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Bianchi Space-Time Metric-I in Landau and Lifshitz Energy Tensor, Including Linearly Varying Deceleration Parameter with Saez-Ballester Theory of Gravitation

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Abstract

In this study, the Bianchi Space Metric-I cosmological model is introduced within the context of the Saez-Ballester theory of gravitation, including the incorporation of scale factors in the framework of gravity. The objective is to derive an exact solution for the cosmological field equation, with consideration given to the Landau and Lifshitz energy tensor. This solution includes a metric potential comprising commoving vectors and an energy conservation equation. Within this framework, a set of relations between the deceleration parameter, Hubble parameter, and average scale factor is established. By adopting a probable set of relations, the influence of the dynamics of the deceleration parameter on energy density and isotropic pressure, particularly in exponential form, is explored. A key proposal involves the utilization of a linearly decelerating parameter alongside exponential scale factors. The results are presented graphically, offering insights into potential future cosmological models. These graphical representations are intended to facilitate the understanding of numerous physical and kinematical properties inherent in the cosmological model under investigation.

Keywords

Bianchi Space Metric-I, Saez-Ballester Theory,
Landu and Lifshitz Energy Tensor, Bulk Viscosity,
Hubble and Deceleration Parameter

1. Introduction

Bianchi Type-I universes are characterized by different scale factors, making them one of the most straightforward and elegant descriptions of anisotropic flat universes. Behavior akin to that of the Kansor universe is exhibited near the singularity. In the presence of matter, the initial anisotropy in a Bianchi Type-I universe is typically diminished rapidly, resulting in an evolution resembling that of a Friedman-Robertson-Walker (FRW) universe. These universes have been extensively explored from various perspectives due to their intriguing properties. Unlike FRW universes, directional scale factors are featured in Bianchi Type-I universes.

In the current stage of cosmic evolution, the universe is considered spherically symmetric, with scalar fields constituting the primary matter distribution, which is overall isotropic and homogeneous. Scalar fields, being among the simplest classical fields, have been extensively studied in the context of solving the Einstein equations with minimal coupling to the gravitational field. Spatially self-interacting scalar fields are recognized as playing a significant role in cosmological models within gravitational fields. At present, the universe is observed to be experiencing accelerated expansion, a phenomenon supported by numerous studies in astrophysics and cosmology based on observational evidence. The work of Ananda and Bruni [1] on the study of relativistic dynamics using Robertson-Walker models with a non-linear quadratic equation of state is regarded as noteworthy and is seen as opening new avenues for research.

It has been demonstrated by Tegmark [2] and other authors that the universe is highly homogeneous and isotropic, a conclusion supported by research conducted by Benet *et al.* [3] and Spergel *et al.* [4] through observations of the cosmic microwave background radiation. An inhomogeneous Hubble parameter term was utilized by Nojiri and Odintsov [5] to discuss variations in the dark energy content of the universe in relation to different equations of state. The proposal by Misner [6] [7] suggested that strong dissipation due to neutrino viscosity could significantly reduce the anisotropy of black body radiation, emphasizing the importance of bulk viscosity in studies of the early universe. Bulk viscosity is recognized to arise under various circumstances during the universe's evolution, leading to investigations by many researchers [8]-[11] of bulk viscous cosmological models within the framework of general relativity by numerous researchers.

The metric [12] field equations are

$$G_{ij} - w\varnothing^n\left(\varnothing_{,i}\varnothing_{,j} - \frac{1}{2}g_{ij}\varnothing_{,k}\varnothing^{,k}\right) = -8\pi T_{ij} \tag{1.1}$$

where $\varnothing$ satisfies the following conditions

$$2\varnothing^n\varnothing^i_{,i} + n\varnothing^{n-1}\varnothing^{,k}\varnothing_{,k} = 0 \tag{1.2}$$

where

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \tag{1.3}$$

The above Equation (1.3) is referred to as the Einstein tensor, while T_{ij} identified as the stress-energy tensor of the matter. The constants w and n are included. The comma (,) and semicolon (;) are used to denote partial and covariant differentiation, respectively. A detailed explanation of the Saez-Ballester cosmological model is formulated in the work of [13]-[15]. Based on this paper, the Bianchi Type-I cosmological model in the self-creation theory of gravitation is obtained as formulated by [16].

Furthermore, the organization of the paper is as follows: In Section 2, the metric and field equations are derived. In Section 3, the solution of the field equation in the presence of bulk viscous fluid and the Hubble parameter is addressed. Section 4 provides the solution for the cosmological model with a generalized linearly varying deceleration parameter. Section 5 presents the solution of field equations for energy density and anisotropic pressure. Section 6 focuses on the discussion of physical and kinematical properties. The concluding section provides a summary of the findings.

The metric potential and deceleration parameter are recognized as playing important and interesting roles in numerous areas, including bulk viscosity and the dynamics of different models in general relativity. In the study of anisotropic problems, particularly Bianchi Type-I homogeneous cosmological models containing perfect fluid, the use of the Hubble parameter is considered an intriguing concept that motivates further research into exploring new cosmological universes.

Based on the outlined framework, it appears that the paper will contribute to the understanding of cosmological models and their implications for the evolution and dynamics of the universe, particularly in scenarios involving anisotropic conditions and bulk viscosity effects. Both theoretical aspects, such as the derivation of field equations and the analysis of solutions, and their physical interpretations are explored, aiming to provide comprehensive insights into the subject matter.

This research inspired me by the profound contributions of two legendary scientists, Albert Einstein and Stephen Hawking. The theory of relativity, formulated by Albert Einstein, led to a revolution in the understanding of space, time, and gravity. Groundbreaking discoveries in the study of black holes and the origins of the universe were made by Stephen Hawking, who is widely regarded as one of the most brilliant theoretical physicists. His work on Hawking radiation and the Big Bang theory has been extensively studied. The pioneering research of both scientists has been used as a foundational resource for this study.

2. The Metric and Field Equations

We consider anisotropic Bianchi type-1 space time metric given by [17],

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + B^2(t)\mathrm{d}y^2 + C^2(t)\mathrm{d}z^2 \tag{2.1}$$

where A, B, C are the directional scale factors and are the functions of cosmic time t. The Bianchi type-1 space time becomes isotropic if the entire directional scale factor becomes equal and we get usual FRW space time. The stress energy tensor by Landu and Lifshitz [18] is as follows:

$$\begin{aligned} T_i^j &= (\rho+p)v^i v_j + pg_j^i - \eta\left(v_{;i}^j + v_{i;}^j + v^j v^l v_{i;l} + v_i v^l v_{;l}^j\right) \\ &-\left(\xi - \frac{2}{3}\eta\right)\theta\left(g_i^j + v_i v^j\right) \end{aligned} \tag{2.2}$$

ξ is the Bulk viscous coefficient, θ is the expansion scalar of the cosmological model, ρ is the energy density, p is the isotropic pressure and η is the coefficient of shear velocity.

In the commoving coordinates

$$v^4 = -1, v^1 = v^2 = v^3 = 0 \tag{2.3}$$

Also, energy conservation equation

$$T_{,j}^{ij} = 0 \tag{2.4}$$

In the commoving system, the field Equations (1.1), (1.2), (1.3) for the metric (2.1) with the help of Equations (2.2), (2.3) can be explicitly written as,

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{w}{2}\varnothing^n\dot{\varphi}^2 = 8\pi\left\{p - \left(\xi - \frac{2}{3}\eta\right)\theta\right\} \tag{2.5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{w}{2}\varnothing^n\dot{\varphi}^2 = 8\pi\left\{p - \left(\xi - \frac{2}{3}\eta\right)\theta\right\} \tag{2.6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{B}\dot{A}}{BA} - \frac{w}{2}\varnothing^n\dot{\varphi}^2 = 8\pi\left\{p - \left(\xi - \frac{2}{3}\eta\right)\theta\right\} \tag{2.7}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{AC} - \frac{w}{2}\varnothing^n\dot{\varphi}^2 = -8\pi\rho \tag{2.8}$$

From Equation (2.4) we can write,

$$T_{,j}^{ij} + \Gamma_{kj}^i T^{kj} + \Gamma_{kj}^j T^{ik} = 0$$

Hence,

$$\dot{\rho} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\left\{p - \left(\xi - \frac{2}{3}\eta\right)\theta\right\} = 0 \tag{2.9}$$

3. Solution of the Field Equations

To obtain the expression of Saez-Ballester scalar field $\varnothing$, we obtain the equation from (1.2)

$$\varnothing^n\varnothing_{,i}^i + \frac{n}{2}\varnothing^{n-1}\varnothing^{,k}\varnothing_{,k} = 0 \tag{3.1}$$

or,

$$\ddot{\varphi} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{\varphi} + \frac{n}{2}\frac{\dot{\varphi}^2}{\varphi} = 0$$

$$\therefore \frac{\ddot{\varphi}}{\dot{\varphi}}+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{n\dot{\varphi}}{2\varphi}=0$$

Hence,

$$\dot{\varphi}\left(ABC\right)\varphi^{\frac{n}{2}}=c \tag{3.2}$$

where, c is the integrating constant and we denote the average scale factor of the Bianchi-I universe by $a\left(t\right)$ which is given by

$$a\left(t\right)=\left(ABC\right)^{\frac{1}{3}}=V \tag{3.3}$$

or,

$$a^3=ABC$$

$$\therefore \frac{\dot{a}}{a}=\frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)$$

So,

$$H=\frac{\dot{a}}{a}=\frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=\frac{1}{3}\left(H_1+H_2+H_3\right)$$

$$\frac{\dot{V}}{V}=\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}$$

where, $\left(H_1=\frac{\dot{A}}{A}, H_2=\frac{\dot{B}}{B}, H_3=\frac{\dot{C}}{C}\right)$ are the Hubble Parameters in different and usual direction by [5].

So, Equation (3.2) can be written as,

$$\dot{\varphi}a^3\left(\varphi\right)^{\frac{n}{2}}=c \tag{3.4}$$

where, c is the integrating constant.

In terms of the Hubble parameter in the axial direction Equations (2.5)-(2.8) can be expresses by [19],

$$H_1H_2+H_2H_3+H_3H_1-\frac{w}{2}\varnothing^n\dot{\varphi}^2=8\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\} \tag{3.5}$$

$$\dot{H}_2+\dot{H}_3+H_2^2+H_3^2+H_2H_3-\frac{w}{2}\varnothing^n\dot{\varphi}^2=-8\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\} \tag{3.6}$$

$$\dot{H}_1+\dot{H}_3+H_3^2+H_1^2+H_1H_3-\frac{w}{2}\varnothing^n\dot{\varphi}^2=-8\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\} \tag{3.7}$$

$$\dot{H}_1+\dot{H}_2+H_2^2+H_1^2+H_2H_1-\frac{w}{2}\varnothing^n\dot{\varphi}^2=-8\pi\rho \tag{3.8}$$

Subtract Equation (2.5) from (2.6)

$$\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{C}}{AC}-\frac{\ddot{B}}{B}-\frac{\dot{B}\dot{C}}{BC}=0 \tag{3.9}$$

Again, subtract Equation (2.6) from (2.7)

$$\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{C}}{C}-\frac{\dot{A}\dot{C}}{AC}=0 \tag{3.10}$$

Now by adding Equation (3.9) and (3.10)

$$\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{B}}{AB}-\frac{\ddot{C}}{C}-\frac{\dot{B}\dot{C}}{BC}=0 \tag{3.11}$$

By integrating,

$$B\dot{A}-A\dot{B}=\frac{d_1}{C},$$

where d_1 is the integrating constant.

$$\therefore \frac{\dot{A}}{A}-\frac{\dot{C}}{C}=\frac{d_1}{V},$$

where $V=ABC$.

Again, integrating both sides

$$\frac{A}{B}=d_1\int\frac{\mathrm{d}t}{V}+d_2$$

$$A=B\left[x_1\int\frac{\mathrm{d}t}{V}+d_1\right] \tag{3.12}$$

Similarly,

$$B=C\left[x_2\int\frac{\mathrm{d}t}{V}+d_2\right] \tag{3.13}$$

And,

$$A=C\left[x_3\int\frac{\mathrm{d}t}{V}+d_3\right] \tag{3.14}$$

where, d_1,d_2,d_3,x_1,x_2,x_3 are the integrating constant.

4. Solution of Cosmological Model with Deceleration Parameter

In order to solve the system of Equations (3.2), (3.3), (3.12), (3.13), (4.2) completely, we use a generalized linearly varying deceleration parameter q by [20] defined as,

$$q=-\frac{a\dot{a}}{a^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right)-1=kt+m-1 \tag{4.1}$$

where $k\geq 0, m\geq 0$ are constants. The universe exhibits decelerating expansion if $q>0$, expand with constant rate if $q=0$, accelerating power law expansion if $-1<q<0$ and exponential expansion if $q=-1$.

Now for $k=0$ and $m=0$, we obtain

$$-\frac{a\dot{a}}{a^2}=-1$$

So

$$a = a_1 e^{l_1 t} \tag{4.2}$$

$$V = a^3 = a_1^3 e^{3l_1 t} \tag{4.3}$$

From Equation (3.12) we can express,

$$A^2 = AB\left[x_1 \int \frac{dt}{V} + d_1\right]$$

or,

$$C^2\left[x_3 \int \frac{dt}{V} + d_3\right]^2 = AB\left[x_1 \int \frac{dt}{V} + d_1\right]$$

or,

$$C^3\left[x_3 \int \frac{dt}{V} + d_3\right]^2 = V\left[x_1 \int \frac{dt}{V} + d_1\right]$$

or,

$$C^3\left[d_3 - \frac{x_3 a_1^{-3}}{3l_1} e^{-3l_1 t}\right]^2 = a_1^3 e^{3l_1 t}\left[d_1 - \frac{x_1 a_1^{-3}}{3l_1} e^{-3l_1 t}\right]$$

or,

$$C^3\left[d_3 + \frac{k}{V}\right]^2 = V\left[d_1 + \frac{k_1}{V}\right]$$

$$\therefore C = \left[\frac{V^2\left(Vd_1 + k_1\right)}{\left(Vd_3 + k\right)^2}\right]^{\frac{1}{3}} \tag{4.4}$$

The relationship between C and V is illustrated in Figure 1.

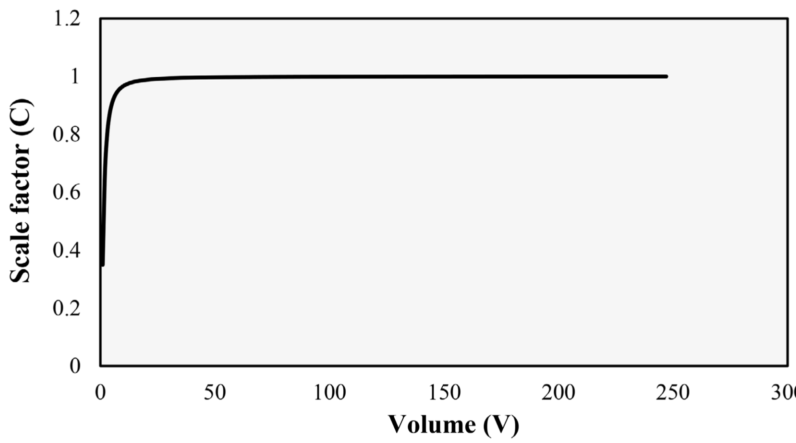

Figure 1. Phase plot for the and Scale Factor (C) and Volume (V) of equation (4.4), where $d_1 = 3$, $k_1 = 5$, $d_3 = 3$, $k = 5$.

Similarly, by using (3.14) we can write,

$$A = \left[\frac{V^2\left(Vd_1 + k_1\right)}{\left(Vd_3 + k\right)^2}\right]^{\frac{1}{3}}\left[d_3 - \frac{x_3 a_1^{-3}}{3l_1} e^{-3l_1 t}\right]$$

Hence,

$$A=\left[\frac{V^2\left(Vd_1+k_1\right)}{\left(Vd_3+k\right)^2}\right]^{\frac{1}{3}}\left[d_3+\frac{K}{V}\right]$$

$$\therefore A=\left[\frac{Vd_1+k_1}{V\left(Vd_3+k\right)^2}\right]^{\frac{1}{3}}\left[k+Vd_3\right] \tag{4.5}$$

The relationship between A and V is illustrated in Figure 2.

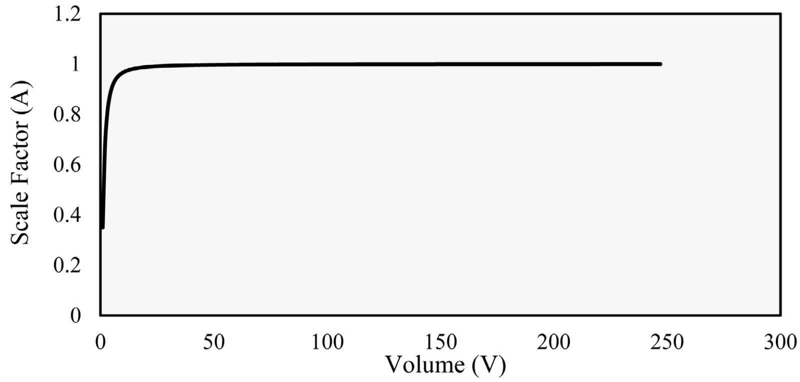

Figure 2. Phase plot for the Scale Factor (A) and Volume (V) of Equation (4.5), where d_1 = 0.01, k_1 = 0.2, d_3 = 0.03, k = 0.4.

Now by using (3.13) we can get,

$$B=\left[\frac{V^2\left(Vd_1+k_1\right)}{\left(Vd_3+k\right)^2}\right]^{\frac{1}{3}}\left[x_2\int\frac{\mathrm{d}t}{V}+d_2\right]$$

$$\therefore B=\left[\frac{Vd_1+k_1}{V\left(Vd_3+k\right)^2}\right]^{\frac{1}{3}}\left[Vd_2+k_2\right] \tag{4.6}$$

where, $k=-\frac{x_3}{3I_1}$, $k_1=-\frac{x_1}{3I_1}$ and $k_2=-\frac{x_2}{3I_1}$ all are constants. The relationship between B and V is illustrated in Figure 3.

After a suitable transformation of coordinates, the metric (2.1) reduces to

$$\begin{aligned}\mathrm{d}s^2=&-\mathrm{d}t^2+\left[\frac{Vd_1+k_1}{V\left(Vd_3+k\right)^2}\right]^{\frac{2}{3}}\left[k+Vd_3\right]^2\mathrm{d}x^2\\&+\left[\frac{Vd_1+k_1}{V\left(Vd_3+k\right)^2}\right]^{2}\left[Vd_2+k_2\right]^2\mathrm{d}y^2+\left[\frac{V^2\left(Vd_1+k_1\right)}{\left(Vd_3+k\right)^2}\right]^{\frac{2}{3}}\mathrm{d}z^2\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathrm{d}s^2=&-V^{\frac{2}{3}}\left(Vd_3+k\right)^{\frac{4}{3}}\mathrm{d}t^2+\left[Vd_1+k_1\right]^{\frac{2}{3}}\Big[\left(k+Vd_3\right)^2\mathrm{d}x^2\\&+\left(k_2+Vd_2\right)^2\mathrm{d}y^2+V^2\mathrm{d}z^2\Big]\end{aligned} \tag{4.8}$$

From Equation (4.3) we can write,

$$\frac{\dot{V}}{V}=3I_1 \quad \text{and} \quad \frac{\ddot{V}}{V}=9I_1^2 \tag{4.9}$$

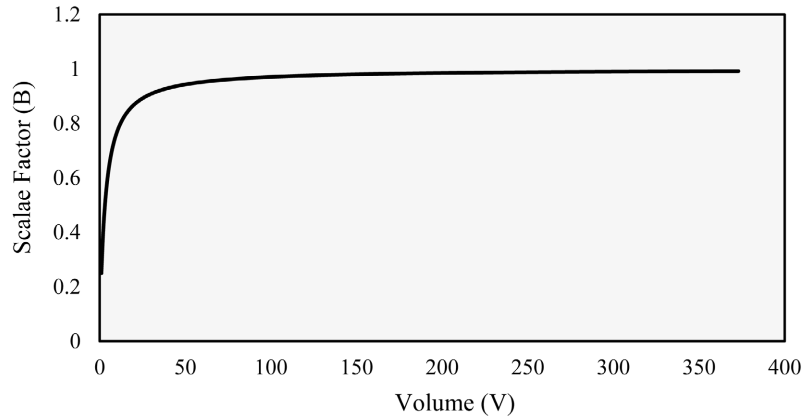

Figure 3. Phase plot for the Scale Factor (B) and Volume (V) of equation (4.6), where d_1 = 2, k_1 = 0.3, d_3 = 0.4, k = 0.5, d_2 = 0.62, k_2 = 0.6.

5. Solution of Cosmological Model for Energy Density and Scalar Field

By adding Equations (2.5)-(2.8) we can write

$$2\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right)+\left(\frac{\ddot{B}}{B}+\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}\right)-2w\varphi^n\dot{\varphi}^2$$
$$=-8\pi\rho+24\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\}$$

Now by using Equation (2.8) the above equation becomes, or,

$$\left(\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{C}\dot{A}}{CA}\right)+2\left(\frac{\ddot{B}}{B}+\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}\right)-\frac{3}{2}w\varphi^n\dot{\varphi}^2=24\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\} \tag{5.1}$$

with the help of Equation (3.3) the above Equation (5.1) can be written as,

$$\frac{\ddot{V}}{V}=\frac{3}{2}w\varphi^n\dot{\varphi}^2+24\pi\left\{p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\} \tag{5.2}$$

Now Equation (2.9) can be expresses as

$$\dot{\rho}+\frac{\dot{V}}{V}\left\{\rho+p-\left(\xi-\frac{2}{3}\eta\right)\theta\right\}=0$$

$$\therefore p-\left(\xi-\frac{2}{3}\eta\right)\theta=-\left(\rho+\frac{\dot{\rho}V}{\dot{V}}\right) \tag{5.3}$$

Equations (5.1) and (5.3) can be written as,

$$\frac{\ddot{V}}{V}=\frac{3}{2}w\varphi^n\dot{\varphi}^2-24\pi\left(\rho+\frac{\dot{\rho}V}{\dot{V}}\right) \tag{5.4}$$

After simplifying Equation (5.4) it can be expressed as by the help of equation

(3.2) and (3.3)

$$2\frac{\ddot{V}}{V}+48\pi\rho+48\pi\dot{\rho}\frac{V}{\dot{V}}=\frac{3wc^2}{V^2} \tag{5.5}$$

By applying Equation (4.3), Equation (5.2) transforms into a new form with the assistance of Equation (4.10).

$$9I_1^2=\frac{3wc^2}{2a_1^9\mathrm{e}^{9I_1t}}-\frac{8\pi\dot{\rho}}{I_1}$$

Hence,

$$\dot{\rho}=\frac{3wc^2I_1}{16\pi a_1^9\mathrm{e}^{9I_1t}}-\frac{9I_1^3}{8\pi} \tag{5.6}$$

After integrating,

$$\rho=\frac{\beta_1}{\mathrm{e}^{9I_1t}}+\beta_2t+\beta \tag{5.7}$$

where, $\beta_1=\frac{-3wc^2I_1}{16\pi a_1^9}$, $\beta_2=\frac{-9I_1^3}{8\pi}$, β all are constants.

So, Equation (5.5) can be rewritten using Equation (4.10) as follows:

$$\frac{9wc^2I_1}{a_1^9\mathrm{e}^{9I_1t}}=48\pi\beta-54I_1^3t \tag{5.8}$$

$$\therefore w=\left(tk_3+k_4\right)\mathrm{e}^{9I_1t} \tag{5.9}$$

where, $k_3=\frac{48\pi\beta a_1^9}{9c^2}, k_4=-\frac{6I_1a_1^9}{c^2}$ all are constants.

Equation (3.2) can be written as by the help of Equation (4.3)

$$\varphi^{\frac{n}{2}}\mathrm{d}\varphi=-\frac{c\mathrm{e}^{-3I_1t}}{a_1^3}\mathrm{d}t \tag{5.10}$$

After integrating both sides,

$$\varphi^{\frac{n+4}{2}}=\frac{n+4}{2}\left(k_5-\frac{c\mathrm{e}^{-3I_1t}}{3I_1a_1^3}\right) \tag{5.11}$$

$$\therefore\varphi=\left\{\frac{n+4}{2}\left(k_5-\frac{c\mathrm{e}^{-3I_1t}}{3I_1a_1^3}\right)\right\}^{\frac{2}{n+4}} \tag{5.12}$$

The relationship between φ and t is illustrated in **Figure 4**.

6. Physical and Kinematical Properties

Shear Scalar

$$\sigma^2=\frac{1}{2}\sigma_{ij}\sigma^{ij}=\frac{1}{2}\sum_{i=1}^{3}\left(H_i-H\right)^2=\frac{1}{2}\left(H_1^2+H_2^2+H_3^2-3H^2\right)=k_6+k_7\mathrm{e}^{-9I_1t} \tag{6.1}$$

where, $k_6=27I_1^2+8\pi\beta$ and $k_7=8\pi\beta_1-\frac{wc}{2a_1^9}$ all are constant.

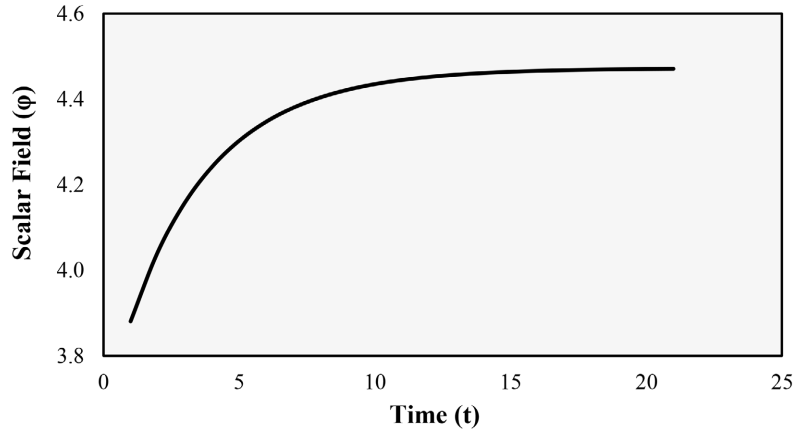

Figure 4. Phase plot for the scalar Field ($\varnothing$) and cosmic time (t) for Equation (5.12) where, $n = 2$, $k_5 = 10$, $c = 0.2$, $I_1 = 0.1$, $a_1 = 0.2$.

Scalar Expansion,

$$\theta = 3I_1 \tag{6.2}$$

Spatial Volume,

$$V = \sqrt{-g} = M^{\frac{5}{3}}\left(k + Vd_3\right)\left(k_2 + Vd_2\right)V \tag{6.3}$$

Hubble Parameter,

$$H = I_1 \tag{6.4}$$

7. Discussion

In this paper we summarize our findings throughout Sections 1 to 6. By using energy momentum tensor with Bulk Viscosity, energy conservation equation and commoving vector we get new cosmological field equation. Meanwhile, we use Hubble parameter, so new special solution of field equation is created. In future, this research work will help to investigate more realistic cosmology. It may explain the phase transition of the universe [21] more effectively. We have graphically depicted the cosmological scale factor represented by (A, B, C, φ) with respect to time (t). Moreover, we have illustrated the variations of energy density (ρ), Hubble parameter (H) and isotropic pressure (P) with respect to time (t).

In this paper, the findings from Sections 1 to 6 are summarized. By employing the energy-momentum tensor with bulk viscosity, the energy conservation equation, and a commoving vector, a new cosmological field equation is obtained. Additionally, the use of the Hubble parameter results in the creation of a new special solution to the field equation. In the future, this research work is expected to assist in investigating more realistic cosmological models and may provide a more effective explanation of the phase transition of the universe [21]. The cosmological scale factor, represented by (A, B, C, ϕ) has been graphically depicted with respect to time (t). Furthermore, variations in energy density (ρ), the Hubble parameter (H), and isotropic pressure (P) with respect to time (t) have been illustrated.

8. Conclusion

The scale factors *A*, *B*, *C* are singular at $t = 0$. As $t \to \infty$ so, $A \to \infty$, $B \to \infty$, $C \to \infty$, $V \to \infty$. Over time, the scale factors *A*, *B* and *C* increase, and therefore, we can conclude that the universe is expanding. Both the energy density ρ and Saez-Ballester scalar field φ exhibit singularities at $t = 0$. Moreover, as $t \to \infty$ so, $\rho \to \infty$, $\varphi \to \infty$. Furthermore, we have seen that Hubble parameter (H) and scalar expansion (θ) are constant.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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