last statement holds.

□

2.2. The delay model

When the time delay $\tau \neq 0$. The characteristic equation is

$$\Delta_n(\lambda,\tau)=\lambda^2+\lambda X_n+Y_n+s(\lambda+Z_n)e^{-\lambda\tau}=0 \tag{11}$$

where

$$X_n=(d_1+d_2)\mu_n-a_1, Y_n=d_2\mu_n(d_1\mu_n-a_1), Z_n=d_1\mu_n-(a_1+a_2\gamma).$$

In this section, we assume $a_1+a_2\gamma<0$ and $s>\max\left\{\frac{d_2}{d_1}a_1,a_1\right\}$ hold. When $\tau=0$, we can easily get $\Delta_n(0,\tau)=Y_n+Z_n=\nu_n>0$, then 0 is not a characteristic root of (11). Then we have the following lemma.

Lemma 3. Suppose $a_1+a_2\gamma<0$ and $s>\max\left\{\frac{d_2}{d_1}a_1,a_1\right\}$ hold, then Eq. (11) has a couple of purely imaginary roots $\pm i\omega_n^+$ ($0\le n\le N_1$) at $\tau_n^{+,j}$, where $\tau_n^j=\tau_n^{+,0}+\frac{2j\pi}{\omega_n}, j\in\mathbb{N}_0$, $\tau_n^0=\frac{1}{\omega_n}\arccos\frac{\omega_n^2(Z_n-X_n)-Y_nZ_n}{s(\omega_n^2+Z_n^2)}$,

$$\omega_n=\sqrt{\frac{1}{2}\left[-\left(X_n^2-2Y_n-s^2\right)+\sqrt{\left(X_n^2-2Y_n-s^2\right)^2-4\left(Y_n^2-s^2Z_n^2\right)}\right]}.$$

Proof. $i\omega$ ($\omega>0$) is a root of Eq. (11) if and only if ω satisfies

$$-\omega^2+i\omega X_n+Y_n+s(i\omega+Z_n)(\cos\omega\tau-i\sin\omega\tau)=0.$$

Then we have

$$\begin{cases}-\omega^2+Y_n+sZ_n\cos\omega\tau+s\omega\sin\omega\tau=0,\\ \omega X_n-sZ_n\sin\omega\tau+s\omega\cos\omega\tau=0\end{cases}$$

which lead to

$$\omega^4+\omega^2\left(X_n^2-2Y_n-s^2\right)+Y_n^2-Z_n^2s^2=0 \tag{12}$$

and the roots of (12) are

$$\omega_\pm^2=\frac{1}{2}\left[-\left(X_n^2-2Y_n-s^2\right)\pm\sqrt{\left(X_n^2-2Y_n-s^2\right)^2-4\left(Y_n^2-s^2Z_n^2\right)}\right].$$

Obviously, $Y_n+sZ_n>0$ and $Y_n-sZ_n=d_1d_2\mu_n^2-(a_1d_2+sd_1)\mu_n+s(a_1+a_2\gamma)$. Obviously, there is a $n_1\in\mathbb{N}_0$ such $Y_n-sZ_n<0$ for $0\le n\le N_1$. Then $\omega_-^2<0$ and $\omega_+^2>0$ for $0\le n\le N_1$. Based on the above discussion, the lemma holds.

□

Denote $\tau_*^0=\min_{0\le i\le N_1}\left\{\tau_i^0\right\}$. Based on the above analysis, we have the following theorem.

Theorem 4. Suppose $a_1+a_2\gamma<0$ and $s>\max\left\{\frac{d_2}{d_1}a_1,a_1\right\}$ hold, for system (2), the following statements are true.

i) If $\tau\in\left[0,\tau_*^0\right)$, then $E_*(u_*,v_*)$ is local asymptotically stable;

ii) If $\tau>\tau_*^0$, then equilibrium $E_*(u_*,v_*)$ is unstable;

iii) $\tau=\tau_0^j$ ($j\in\mathbb{N}_0$) are Hopf bifurcation values of system (2).

Proof. Denote $\lambda_n(\tau)=\alpha_n(\tau)+i\omega_n(\tau)$ is the root of (11), which satisfy

$\alpha_n(\tau_n^j)=0$ and $\omega_n(\tau_n^j)=\omega_n$ when τ is near τ_n^j. Then we can obtain the following transversality condition. Differentiating two sides of (11) with respect τ, we have

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}=\frac{2\lambda+X_n+s\mathrm{e}^{-\lambda\tau}}{\lambda s(\lambda+Z_n)\mathrm{e}^{-\lambda\tau}}-\frac{\tau}{\lambda}.$$

Then

$$\begin{aligned}\left[\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right]^{-1}_{\tau=\tau_n^j}&=\left[\frac{2\lambda+X_n+s\mathrm{e}^{-\lambda\tau}}{\lambda s(\lambda+Z_n)\mathrm{e}^{-\lambda\tau}}-\frac{\tau}{\lambda}\right]_{\tau=\tau_n^j}\\&=\left[\frac{s+X_n\cos\omega\tau-2\omega\sin\omega\tau+\mathrm{i}(2\omega\cos\omega\tau)+X_n\sin\omega\tau}{-s\omega^2+\mathrm{i}Z_ns\omega}-\frac{\tau}{\mathrm{i}\omega}\right]_{\tau=\tau_n^j}\\&=\frac{1}{\Lambda}\omega^2\left(2\omega^2-2Y_n+X_n^2-s^2\right)\\&=\frac{1}{\Lambda}\omega^2\sqrt{\left(X_n^2-2Y_n-s^2\right)^2-4\left(Y_n^2-s^2Z_n^2\right)}>0,\end{aligned}$$

where $\Lambda=\omega^4c^2+Z_n^2c^2\omega^2>0$. Therefore the transversal condition hold. The content of the theorem is obviously valid.

□

3. Direction and Stability of Hopf bifurcation

In this section, we use center manifold theorem and normal form theorem of partial functional differential equations to analyze the stability of the bifurcating periodic solution and direction of Hopf bifurcation. Denote $\tilde{\tau}=\tau-\mu$, $v_1(t)=u(\cdot,t)$, $v_2(t)=v(\cdot,t)$ and $V=(v_1,v_2)^{\mathrm{T}}$. Thus, in the phase space $\mathbb{C}_1:=C([-1,0],X)$, (2) can be transformed into an abstract form

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}=\tilde{\tau}d\Delta V(t)+L_{\tilde{\tau}}(V_t)+F(V_t,\mu),\tag{13}$$

with $L_\mu(\varphi)=\mu\begin{pmatrix}a_1\varphi_1(0)+a_2\varphi_2(0)\\ s\gamma\varphi_1(-1)-s\varphi_2(-1)\end{pmatrix}$, and $F(\varphi,\mu)=\mu d\Delta\varphi+L_\mu(\varphi)+h(\varphi,\mu)$, where $h(\varphi,\mu)=(\tilde{\tau}+\mu)(F_1(\varphi,\mu),F_2(\varphi,\mu))^{\mathrm{T}}$, with $\varphi=(\varphi_1,\varphi_2)^{\mathrm{T}}\in\mathbb{C}_\tau$, and

$$\begin{aligned}F_1(\varphi,u)=r(\varphi_1(0)+u_0)\left(1-\frac{\varphi_1(0)+u_0}{K}\right)\left(1-\frac{f+\theta(\varphi_2(0)+v_0)}{f+(\varphi_1(0)+u_0)}\right)\\-\frac{m(\varphi_1(0)+u_0)(\varphi_2(0)+v_0)}{a+(\varphi_1(0)+u_0)}-a_1\varphi_1(0)-a_2\varphi_2(0),\end{aligned}\tag{14}$$

$$F_2(\varphi,v)=s(\varphi_2(0)+v_0)\left(1-\frac{\varphi_2(-1)+v_0}{\beta+\gamma(\varphi_1(-1)+u_0)}\right)-s\gamma\varphi_1(-1)+s\varphi_2(-1).\tag{15}$$

Next, we study the linear equation of (13).

From the subsection 2.2, $\Lambda_n:=\{\mathrm{i}\omega_n\tilde{\tau},-\mathrm{i}\omega_n\tilde{\tau}\}$ are characteristic eigenvalues of equation

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t}=-\tilde{\tau}d\frac{n^2}{l^2}z(t)+L_{\tilde{\tau}}(z_t). \tag{16}$$

From Riesz representation, we can get a 2×2 matrix function $\eta^n(\varepsilon,\tilde{\tau})$, $-1\le\varepsilon\le 0$, such that $-\tilde{\tau}d\frac{n^2}{l^2}\varphi(0)+L_{\tilde{\tau}}(\varphi)=\int_{-1}^{0}\mathrm{d}\eta^n(\varepsilon,\tau)\varphi(\varepsilon)$, for $\varphi\in C([-1,0],\mathbb{R}^2)$.

We can choose

$$\eta^n(\varepsilon,\tau)=\begin{cases}\tau\begin{pmatrix}a_1-d_1\frac{n^2}{l^2} & a_2\\ 0 & 0\end{pmatrix} & \varepsilon=0,\\ 0 & \varepsilon\in(-1,0),\\ \tau\begin{pmatrix}0 & 0\\ s\gamma & -s-d_2\frac{n^2}{l^2}\end{pmatrix} & \varepsilon=-1,\end{cases} \tag{17}$$

Let $B(\tilde{\tau})$ be the infinitesimal generators of semigroup included by the solutions of equation (16) and B^* be the formal adjoint of $B(\tilde{\tau})$ under the bilinear paring

$$\begin{aligned}(\phi,\varphi)_n&=\phi(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\varepsilon}\phi(\xi-\varepsilon)\mathrm{d}\eta^n(\varepsilon,\tilde{\tau})\varphi(\xi)\mathrm{d}\xi\\&=\phi(0)\varphi(0)+\tilde{\tau}\int_{-1}^{0}\phi(\xi+1)F\varphi(\xi)\mathrm{d}\xi\end{aligned} \tag{18}$$

for $\varphi\in C([-1,0],\mathbb{R}^2),\phi\in C([0,1],\mathbb{R}^2)$. $B(\tilde{\tau})$ has a pair of purely imaginary eigenvalues $\pm\mathrm{i}\omega_n\tilde{\tau}$, Let P and p^* be the characteristic subspaces of Λ_n respectively, and they are also characteristic subspaces of $B(\tilde{\tau})$ and B^*. Therefore, p^* is the adjoint matrix of P, we have $\dim P=\dim P^*=2$. Obviously, $p_1(\varepsilon)=(1,\xi)^{\mathrm{T}}\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\varepsilon}(\varepsilon\in[-1,0])$ and $p_2(\varepsilon)=\overline{p_1(\varepsilon)}$ are bases of $B(\tilde{\tau})$ corresponding to Λ_n, $q_1(\theta)=(1,\eta)\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\theta}(\theta\in[0,1])$ and $q_2(\theta)=\overline{q_1(\theta)}$ are bases of B^* corresponding to Λ_n, where

$$\xi=\frac{-a_1+d_1\mu_n+\mathrm{i}\omega}{a_2},\eta=\frac{a_2}{s+d_2\mu_n+\mathrm{i}\omega}.$$

Let $\Phi=(\Phi_1,\Phi_2)^{\mathrm{T}}$ and $\dot{\Psi}=(\dot{\Psi}_1,\dot{\Psi}_2)^{\mathrm{T}}$, for $\varepsilon\in[-1,0]$, we have

$$\Phi_1(\varepsilon)=\frac{p_1(\varepsilon)+p_2(\varepsilon)}{2}=\begin{pmatrix}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\varepsilon})\\ \mathrm{Re}(\xi\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\varepsilon})\end{pmatrix}=\begin{pmatrix}\cos\omega_n\tilde{\tau}\varepsilon\\ \left(\frac{d_1\mu_n-a_1}{a_2}\right)\cos\varepsilon\tau\omega_n-\frac{\omega_n^2\sin\varepsilon\tau\omega_n}{a_2}\end{pmatrix},$$

$$\Phi_2(\varepsilon)=\frac{p_1(\varepsilon)-p_2(\varepsilon)}{2\mathrm{i}}=\begin{pmatrix}\mathrm{Im}(\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\varepsilon})\\ \mathrm{Im}(\xi\mathrm{e}^{\mathrm{i}\omega_n\tilde{\tau}\varepsilon})\end{pmatrix}=\begin{pmatrix}\sin\omega_n\tilde{\tau}\varepsilon\\ -\frac{\omega_n}{a_2}\cos\varepsilon\tau\omega_n+\frac{(a_1-d_1\mu_n)\omega_n}{a_2}\sin\varepsilon\tau\omega_n\end{pmatrix}.$$

For $\theta \in [-1,0]$, we have

$$\dot{\Psi}_1(\theta)=\frac{q_1(\theta)+q_2(\theta)}{2}=\begin{pmatrix}\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\theta}\right)\\ \mathrm{Re}\left(\eta\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\theta}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\cos\theta\tau\omega_n\\ \dfrac{a_2(s+d_2\mu_n)}{(s+d_2\mu_n)^2+\omega_n^2}\cos\theta\tau\omega_n-\dfrac{a_2\omega_n^2}{(s+d_2\mu_n)^2+\omega_n^2}\sin\theta\tau\omega_n\end{pmatrix}.$$

$$\dot{\Psi}_2(\theta)=\frac{q_1(\theta)-q_2(\theta)}{2\mathrm{i}}=\begin{pmatrix}\mathrm{Im}\left(\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\theta}\right)\\ \mathrm{Im}\left(\eta\mathrm{e}^{-\mathrm{i}\omega_n\tilde{\tau}\theta}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\sin\theta\tau\omega_n\\ \dfrac{a_2\omega_n}{(s+d_2\mu_n)^2+\omega_n^2}\cos\theta\tau\omega_n+\dfrac{a_2\omega_n(s+d_2\mu_n)}{(s+d_2\mu_n)^2+\omega_n^2}\sin\theta\tau\omega_n\end{pmatrix}.$$

Then we can obtain the following equation by (18)

$$D_1:=(\dot{\Psi}_1,\Phi_1),D_2:=(\dot{\Psi}_1,\Phi_2),D_3:=(\dot{\Psi}_2,\Phi_1),D_4:=(\dot{\Psi}_2,\Phi_2).$$

Define $(\dot{\Psi},\Phi)=(\dot{\Psi}_j,\Phi_k)=\begin{pmatrix}D_1 & D_2\\ D_3 & D_4\end{pmatrix}$ and construct a new basis Ψ for P^* by $\Psi=(\Psi_1,\Psi_2)^{\mathrm{T}}=(\dot{\Psi},\Phi)^{-1}\dot{\Psi}$. Then $(\Psi,\Phi)=I_2$, Besides, define $l_n:=(\gamma_n^1,\gamma_n^2)$, with

$$\gamma_n^1=\begin{pmatrix}\cos\frac{n}{l}x\\ 0\end{pmatrix},\gamma_n^2=\begin{pmatrix}0\\ \cos\frac{n}{l}x\end{pmatrix},$$

and $\gamma\cdot l_n=\gamma_1\gamma_n^1+\gamma_2\gamma_n^2$, for $\gamma=(\gamma_1,\gamma_2)^{\mathrm{T}}\in\mathbb{C}_1$. Hence, we have $\langle u,v\rangle:=\frac{1}{l\pi}\int_0^{l\pi}u_1\overline{v_1}\mathrm{d}x+\frac{1}{l\pi}\int_0^{l\pi}u_2\overline{v_2}\mathrm{d}x$ for $u=(u_1,u_2)$, $v=(v_1,v_2)$, $u,v\in X$ and $\langle\varphi,f_0\rangle=\left(\langle\varphi,f_0^1\rangle,\langle\varphi,f_0^2\rangle\right)^{\mathrm{T}}$. From [15], we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t}=B_{\tilde{\tau}}V_t+R(V_t,\mu) \tag{19}$$

where

$$R(V_t,\mu)=\begin{cases}0 & \varepsilon\in[-1,0)\\ F(V_t,\mu) & \varepsilon=0\end{cases}$$

The solution of (19) can be expressed as

$$V_t=\Phi\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}\cdot l_n+h(\dot{y}_1,\dot{y}_2,\mu) \tag{20}$$

with

$$(\dot{y}_1,\dot{y}_2)^{\mathrm{T}}=(\Psi,\langle V_t,l_n\rangle),h(\dot{y}_1,\dot{y}_2,\mu)\in P_S\mathbb{C}_1,h(0,0,0)=0,Dh(0,0,0)=0.$$

From center manifold theorem, the solution of (13) can be expressed as

$$V_t=\Phi(\dot{y}_1(t)\dot{y}_2(t))^{\mathrm{T}}\cdot l_n+h(\dot{y}_1,\dot{y}_2,0). \tag{21}$$

Denote $c_* = \dot{y}_1 - i\dot{y}_2$, and notice that $p_1 = \Phi_1 + i\Phi_2$. Then we have

$$\Phi\begin{pmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{pmatrix}\cdot l_n = (\Phi_1,\Phi_2)\begin{pmatrix}\dfrac{c_*+\overline{c}_*}{2}\\ \dfrac{i(c_*-\overline{c}_*)}{2}\end{pmatrix}\cdot l_n = \frac{1}{2}\left(p_1c_* + \overline{p}_1\overline{c}_*\right)\cdot l_n,$$

$$h(\dot{y}_1,\dot{y}_2,0) = h\left(\frac{c_*+\overline{c}_*}{2},\frac{(c_*-\overline{c}_*)i}{2},0\right).$$

Thus, Equation (21) become

$$\begin{aligned} V_t &= \frac{1}{2}\left(p_1c_* + \overline{p}_1\overline{c}_*\right)\cdot l_n + h\left(\frac{c_*+\overline{c}_*}{2},\frac{(c_*-\overline{c}_*)i}{2},0\right) \\ &= \frac{1}{2}\left(p_1c_* + \overline{p}_1\overline{c}_*\right)\cdot l_n + W(c_*,\overline{c}_*) \end{aligned} \tag{22}$$

where

$$W(c_*,\overline{c}_*) = h\left(\frac{c_*+\overline{c}_*}{2},\frac{(c_*-\overline{c}_*)i}{2},0\right)$$

From [16], we can know

$$\dot{c}_* = i\omega_n\tilde{\tau}c_* + g(c_*,\overline{c}_*), \tag{23}$$

with

$$g(c_*,\overline{c}_*) = \left(\Psi_1(0) - i\Psi_2(0)\right)\left\langle F(V_t,0), l_n\right\rangle. \tag{24}$$

Denote

$$W(c_*,\overline{c}_*) = W_{20}\frac{c_*^2}{2} + W_{11}c_*\overline{c}_* + W_{02}\frac{\overline{c}_*^2}{2} + \cdots, \tag{25}$$

$$g(c_*,\overline{c}_*) = g_{20}\frac{c_*^2}{2} + g_{11}c_*\overline{c}_* + g_{02}\frac{\overline{c}_*^2}{2} + g_{21}\frac{c_*^2\overline{c}_*}{2} + \cdots. \tag{26}$$

From (23) and (26), we have

$$\begin{aligned} u_t(0) &= \frac{1}{2}(c_*+\overline{c}_*)\cos\left(\frac{nx}{l}\right) + W_{20}^1(0)\frac{c_*^2}{2} + W_{11}^1(0)c_*\overline{c}_* + W_{02}^1(0)\frac{\overline{c}_*^2}{2} + \cdots \\ v_t(0) &= \frac{1}{2}\left(\xi c_* + \overline{\xi}\overline{c}_*\right)\cos\left(\frac{nx}{l}\right) + W_{20}^2(0)\frac{c_*^2}{2} + W_{11}^2(0)c_*\overline{c}_* + W_{02}^2(0)\frac{\overline{c}_*^2}{2} + \cdots \\ u_t(-1) &= \frac{1}{2}\left(c\mathrm{e}^{-\mathrm{i}\omega_n\tau} + \overline{c}_*\mathrm{e}^{\mathrm{i}\omega_n\tau}\right)\cos\left(\frac{nx}{l}\right) + W_{20}^1(-1)\frac{c_*^2}{2} \\ &\quad + W_{11}^1(-1)c_*\overline{c}_* + W_{02}^1(-1)\frac{\overline{c}_*^2}{2} + \cdots \end{aligned} \tag{27}$$

and

$$\overline{F}_1(V_t,0) = \frac{1}{\tilde{\tau}}F_1 = \frac{1}{2}f_{uu}u_t^2(0) + f_{uv}u_t(0)v_t(0) + \frac{1}{6}f_{uuu}u_t^3(0) + \cdots$$

$$\overline{F}_2(V_t,0)=\frac{1}{\tilde{\tau}}F_2=g_{uv}u_t(-1)v_t(0)+\frac{1}{2}g_{vv}v_t^2(0)+\cdots$$

with

$$\begin{aligned}
f_{uu}&=\frac{fr\left(-3u_0^2+\theta Kv_0+f\left(K-3u_0+\theta v_0\right)\right)}{K\left(f+u_0\right)^3},\\
f_{uv}&=-\frac{am}{\left(a+u_0\right)^2}+\frac{r\theta\left(-fK+2fu_0+u_0^2\right)}{K\left(f+u_0\right)^2},\\
f_{uuu}&=6\left(-\frac{f^2\left(f+K\right)r}{K\left(f+u_0\right)^4}+v_0\left(-\frac{am}{\left(a+u_0\right)^4}-\frac{fr\theta\left(f+K\right)}{K\left(f+u_0\right)^4}\right)\right),\\
f_{uuv}&=\frac{2am}{\left(a+u_0\right)^3}+\frac{2fr\theta\left(f+K\right)}{K\left(f+u_0\right)^3},g_{uu}=-\frac{2s\gamma^2}{\beta+u_0\gamma},g_{uv}=\frac{2s\gamma}{\beta+u_0\gamma},\\
g_{vv}&=-\frac{2s}{\beta+u_0\gamma},g_{uuu}=\frac{6s\gamma^3}{\left(\beta+u_0\gamma\right)^2},g_{uuv}=-\frac{4s\gamma^2}{\left(\beta+u_0\gamma\right)^2},\\
g_{uvv}&=\frac{2s\gamma}{\left(\beta+u_0\gamma\right)^2},f_{vv}=f_{uvv}=f_{vvv}=g_{vvv}=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{F}_1(V_t,0)=&\frac{c_*^2}{2}\left[\frac{1}{4}\cos^2\mu_n x\left(f_{uu}+2\xi f_{uv}\right)\right]\\
&+c_*\overline{c}_*\left[\frac{1}{4}\cos^2\mu_n x\left(f_{uu}+\left(\overline{\xi}+\xi\right)f_{uv}\right)\right]\\
&+\frac{\overline{c}_*^2}{2}\left[\frac{1}{4}\cos^2\mu_n x\left(f_{uu}+2\overline{\xi}f_{uv}\right)\right]\\
&+\frac{c_*^2\overline{c}_*}{2}\left[\frac{1}{2}\cos\mu_n x\left(\left(2W_{11}^1(0)+W_{20}^1(0)\right)f_{uu}\right.\right.\\
&\left.\left.+\left(2W_{11}^2(0)+W_{20}^2(0)+W_{20}^1(0)\overline{\xi}+2W_{11}^1(0)\xi\right)f_{uv}\right)\right],
\end{aligned}\tag{28}$$

$$\begin{aligned}
\overline{F}_2(V_t,0)=&\frac{c_*^2}{2}\left[\frac{1}{4}\cos^2\mu_n x\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\right]\\
&+c_*\overline{c}_*\left[\frac{1}{4}\cos^2\mu_n x\left(g_{uu}+\left(\mathrm{e}^{-\mathrm{i}\tau\omega_n}\overline{\xi}+\mathrm{e}^{\mathrm{i}\tau\omega_n}\xi\right)g_{uv}+\overline{\xi}\xi g_{vv}\right)\right]\\
&+\frac{\overline{c}_*^2}{2}\left[\frac{1}{4}\cos^2\mu_n x\left(\mathrm{e}^{2\mathrm{i}\tau\omega_n}g_{uu}+\overline{\xi}\left(2\mathrm{e}^{\mathrm{i}\tau\omega_n}g_{uv}+\overline{\xi}g_{vv}\right)\right)\right]\\
&+\frac{c_*^2\overline{c}_*}{2}\left[\frac{1}{2}\mathrm{e}^{-\mathrm{i}\tau\omega_n}\cos\mu_n x\left(\left(2W_{11}^1(-1)+\mathrm{e}^{2\mathrm{i}\tau\omega_n}W_{20}^1(-1)\right)g_{uu}\right.\right.\\
&+\left(2W_{11}^2(0)+\mathrm{e}^{\mathrm{i}\tau\omega_n}\left(\mathrm{e}^{\mathrm{i}\tau\omega_n}W_{20}^2(0)+W_{20}^1(-1)\overline{\xi}+2W_{11}^1(-1)\xi\right)\right)g_{uv}\\
&\left.\left.+\mathrm{e}^{\mathrm{i}\tau\omega_n}\left(W_{20}^2(0)\overline{\xi}+2W_{11}^2(0)\xi\right)g_{vv}\right)\right].
\end{aligned}\tag{29}$$

$$\begin{aligned}\left\langle F\left(V_t,0\right),l_n\right\rangle &= \tilde{\tau}\left(\overline{F}_1\left(V_t,0\right)l_n^1+\overline{F}_2\left(V_t,0\right)l_n^2\right)\\ &= \frac{c_*^2}{2}\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+2\xi f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix}\Gamma\\ &\quad+c_*\overline{c}_*\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+\left(\overline{\xi}+\xi\right)f_{uv}\right)\\ \frac{1}{4}\left(g_{uu}+\left(\mathrm{e}^{-\mathrm{i}\tau\omega_n}\overline{\xi}+\mathrm{e}^{\mathrm{i}\tau\omega_n}\xi\right)g_{uv}+\overline{\xi}\xi g_{vv}\right)\end{pmatrix}\Gamma\\ &\quad+\frac{\overline{c}_*^2}{2}\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+2\overline{\xi}f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{2\mathrm{i}\tau\omega_n}g_{uu}+\overline{\xi}\left(2\mathrm{e}^{\mathrm{i}\tau\omega_n}g_{uv}+\overline{\xi}g_{vv}\right)\right)\end{pmatrix}\Gamma\\ &\quad+\frac{c_*^2\overline{c}_*}{2}\tilde{\tau}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix}+\cdots\end{aligned}\tag{30}$$

where

$$\Gamma=\frac{1}{l\pi}\int_0^{l\pi}\cos^3\left(\frac{nx}{l}\right)\mathrm{d}x.$$

$$\begin{aligned}\vartheta_1=\frac{1}{2}&\Big(\Big(2\left\langle W_{11}^1(0)\cos\mu_n x,\cos\mu_n x\right\rangle+\left\langle W_{20}^1(0)\cos\mu_n x,\cos\mu_n x\right\rangle\Big)f_{uu}\\ &+\Big(2\left\langle W_{11}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle+\left\langle W_{20}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle\\ &+\left\langle W_{20}^1(0)\cos\mu_n x,\cos\mu_n x\right\rangle\overline{\xi}+2\left\langle W_{11}^1(0)\cos\mu_n x,\cos\mu_n x\right\rangle\xi\Big)f_{uv}\Big),\end{aligned}$$

$$\begin{aligned}\vartheta_2=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\tau\omega_n}&\Big(\Big(2\left\langle W_{11}^1(-1)\cos\mu_n x,\cos\mu_n x\right\rangle+\mathrm{e}^{2\mathrm{i}\tau\omega_n}\left\langle W_{20}^1(-1)\cos\mu_n x,\cos\mu_n x\right\rangle\Big)g_{uu}\\ &+\Big(2\left\langle W_{11}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle+\mathrm{e}^{\mathrm{i}\tau\omega_n}\Big(\mathrm{e}^{\mathrm{i}\tau\omega_n}\left\langle W_{20}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle\\ &+\left\langle W_{20}^1(-1)\cos\mu_n x,\cos\mu_n x\right\rangle\overline{\xi}+2\left\langle W_{11}^1(-1)\cos\mu_n x,\cos\mu_n x\right\rangle\xi\Big)\Big)g_{uv}\\ &+\mathrm{e}^{\mathrm{i}\tau\omega_n}\Big(\left\langle W_{20}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle\overline{\xi}+2\left\langle W_{11}^2(0)\cos\mu_n x,\cos\mu_n x\right\rangle\xi\Big)g_{vv}\Big).\end{aligned}$$

Denote $\Psi_1(0)-i\Psi_2(0):=(\varepsilon_1\ \ \varepsilon_2)$, observe that $\frac{1}{l\pi}\int_0^{l\pi}\cos^3\mu_n x\mathrm{d}x=0,n=1,2,3,\cdots$.

Thus, we have

$$\begin{aligned}&\left(\Psi_1(0)-i\Psi_2(0)\right)\left\langle F\left(V_t,0\right),l_n\right\rangle\\ &=\frac{c_*^2}{2}\left[\frac{\varepsilon_1}{4}\left(f_{uu}+2\xi f_{uv}\right)+\frac{\varepsilon_2}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\right]\varepsilon\tilde{\tau}\\ &\quad+c_*\overline{c}_*\left[\frac{\varepsilon_1}{4}\left(f_{uu}+(\overline{\xi}+\xi)f_{uv}\right)+\frac{\varepsilon_2}{4}\left(g_{uu}+\left(\mathrm{e}^{-\mathrm{i}\tau\omega_n}\overline{\xi}+\mathrm{e}^{\mathrm{i}\tau\omega_n}\xi\right)g_{uv}+\overline{\xi}\xi g_{vv}\right)\right]\varepsilon\tilde{\tau}\\ &\quad+\frac{\overline{c}_*^2}{2}\left[\frac{\varepsilon_1}{4}\left(f_{uu}+2\overline{\xi}f_{uv}\right)+\frac{\varepsilon_2}{4}\left(\mathrm{e}^{2\mathrm{i}\tau\omega_n}g_{uu}+\overline{\xi}\left(2\mathrm{e}^{\mathrm{i}\tau\omega_n}g_{uv}+\overline{\xi}g_{vv}\right)\right)\right]\varepsilon\tilde{\tau}\\ &\quad+\frac{c_*^2\overline{c}_*}{2}\tilde{\tau}\left[\varepsilon_1\vartheta_1+\varepsilon_2\vartheta_2\right]+\cdots\end{aligned}\tag{31}$$

Hence, from (24) (26) (31), we can get $g_{20}=g_{11}=g_{02}=0$, for $n\in\mathbb{N}$. When $n=0$, yields

$$g_{20}=\frac{\varepsilon_1}{4}\left(f_{uu}+2\xi f_{uv}\right)+\frac{\varepsilon_2}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right),$$

$$g_{11}=\frac{\varepsilon_1}{4}\left(f_{uu}+\left(\xi+\overline{\xi}\right)f_{uv}\right)+\frac{\varepsilon_2}{4}\left(g_{uu}+\left(\mathrm{e}^{-\mathrm{i}\tau\omega_n}\overline{\xi}+\mathrm{e}^{\mathrm{i}\tau\omega_n}\xi\right)g_{uv}+\overline{\xi}\xi g_{vv}\right),$$

$$g_{02}=\frac{\varepsilon_1}{4}\left(f_{uu}+2\overline{\xi}f_{uv}\right)+\frac{\varepsilon_2}{4}\left(\mathrm{e}^{2\mathrm{i}\tau\omega_n}g_{uu}+\overline{\xi}\left(2\mathrm{e}^{\mathrm{i}\tau\omega_n}g_{uv}+\overline{\xi}g_{vv}\right)\right).$$

And $g_{21}=\tilde{\tau}\left(\varepsilon_1\vartheta_1+\varepsilon_2\vartheta_2\right)$, for $n\in\mathbb{N}_0$. For the next part, we compute $W_{20}(\varepsilon)$ and $W_{11}(\varepsilon)$ to get g_{21}. We have

$$\dot{W}\left(c_*,\overline{c}_*\right)=W_{20}c_*\dot{c}_*+W_{11}\dot{c}_*\overline{c}_*+W_{11}c_*\dot{\overline{c}}_*+W_{02}\overline{c}_*\dot{\overline{c}}_*+\cdots,$$

$$B_{\tilde{\tau}}W\left(c_*,\overline{c}_*\right)=B_{\tilde{\tau}}W_{20}\frac{c_*^2}{2}+B_{\tilde{\tau}}W_{11}c_*\overline{c}_*+B_{\tilde{\tau}}W_{02}\frac{\overline{c}_*^{\,2}}{2}+\cdots,$$

and $\dot{W}\left(c_*,\overline{c}_*\right)$ satisfies $\dot{W}\left(c_*,\overline{c}_*\right)=B_{\tilde{\tau}}W+H\left(c_*,\overline{c}_*\right)$, with

$$\begin{aligned}H\left(c_*,\overline{c}_*\right)&=H_{20}\frac{c_*^2}{2}+W_{11}c_*\overline{c}_*+H_{02}\frac{\overline{c}_*^2}{2}+\cdots\\&=X_0F\left(V_t,0\right)-\Phi\left(\Psi,\left\langle X_0F\left(V_t,0\right),l_n\right\rangle\cdot l_n\right).\end{aligned}\tag{32}$$

We have

$$\left(2\mathrm{i}\omega_n\tilde{\tau}-B_{\tilde{\tau}}\right)W_{20}=H_{20},\quad -B_{\tilde{\tau}}W_{11}=H_{11},\quad \left(-2\mathrm{i}\omega_n\tilde{\tau}-B_{\tilde{\tau}}\right)W_{02}=H_{02},\tag{33}$$

From (31), we can get

$$\begin{aligned}H\left(c_*,\overline{c}_*\right)&=-\Phi(0)\Psi(0)\left\langle F\left(V_t,0\right),l_n\right\rangle\cdot l_n\\&=-\left(\frac{p_1(\varepsilon)+p_2(\varepsilon)}{2},\frac{p_1(\varepsilon)-p_2(\varepsilon)}{2\mathrm{i}}\right)\begin{pmatrix}\Psi_1(0)\\ \Psi_2(0)\end{pmatrix}\left\langle F\left(V_t,0\right),l_n\right\rangle\cdot l_n\\&=-\frac{1}{2}\left[p_1(\varepsilon)\left(\Psi_1(0)-i\Psi_2(0)\right)+p_2(\varepsilon)\left(\Psi_1(0)+i\Psi_2(0)\right)\right]\left\langle F\left(V_t,0\right),l_n\right\rangle\cdot l_n\\&=-\frac{1}{2}l_n\left[\left(p_1(\varepsilon)g_{20}+p_2(\varepsilon)\overline{g}_{02}\right)\frac{c_*^2}{2}+\left(p_1(\varepsilon)g_{11}+p_2(\varepsilon)\overline{g}_{11}\right)c_*\overline{c}_*\right.\\&\quad\left.+l_n\left(p_1(\varepsilon)g_{02}+p_2(\varepsilon)\overline{g}_{20}\right)\frac{\overline{c}_*^2}{2}\right]+\cdots.\end{aligned}$$

Hence, by (32), we have

$$H_{20}(\varepsilon)=\begin{cases}0 & n\in\mathbb{N}\\ -\frac{1}{2}\left(p_1(\varepsilon)g_{20}+p_2(\varepsilon)\overline{g}_{02}\right)\cdot l_0 & n=0,\end{cases}$$

$$H_{11}(\varepsilon)=\begin{cases}0 & n\in\mathbb{N}\\ -\frac{1}{2}\left(p_1(\varepsilon)g_{11}+p_2(\varepsilon)\overline{g}_{11}\right)\cdot l_0 & n=0,\end{cases}$$

$$H_{02}(\varepsilon)=\begin{cases}0 & n\in\mathbb{N}\\ -\frac{1}{2}\left(p_1(\varepsilon)g_{02}+p_2(\varepsilon)\overline{g}_{20}\right)\cdot l_0 & n=0,\end{cases}$$

and

$$H\left(c_*,\overline{c_*}\right)(0)=F\left(V_t,0\right)-\Phi\left(\Psi,\left\langle F\left(V_t,0\right),l_n\right\rangle\right)\cdot l_n,$$

with

$$H_{20}(0)=\begin{cases}\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+2\xi f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix}\cos^2\mu_n x, & n\in\mathbb{N}\\ \tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+2\xi f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix} & n=0,\\ -\frac{1}{2}\left(p_1(\varepsilon)g_{20}+p_2(\varepsilon)\overline{g}_{02}\right)\cdot l_0, & \end{cases}$$

$$H_{11}(0)=\begin{cases}\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+\left(\overline{\xi}+\xi\right)f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix}\cos^2\mu_n x, & n\in\mathbb{N}\\ \tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+\left(\overline{\xi}+\xi\right)f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix} & n=0,\\ -\frac{1}{2}\left(p_1(\varepsilon)g_{11}+p_2(\varepsilon)\overline{g}_{11}\right)\cdot l_0. & \end{cases}$$

By the definition of $B_{\tilde{\tau}}$ and (33), we have

$$\dot{W}_{20}=B_{\tilde{\tau}}W_{20}=2\mathrm{i}\omega_n\tilde{\tau}W_{20}+\frac{1}{2}\left(p_1(\varepsilon)g_{20}+p_2(\varepsilon)\overline{g}_{02}\right)\cdot l_n,\quad -1\le\theta<0.$$

Thus, $W_{20}(\varepsilon)=\frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}\left(g_{20}p_1(\varepsilon)+\frac{\overline{g}_{02}}{3}p_2(\varepsilon)\right)\cdot l_n+G_1\mathrm{e}^{2\mathrm{i}\omega_n\tilde{\tau}\varepsilon}$,

where

$$G_1=\begin{cases}W_{20}(0) & n\in\mathbb{N},\\ W_{20}(0)-\frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}\left(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0)\right)\cdot l_0 & n=0.\end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (33), we have

$$\begin{aligned}&-\left(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0)\right)\cdot l_0+2\mathrm{i}\omega_n\tilde{\tau}G_1-A_{\tilde{\tau}}\left(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0)\right)\cdot l_0\right)\\&-A_{\tilde{\tau}}G_1-L_{\tilde{\tau}}\left(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20}p_1(0)+\frac{\overline{g}_{02}}{3}p_2(0)\right)\cdot l_n+G_1\mathrm{e}^{2\mathrm{i}\omega_n\tilde{\tau}\theta}\right)\\&=\tilde{\tau}\begin{pmatrix}\frac{1}{4}\left(f_{uu}+2\xi f_{uv}\right)\\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tau\omega_n}g_{uu}+\xi\left(2\mathrm{e}^{-\mathrm{i}\tau\omega_n}g_{uv}+\xi g_{vv}\right)\right)\end{pmatrix}-\frac{1}{2}\left(p_1(0)g_{20}+p_2(0)\overline{g}_{02}\right)\cdot l_0.\end{aligned}$$

As

$$B_{\tilde{\tau}}p_1(0)\cdot l_0+L_{\tilde{\tau}}\left(p_1\cdot l_0\right)=\mathrm{i}\omega_0 p_1(0)\cdot l_0,$$

$$B_{\tilde{\tau}} p_2(0)\cdot l_0 + L_{\tilde{\tau}}(p_2\cdot l_0) = -\mathrm{i}\omega_0 p_2(0)\cdot l_0.$$

We get for $n = 0,1,2,\cdots$,

$$2\mathrm{i}\omega_n G_1 - A_{\tilde{\tau}} G_1 - L_{\tilde{\tau}} E_1 \mathrm{e}^{2\mathrm{i}\omega_n} = \tilde{\tau}\begin{pmatrix} \frac{1}{4}\left(f_{uu} + 2\xi f_{uv}\right) \\ \frac{1}{4}\left(\mathrm{e}^{-2\mathrm{i}\tilde{\tau}\omega_n} g_{uu} + \xi\left(2\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_n} g_{uv} + \xi g_{vv}\right)\right) \end{pmatrix} \cos^2\left(\frac{nx}{l}\right)$$

That is

$$G_1 = \tilde{\tau} G \begin{pmatrix} \frac{1}{4}\left(f_{uu} + 2\xi f_{uv}\right) \\ \frac{1}{4}\bar{\xi}\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_n}\left(2g_{uv} + \xi \mathrm{e}^{\mathrm{i}\tilde{\tau}\omega_n} g_{vv}\right) \end{pmatrix} \cos^2\left(\frac{nx}{l}\right)$$

with

$$G = \begin{pmatrix} 2i\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - a_1 & a_2 \\ -s\gamma \mathrm{e}^{-2\mathrm{i}\omega_n\tilde{\tau}} & 2i\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2} + s \end{pmatrix}^{-1}.$$

Similarly, from (33), we have

$$-W_{11} = \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(p_1(\varepsilon) g_{11} + p_2(\varepsilon)\bar{g}_{11}\right)\cdot l_n, \quad -1 \le \varepsilon < 0.$$

$$G_2 = \tilde{\tau} G^* \begin{pmatrix} \frac{1}{4}\left(f_{uu} + \left(\bar{\xi} + \xi\right) f_{uv}\right) \\ \frac{1}{4}\mathrm{e}^{-\mathrm{i}\tilde{\tau}\omega_n}\left(\left(\bar{\xi} + \xi \mathrm{e}^{2\mathrm{i}\tilde{\tau}\omega_n}\right) g_{uv} + \xi\bar{\xi}\mathrm{e}^{\mathrm{i}\tilde{\tau}\omega_n} g_{vv}\right) \end{pmatrix} \cos^2\left(\frac{nx}{l}\right),$$

with

$$G^* = \begin{pmatrix} d_1\frac{n^2}{l^2} - a_1 & a_2 \\ -s\gamma & d_2\frac{n^2}{l^2} + s \end{pmatrix}^{-1}$$

Therefore, we can calculate the relevant quantities that govern the stability and direction of the branching periodic orbits.

$$\mu_1(0) = \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}\left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{1}{2} g_{21}, \quad \mu_2 = -\frac{\mathrm{Re}\left(\mu_1(0)\right)}{\mathrm{Re}\left(\lambda'\left(\tau_n^j\right)\right)},$$

$$T_2 = -\frac{1}{\omega_n\tilde{\tau}}\left(\mathrm{Im}\left(\mu_1(0)\right) + \mu_2 \,\mathrm{Im}\left(\lambda'\left(\tau_n^j\right)\right)\right), \quad \beta_2 = 2\,\mathrm{Re}\left(\mu_1(0)\right).$$

Then we have the following theorem.

Theorem 5. For any critical value τ_n^j, we have:

i) if $\mu_2 < 0$ (respectively > 0), then the Hopf bifurcation is backward (respectively forward), in other words, the bifurcating periodic solutions exists for $\tau > \tau_n^j$ (respectively $\tau < \tau_n^j$).

ii) if $\beta_2 < 0$ (respectively > 0), then the bifurcating periodic solutions are orbitally asymptotically stable (respectively unstable).

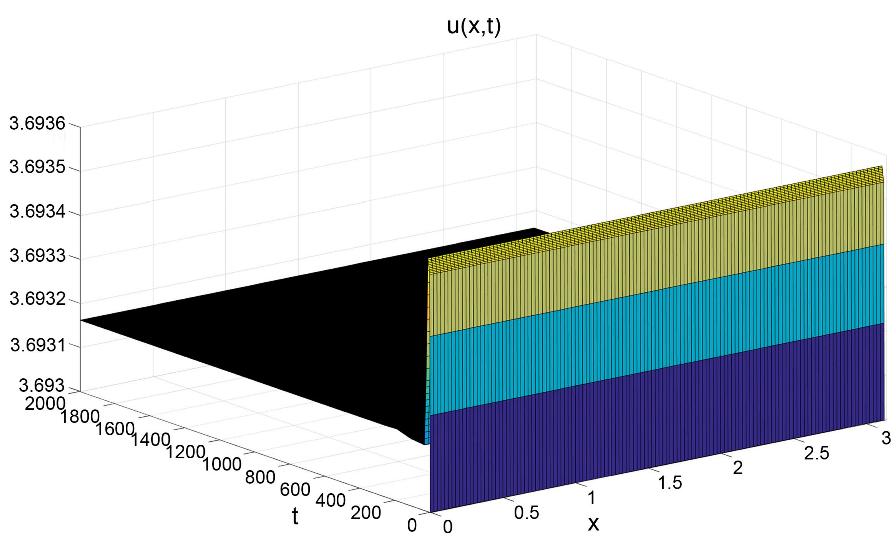
iii) if $T_2 < 0$ (respectively > 0), then the period decreases (respectively increases).

4. Numerical Simulations

Choose the parameters

$$\begin{gathered} r = 0.5,\ K = 10,\ f = 0.2,\ m = 0.5,\ a = 0.8,\ \beta = 0.5, \\ \gamma = 0.5,\ \theta = 0.2,\ s = 0.3,\ d_1 = 0.1,\ d_2 = 0.2,\ l = 1. \end{gathered} \tag{34}$$

The model has positive equilibria $(u_*, v_*) \approx (3.6932, 2.3466)$ and $(u_1, v_1) \approx (0.7678, 0.8839)$, and (u_1, v_1) is always unstable. Therefore, we mainly study the equilibrium $E_*(u_*, v_*)$. By direct calculation, it can be obtained $a_1 \approx 0.1132$, $a_2 \approx -0.4708$, $a_1 + a_2\gamma \approx -0.1223$. Obviously, hypothesis (**H**$_3$) holds, then from the (2) we know that the equilibrium $E_*(u_*, v_*)$ is local asymptotically stable when $\tau = 0$ (shown in **Figure 1**). In addition, we obtain $\tau_*^0 \approx 2.4722$, then then from the (4) we know that the equilibrium $E_*(u_*, v_*)$ is local asymptotically stable when $\tau < \tau_*^0$ (shown in **Figure 2**), and the bifurcating periodic solution exists when $\tau > \tau_*^0$ (shown in **Figure 3**). Moreover, we have $\mu_2 = 6.0319$, $\beta_2 = -1.1299$, $T_2 = 0.6212$, then from theorem (5) we can know that the Hopf bifurcation is forward, the bifurcating periodic solutions are orbitally asymptotically stable and the period of bifurcating periodic solutions increase. Besides, we change s to 0.6 and $\tau = 2$, (shown in **Figure 4**), by comparing with **Figure 2**, we find parameter s is also an important parameter which affect the stability of Hopf bifurcation. By above results, we can know that the gestation delay can affect the stability of population, this kind of influence is not monotonous and manifest as periodic oscillations.



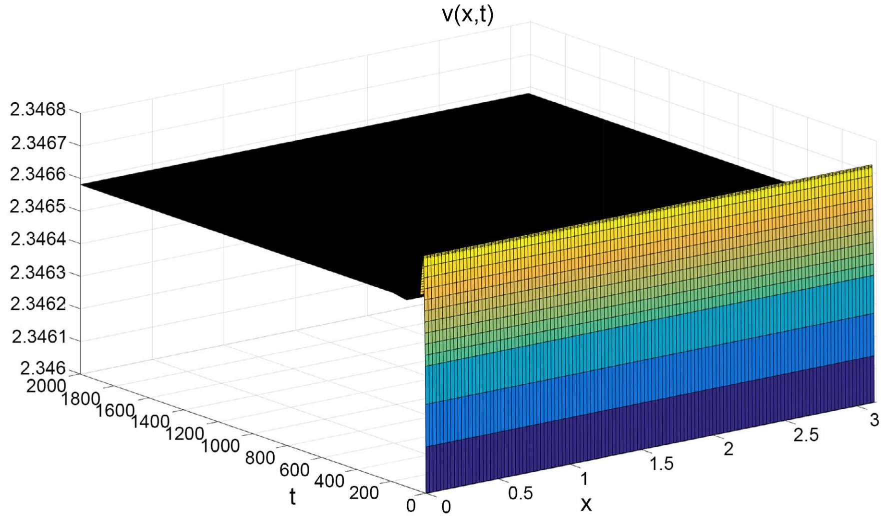


Figure 1. The numerical simulations of syetem (2) with parameters in (34), and $\tau = 0$.

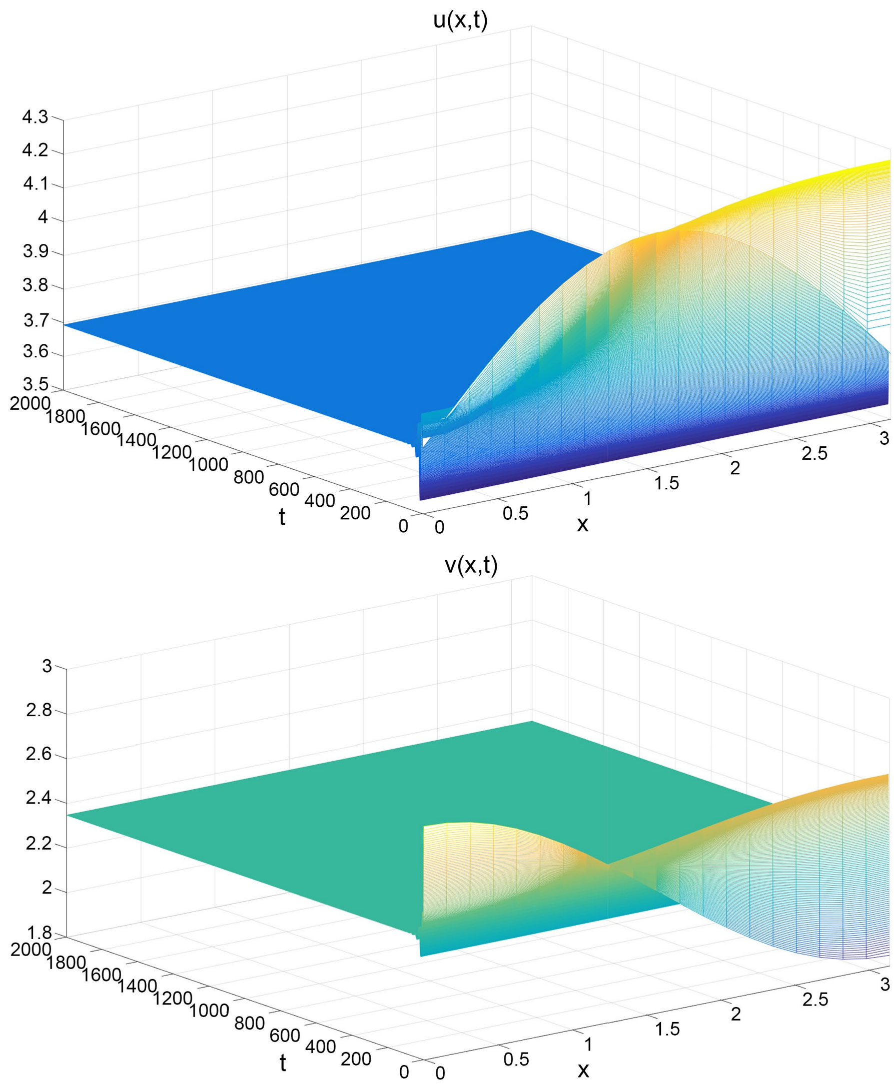


Figure 2. The numerical simulations of syetem (2) with parameters in (34), and $\tau = 2$.

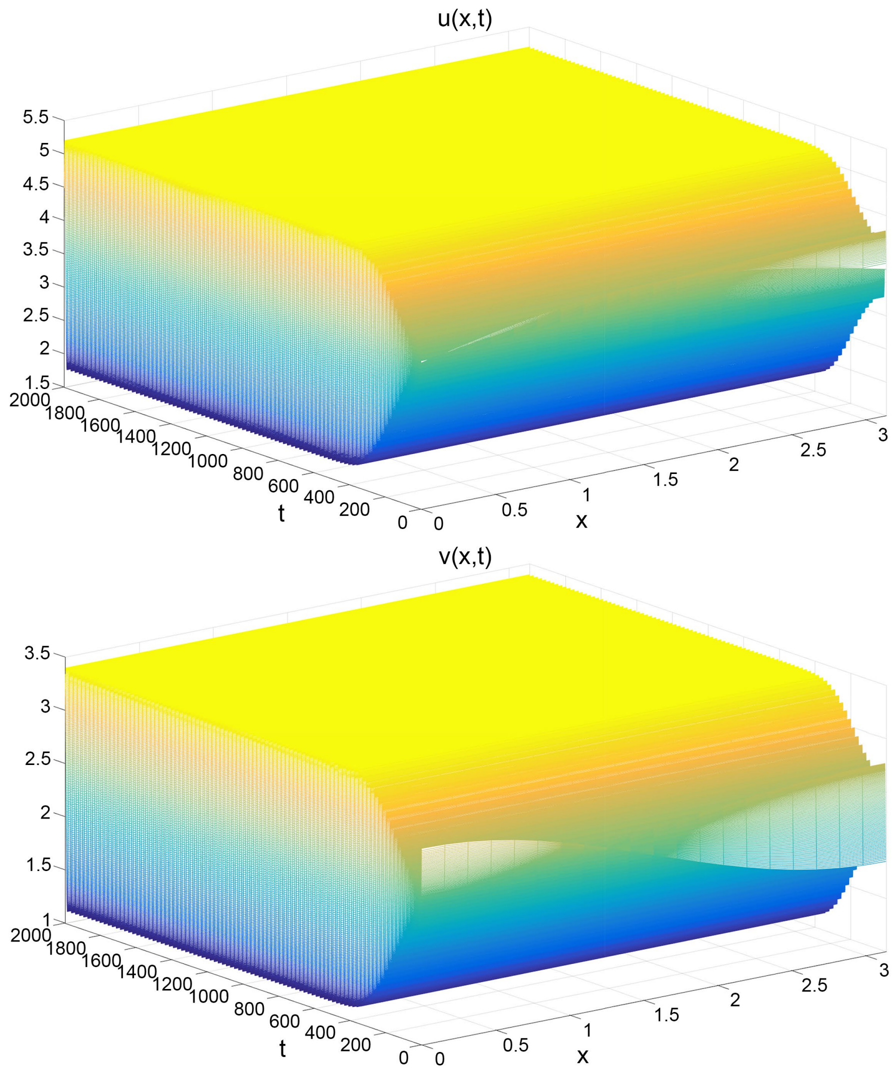
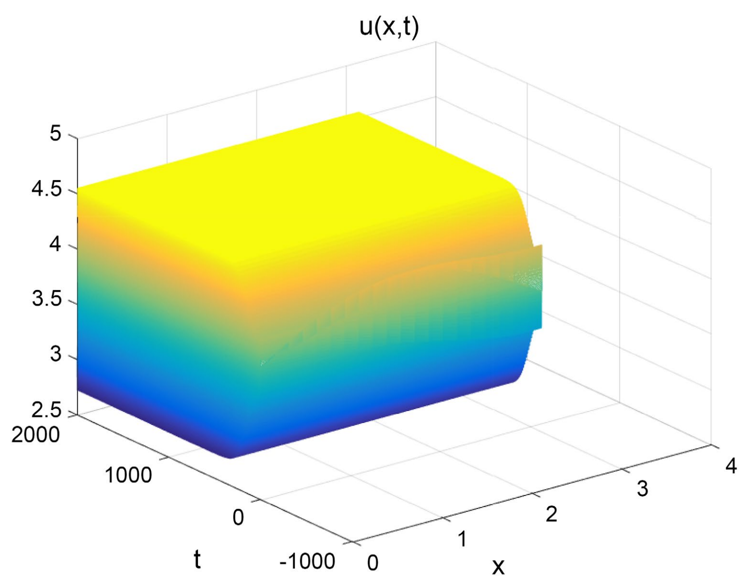


Figure 3. The numerical simulations of syetem (2) with parameters in (34), and $\tau = 2.8$.



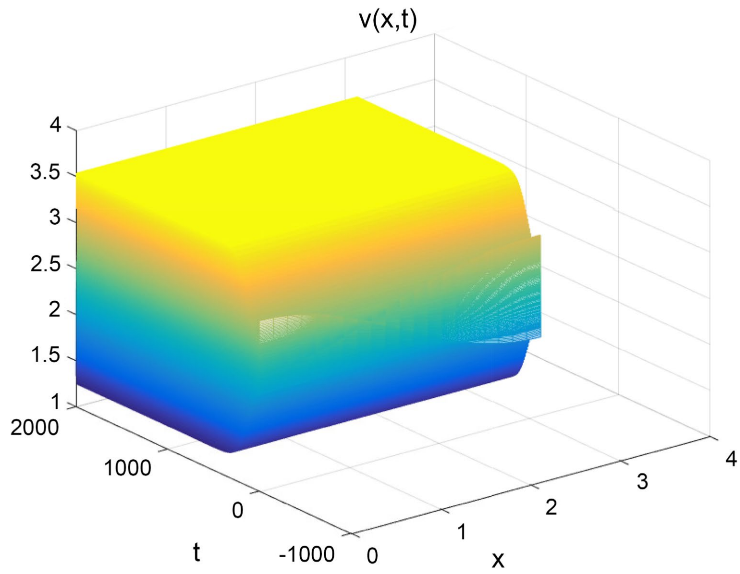


Figure 4. The numerical simulations of syetem (2) with parameters in (34), and $s = 0.6$, $\tau = 2$.

5. Conclusion

In this paper, we devote to exploring a diffusive predator-prey model with Allee effect and gestation time delay. By analyzing the associated characteristic transcendental equation, the linear stability of the positive equilibrium is investigated. We also investigate the phenomenon Turing instability and Hopf bifurcation. Furthermore, we conducted some calculations to determine the stability and direction of Hopf bifurcation. The addition of gestation time delay enables us to gain a more comprehensive understanding of biological processes such as reproductive strategies, population dynamics, environmental adaptability, and biodiversity maintenance in organisms. By considering this time delay, the model can more accurately predict and explain complex phenomena in actual biological systems. The numerical simulations reveal that time delay leads to the instability of population numbers that were originally stable. The population size of predators and prey is evenly distributed in time and space when there is no time delay (**Figure 1**). Further more, we found that there is a critical value for the impact of time delay on population size. When τ is weaker than this value (**Figure 2**), the population size is stable over time. Conversely, when τ is stronger than this value (**Figure 3**), the population size is unstable over time. In summary, the slower predators are born, the more unstable their populations become, which may be influenced by the predator-driven Allee effect. Besides, time delay can lead to non monotonic changes in population size, manifested as periodic oscillations, Periodic oscillations may lead to the disruption of ecological balance. Adding a model with gestation delay can reveal the complexity and potential oscillatory behavior in population dynamics. The emergence of Hopf branches suggests that populations may transition from stable equilibrium states to unstable periodic oscillations, which is of great significance for understanding the stability of ecosystems, survival strategies of species, and the sustainability of ecological services. For example, in the reproductive strategies of some insects, the time interval between larval and adult development can serve as a biological indicator of time delay. By studying these models, scholars can better predict and manage population fluctuations in ecosystems.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Liu, C. and Yang, R. (2021) Bifurcation Analysis of a Diffusive Bimolecular Model with Delayed Feedback. *Advances in Differential Equations and Control Processes*, **24**, 1-9. https://doi.org/10.17654/de024010001

[2] Han, R., Guin, L.N. and Acharya, S. (2022) Complex Dynamics in a Reaction-Cross-Diffusion Model with Refuge Depending on Predator-Prey Encounters. *The European Physical Journal Plus*, **137**, Article No. 134. https://doi.org/10.1140/epjp/s13360-022-02358-7

[3] Wang, F. and Yang, R. (2023) Dynamics of a Delayed Reaction-Diffusion Predator-Prey Model with Nonlocal Competition and Double Allee Effect in Prey. *International Journal of Biomathematics*, **18**, Article 2350097. https://doi.org/10.1142/s1793524523500973

[4] Mandal, G., Guin, L.N., Chakravarty, S., Rojas-Palma, A. and González-Olivares, E. (2024) Allee-Induced Bubbling Phenomena in an Interacting Species Model. *Chaos, Solitons & Fractals*, **184**, Article 114949. https://doi.org/10.1016/j.chaos.2024.114949

[5] Kayal, K., Samanta, S. and Chattopadhyay, J. (2023) Impacts of Predation-Driven Allee Effect in a Predator-Prey Model. *International Journal of Bifurcation and Chaos*, **33**, Article 2350023. https://doi.org/10.1142/s0218127423500232

[6] Farshid, M. and Jalilian, Y. (2022) Steady-State Bifurcation and Hopf Bifurcation in a Cross-Diffusion Prey-Predator System with Ivlev Functional Response. *Mathematical Methods in the Applied Sciences*, **46**, 5328-5348. https://doi.org/10.1002/mma.8837

[7] Ma, T. and Meng, X. (2022) Global Analysis and Hopf-Bifurcation in a Cross-Diffusion Prey-Predator System with Fear Effect and Predator Cannibalism. *Mathematical Biosciences and Engineering*, **19**, 6040-6071. https://doi.org/10.3934/mbe.2022282

[8] Mi, Y., Song, C. and Wang, Z. (2023) Global Boundedness and Dynamics of a Diffusive Predator-Prey Model with Modified Leslie-Gower Functional Response and Density-Dependent Motion. *Communications in Nonlinear Science and Numerical Simulation*, **119**, Article 107115. https://doi.org/10.1016/j.cnsns.2023.107115

[9] Yang, W. (2013) Global Asymptotical Stability and Persistent Property for a Diffusive Predator-Prey System with Modified Leslie-Gower Functional Response. *Nonlinear Analysis: Real World Applications*, **14**, 1323-1330. https://doi.org/10.1016/j.nonrwa.2012.09.020

[10] Guo, S. (2023) Global Hopf Bifurcation of State-Dependent Delay Differential Equa-

tions. *International Journal of Bifurcation and Chaos*, **33**, Article 2350074. https://doi.org/10.1142/s0218127423500748

[11] An, Q., Gu, X. and Zhang, X. (2024) Normal Form and Hopf Bifurcation for the Memory-Based Reaction-Diffusion Equation with Nonlocal Effect. *Mathematical Methods in the Applied Sciences*, **47**, 12883-12904. https://doi.org/10.1002/mma.10185

[12] Hu, Z., Yang, J., Li, Q., Liang, S. and Fan, D. (2024) Mathematical Analysis of Stability and Hopf Bifurcation in a Delayed HIV Infection Model with Saturated Immune Response. *Mathematical Methods in the Applied Sciences*, **47**, 9834-9857. https://doi.org/10.1002/mma.10097

[13] Chen, X., Huang, C., Cao, J., Shi, X. and Luo, A. (2023) Hopf Bifurcation in the Delayed Fractional Leslie-Gower Model with Holling Type II Functional Response. *Journal of Applied Analysis & Computation*, **13**, 2555-2571. https://doi.org/10.11948/20220451

[14] Parwaliya, A., Singh, A., Kumar, A. and Barman, D. (2024) The Impact of Delays on Prey-Predator Dynamics with Predation-Induced Fear. *Journal of Applied Mathematics and Computing*, **70**, 4877-4907. https://doi.org/10.1007/s12190-024-02129-w

[15] Yang, R. and Wei, J. (2015) Bifurcation Analysis of a Diffusive Predator-Prey System with Nonconstant Death Rate and Holling III Functional Response. *Chaos*, *Solitons & Fractals*, **70**, 1-13. https://doi.org/10.1016/j.chaos.2014.10.011

[16] Yang, R. (2015) Hopf Bifurcation Analysis of a Delayed Diffusive Predator-Prey System with Nonconstant Death Rate. *Chaos*, *Solitons & Fractals*, **81**, 224-232. https://doi.org/10.1016/j.chaos.2015.09.021