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A Generalization of $(\mathcal{F},\mathcal{A})$-Gorenstein Flat Modules

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Abstract

Inspired by Bouchiba's work on generalized Gorenstein flat modules, we introduce the generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat modules in this paper. By defining generalized $\mathcal{A}$-complete flat resolution, a new measure of the dimension of generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat modules is given. When $({}^{\perp_1}\mathcal{A},\mathcal{A})$ is a complete cotorsion pair, R is a right coherent ring and $\mathcal{A}$ contains injective right R-modules, it is proved that for each M with finite flat dimension, the generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat dimension of M is equal to the $(\mathcal{F},\mathcal{A})$-Gorenstein flat dimension of M.

Keywords

$(\mathcal{F},\mathcal{A})$-Gorenstein Flat Modules, Generalized $\mathcal{A}$-Complete Flat Resolution, Generalized $(\mathcal{F},\mathcal{A})$-Gorenstein Flat Modules

1. Introduction

Enochs and his collaborators introduced Gorenstein projective, injective and flat modules and developed homological algebras in [1] [2]. In recent 20 years, Gorenstein algebra has been favored by many researchers. Bennis and Mahdou in [3] defined strongly Gorenstein projective, injective, flat modules. Meng and Pan in [4] introduced the concepts of $\mathcal{Y}$-Gorenstein injective right R-module and $\mathcal{Y}$-Gorenstein flat left R-module, proved that some results related to Gorenstein injective modules and flat modules are still true for them. Wang, Yang and Zhu in [5] defined Gorenstein $(\mathcal{X},\mathcal{Y})$-flat modules with respect to a duality pair $(\mathcal{X},\mathcal{Y})$. Holm in [6] gave homological descriptions of the Gorenstein dimensions over associative rings. Bouchiba introduced in [7] the notion of generalized Gorenstein flat modules as a generalization of the Gorenstein flat module. Inspired by the work of Bouchiba, this paper will introduce and study the concept of $(\mathcal{F},\mathcal{A})$-Gorenstein flat modules and its generalization.

This paper is organized as follows. In Section 2, we introduce the notations that will be used in this paper. In Section 3, we focus on $(\mathcal{F},\mathcal{A})$-Gorenstein flat modules. The definitions of generalized $\mathcal{A}$-complete flat resolution and generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat modules and their dimensions are introduced. It is proved that $\mathcal{GF}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GGF}_{(\mathcal{F},\mathcal{A})}$, and when $\mathcal{A}$ contains injective right R-module, it follows that $\mathcal{GGF}^{+}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$, if $({}^{\perp_1}\mathcal{A},\mathcal{A})$ is a complete cotorsion pair and R is a right coherent ring at this point, then the equality $\mathrm{cfd}_{\mathcal{A}}(M) = \mathrm{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^{+}) = \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{fd}_R(M)$ holds for each M with finite flat dimension.

2. Preliminaries

In this section, we will introduce some terms and review some concepts related to this paper.

Throughout this paper, R is an associative ring with identity. All modules, if not otherwise specified, are assumed to be left R-modules. We use $\mathrm{Mod}(R)$ to denote the class of left R-modules, while for a right R-module we will say R^{op}-module and we denote the category by $\mathrm{Mod}(R^{\mathrm{op}})$. In addition, we assume $\mathcal{A} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$. For short, we will use $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ (resp., $\mathcal{P}(R^{\mathrm{op}})$, $\mathcal{I}(R^{\mathrm{op}})$ and $\mathcal{F}(R^{\mathrm{op}})$) to denote the projective, injective and flat left R-modules (resp., right R-modules). Let $M \in \mathrm{Mod}(R)$, we denote by $\mathrm{pd}_R(M)$ (resp., $\mathrm{id}_R(M)$, $\mathrm{fd}_R(M)$) the projective (resp., injective, flat) dimension of M. For any $M \in \mathrm{Mod}(R)$, the character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is denote by M^{+}, where $\mathbb{Z}$ is the additive group of integers and $\mathbb{Q}$ is the additive group of rational numbers.

Orthogonal classes. Let $\mathcal{X}$ be a class of modules in $\mathrm{Mod}(R)$ and for each positive integer i, we define the *left orthogonal* classes by

$$ {}^{\perp_i}\mathcal{X} := \left\{ M \in \mathrm{Mod}(R) : \mathrm{Ext}^i_R(M,-)\big|_{\mathcal{X}} = 0 \right\} \text{ and } {}^{\perp}\mathcal{X} := \bigcap\nolimits_{i>0} {}^{\perp_i}\mathcal{X}. $$

Dually, we have the *right orthogonal* classes $\mathcal{X}^{\perp_i}$ and $\mathcal{X}^{\perp}$.

Given $\mathcal{Y} \subseteq \mathrm{Mod}(R)$, if $\mathrm{Ext}^i_R(X,Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $i > 0$, we write $\mathcal{X} \perp \mathcal{Y}$.

Projectively and injectively resolving. Following [6], we define the following terms for any class $\mathcal{X}$ of $\mathrm{Mod}(R)$:

a) We call $\mathcal{X}$ *projectively resolving* if $\mathcal{P}(R) \in \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

b) We call $\mathcal{X}$ *injectively resolving* if $\mathcal{I}(R) \in \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and

$X'' \in \mathcal{X}$ are equivalent.

Resolution and coresolution dimensions. Following [8], let $\mathcal{X} \subseteq \mathrm{Mod}(R)$ and $M \in \mathrm{Mod}(R)$. The $\mathcal{X}$-*resolution dimension* of M denoted by $\mathrm{resdim}_{\mathcal{X}}(M)$ is the smallest non-negative integer n such that there is an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ with $X_i \in \mathcal{X}$ and $0 \le i \le n$. If such n does not exist, we set $\mathrm{resdim}_{\mathcal{X}}(M) = \infty$. We denote by $\mathcal{X}^{\wedge}$ the class of modules in $\mathrm{Mod}(R)$ with finite $\mathcal{X}$-resolution dimensions. Dually, we have the $\mathcal{X}$-coresolution dimension $\mathrm{coresdim}_{\mathcal{X}}(M)$ of M, and the class $\mathcal{X}^{\vee}$ of modules in $\mathrm{Mod}(R)$ with finite $\mathcal{X}$-coresolution dimensions.

Given $\mathcal{Y} \subseteq \mathrm{Mod}(R)$, we set $\mathrm{resdim}_{\mathcal{X}}(\mathcal{Y}) =: \sup\{\mathrm{resdim}_{\mathcal{X}}(Y)\}$ for all $Y \in \mathcal{Y}$, and $\mathrm{coresdim}_{\mathcal{X}}(\mathcal{Y})$ is defined dually.

Cotorsion pair. Following [8], let $\mathcal{X} \subseteq \mathrm{Mod}(R)$ and $\mathcal{Y} \subseteq \mathrm{Mod}(R)$. The pair $(\mathcal{X}, \mathcal{Y})$ is a *left cotorsion pair* if $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$. Dually, we have a *right cotorsion pair* if $\mathcal{X}^{\perp_1} = \mathcal{Y}$. We say that $(\mathcal{X}, \mathcal{Y})$ is a *cotorsion pair* if it is both a left and right cotorsion pair.

Complete pair. Following [8], let $\mathcal{X} \subseteq \mathrm{Mod}(R)$ and $\mathcal{Y} \subseteq \mathrm{Mod}(R)$. A pair $(\mathcal{X}, \mathcal{Y})$ is called *left complete* if any $M \in \mathrm{Mod}(R)$, then there is an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Dually, we have *right complete* if any $M \in \mathrm{Mod}(R)$, then there is an exact sequence $0 \to M \to Y \to X \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. The pair is called *complete* if it is left and right complete.

Definition 1. ([9], Definition 2.1) A duality pair over R is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X} \subseteq \mathrm{Mod}(R)$ and $\mathcal{Y} \subseteq \mathrm{Mod}(R^{\mathrm{op}})$, satisfying:

1) $X \in \mathcal{X}$ if and only if $X^{+} \in \mathcal{Y}$.

2) $\mathcal{Y}$ is closed under direct summands and direct finite sums.

A duality pair $(\mathcal{X}, \mathcal{Y})$ is called *perfect* if $\mathcal{X}$ contains the module R, and is closed under direct sums and extensions.

3. Generalized $(\mathcal{F}, \mathcal{A})$-Gorenstein Flat Module

In [7] [10] Bouchiba introduced generalized Gorenstein projective(respectivly, flat) modules and defined their dimensions. The main purpose of this section is to characterize the dimension of $(\mathcal{F}, \mathcal{A})$-Gorenstein flat module by generalized $(\mathcal{F}, \mathcal{A})$-Gorenstein flat module.

Definition 2. ([11], Definition 1.1) Assume $(\mathcal{L}, \mathcal{A})$ is a complete duality pair. Let $M \in \mathrm{Mod}(R)$ and $N \in \mathrm{Mod}(R^{\mathrm{op}})$.

1) M is called $(\mathcal{L}, \mathcal{A})$-*Gorenstein projective* if $M = Z_0(\mathbf{P})$ for some exact complex of projective R-modules $\mathbf{P}$ for which $\mathrm{Hom}_R(\mathbf{P}, L)$ is acyclic for all $L \in \mathcal{L}$.

2) N is called $(\mathcal{L}, \mathcal{A})$-*Gorenstein injective* if $N = Z_0(\mathbf{I})$ for some exact complex of injective R^{op}-modules $\mathbf{I}$ for which $\mathrm{Hom}_{R^{\mathrm{op}}}(A, \mathbf{I})$ is acyclic for all $A \in \mathcal{A}$.

3) M is called $(\mathcal{L}, \mathcal{A})$-*Gorenstein flat* if $M = Z_0(\mathbf{F})$ for some exact complex of flat R-modules $\mathbf{F}$ for which $\mathbf{F} \otimes_R A$ is acyclic for all $A \in \mathcal{A}$.

We use $\mathcal{GP}_{(\mathcal{P},\mathcal{L})}$ to denote the $(\mathcal{L},\mathcal{A})$-Gorenstein projective left modules, $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ to denote the $(\mathcal{L},\mathcal{A})$-Gorenstein flat left modules, and $\mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$ to denote the $(\mathcal{L},\mathcal{A})$-Gorenstein injective right modules.

Lemma 1. Let $\mathcal{A} \subseteq \mathrm{Mod}\left(R^{\mathrm{op}}\right)$ such that $\mathcal{I}\left(R^{\mathrm{op}}\right) \subseteq \mathcal{A}$. If $0 \to N \to G_0 \to G_1 \to 0$ is an exact sequence with $G_0, G_1 \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$ and $\mathrm{Ext}_R^1(\mathcal{A}, N) = 0$, then $N \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$.

Proof. Consider the exact sequence $0 \to N \to G_0 \to G_1 \to 0$ with $G_0, G_1 \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$ and $\mathrm{Ext}_R^1(\mathcal{A}, N) = 0$. Since $G_1 \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$, there is an exact sequence $0 \to K \to I \to G_1 \to$ with $I \in \mathcal{I}\left(R^{\mathrm{op}}\right)$ and $K \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$. Consider the following pull back diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & G_0 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

From the exact middle column $0 \to K \to H \to G_0 \to 0$ and ([4], Proposition 2.10), we know that $H \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$. Since $I \in \mathcal{I}\left(R^{\mathrm{op}}\right) \subseteq \mathcal{A}$, we have $\mathrm{Ext}_R^1(I, N) = 0$, therefore the exact sequence $0 \to N \to H \to I \to 0$ splits and N is a direct summand of $\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)$-Gorenstein injective module. Finally, we have $N \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}$.

□

For any $M \in \mathrm{Mod}(R)$, the $(\mathcal{F},\mathcal{A})$-*Gorenstein flat dimension* of M is defined as $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M) := \mathrm{resdim}_{\mathcal{GF}_{(\mathcal{F},\mathcal{A})}}(M)$. For any class $\mathcal{Z} \subseteq \mathcal{A}$, $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(\mathcal{Z}) := \sup\left\{\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(Z) : Z \in \mathcal{Z}\right\}$. Meanwhile, fon any $N \in \mathrm{Mod}\left(R^{\mathrm{op}}\right)$ the $\left(\mathcal{A},\mathcal{I}\left(R^{op}\right)\right)$-*Gorenstein injective dimension* of N is defined as $\mathrm{Gid}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}(N) := \mathrm{coresdim}_{\mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}}(N)$. For any class $W \subseteq \mathcal{A}$,

$$\mathrm{Gid}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}(\mathcal{W}) := \sup\left\{\mathrm{Gid}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}(W) : W \in \mathcal{W}\right\}.$$

Proposition 2. Let $\mathcal{A} \subseteq \mathrm{Mod}\left(R^{\mathrm{op}}\right)$ such that $\mathcal{I}\left(R^{\mathrm{op}}\right) \subseteq \mathcal{A}$ and

$N \in \operatorname{Mod}(R^{\mathrm{op}})$, then the following statements are equivalent for a non-negative integer n:

1) $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(N) \le n$.

2) $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(N) < \infty$ and $\operatorname{Ext}_R^i(A,N)=0$ for all $i>n$ and all $A \in \mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$.

3) $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(N) < \infty$ and $\operatorname{Ext}_R^i(L,N)=0$ for all $i>n$ and all $L \in \mathcal{A}^{\vee} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$.

Proof. (1) $\Rightarrow$ (2) We proceed by induction on n. By definition, it is clear that (2) is true for $n=0$. Suppose that $n \ge 1$. So there exists an exact sequence $0 \to N \to G \to M \to 0$ with $G \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$ and $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M) = n-1$. Then, for all $A \in \mathcal{A}$, $\operatorname{Ext}_R^i(A,G)=0$ for all $i>0$ and $\operatorname{Ext}_R^i(A,M)=0$ for all $i>n-1$ (by induction). By the long exact sequence theorem, we have $\operatorname{Ext}_R^i(A,M) \to \operatorname{Ext}_R^{i+1}(A,N) \to \operatorname{Ext}_R^{i+1}(A,G)$, then $\operatorname{Ext}_R^{i+1}(A,N)=0$ for all $i>n-1$, as desired.

(2) $\Rightarrow$ (3) It follows by standard arguments.

(3) $\Rightarrow$ (1) By hypothesis, let $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(N) = m < \infty$, if $m<n$, there is nothing to prove. So we assume $m>n$. By ([4], Proposition 2.15(4)), we have an exact sequence $0 \to N \to G_0 \to G_1 \to \cdots \to G_m \to 0$ with each $G_i \in \mathcal{I}(R)$ for $0 \le i \le m-1$ and $G_m \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$. Let $K_n = \operatorname{Im}(G_{n-1} \to G_n)$. We truncate the sequence $0 \to K_n \to G_n \to \cdots \to G_m \to 0$ in short sequences $0 \to H_i \to G_i \to H_{i+1} \to 0$ for $m-1 \le i \le n$, where $H_n = K_n$ and $H_m = G_m$. We consider the exact sequence $0 \to H_{m-1} \to G_{m-1} \to H_m (= G_m) \to 0$. We claim that $H_{m-1} \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$. By the exact sequence $0 \to N \to G_0 \to \cdots \to H_{m-1} \to 0$, we have for all $L \in \mathcal{A}^{\vee} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ and $i>0$, we get the isomorphism $\operatorname{Ext}_R^i(L,H_{m-1}) = \operatorname{Ext}_R^{i+m}(L,N)=0$. Therefore, for H_{m-1} there is an exact sequence $0 \to H_{m-1} \to G_{m-1} \to G_m \to 0$ and $\operatorname{Ext}_R^i(L,H_{m-1})=0$ for all $i>0$. Hence, by Lemma 1, $H_{m-1} \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$. Finally, we repeat this argument to conclude that $H_{m-2}, \cdots, H_n = K_n \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$.

□

Definition 3. Let $M \in \operatorname{Mod}(R)$.

1) M is called *copure* $\mathcal{A}$*-flat module* (resp., *copure* $\mathcal{A}$*-injective module*) if $\operatorname{Tor}_1^R(A,M)=0$ (resp., $\operatorname{Ext}_R^1(A,M)=0$) for any $A \in \mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ (resp., $A \in \mathcal{A} \subseteq \operatorname{Mod}(R)$).

2) M is called *strongly copure* $\mathcal{A}$*-flat module* (resp., *strongly copure* $\mathcal{A}$*-injective module*) if $\operatorname{Tor}_n^R(A,M)=0$ (resp., $\operatorname{Ext}_R^n(A,M)=0$) for any $A \in \mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ (resp., $A \in \mathcal{A} \subseteq \operatorname{Mod}(R)$) and any integer $n \ge 1$.

3) The *copure* $\mathcal{A}$*-flat dimension of* M and *copure* $\mathcal{A}$*-injective dimension of* M is defined as:

$$\operatorname{cfd}_{\mathcal{A}}(M) := \sup\left\{n \in \mathbb{N} : \operatorname{Tor}_n^R(A, M) \neq 0 \mid \exists A \in \mathcal{A} \subseteq \operatorname{Mod}\left(R^{\mathrm{op}}\right)\right\}$$

$$\operatorname{cid}_{\mathcal{A}}(M) := \sup\left\{n \in \mathbb{N} : \operatorname{Ext}_R^n(A, M) \neq 0 \mid \exists A \in \mathcal{A} \subseteq \operatorname{Mod}(R)\right\}.$$

The above definitions generalized that of *copure injective module* and *copure flat module* mentioned by Enochs and Jenda in [12].

Proposition 3. Let $\mathcal{A} \subseteq \operatorname{Mod}\left(R^{\mathrm{op}}\right)$ and $n \geq 1$ be an integer.

1) The following assertions are equivalent for any $M \in \operatorname{Mod}(R)$:

a) $\operatorname{cfd}_{\mathcal{A}}(M) \leq n$.

b) For each exact sequence $0 \to K \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$ such that the E_i are strongly copure $\mathcal{A}$-flat modules, K is strongly copure $\mathcal{A}$-flat.

c) For each exact sequence $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that the F_i are flat modules, K is strongly copure $\mathcal{A}$-flat.

2) Let $0 \to N \to E \to M \to 0$ be an exact sequence of R-modules.

a) If E is strongly copure $\mathcal{A}$-flat and $\operatorname{cfd}_{\mathcal{A}}(M) \geq 1$, then $\operatorname{cfd}_{\mathcal{A}}(M) = 1 + \operatorname{cfd}_{\mathcal{A}}(N)$.

b) $\operatorname{cfd}_{\mathcal{A}}(M) \leq 1 + \max\left\{\operatorname{cfd}_{\mathcal{A}}(E), \operatorname{cfd}_{\mathcal{A}}(N)\right\}$.

c) $\operatorname{cfd}_{\mathcal{A}}(E) \leq \max\left\{\operatorname{cfd}_{\mathcal{A}}(M), \operatorname{cfd}_{\mathcal{A}}(N)\right\}$.

3) Let $\cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots$ be an exact sequence of R-modules with $M_i := \operatorname{Im}(f_i)$ for each integer i. Then

$$\sup\left\{\operatorname{cfd}_{\mathcal{A}}(E_i) : i \in \mathbb{Z}\right\} \leq \sup\left\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\right\}$$

with equality if $\sup\left\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\right\}$ is finite.

Proof. By dimension shifting, it is easy to prove 1.

2) a) Assume that $\operatorname{cfd}_{\mathcal{A}}(N) = n \geq 0$ is finite. Then there exists $A \in \mathcal{A}$ such that $\operatorname{Tor}_n^R(A, N) \neq 0$, applying the long exact sequence theorem to the exact sequence $0 \to N \to E \to M \to 0$, we get that

$$\begin{aligned} &\cdots \to \operatorname{Tor}_i^R(A, E) \to \operatorname{Tor}_i^R(A, M) \to \operatorname{Tor}_{i-1}^R(A, N) \to \\ &\operatorname{Tor}_{i-1}^R(A, E) \to \cdots \to \operatorname{Tor}_{n+1}^R(A, M) \to \cdots \end{aligned}$$

for $i \geq n+2$. Since $\operatorname{cfd}_{\mathcal{A}}(M) \geq 1$, we have $\operatorname{Tor}_{n+1}^R(A, M) \neq 0$. Since E is strongly copure $\mathcal{A}$-flat, $\operatorname{Tor}_i^R(A, M) = 0$ for $i \geq n+2$. Consequently, $\operatorname{cfd}_{\mathcal{A}}(M) = 1 + n = 1 + \operatorname{cfd}_{\mathcal{A}}(N)$.

b) Suppose that $\operatorname{cfd}_{\mathcal{A}}(E) = r$, $\operatorname{cfd}_{\mathcal{A}}(N) = n$ are finite and $\operatorname{cfd}_{\mathcal{A}}(E) > \operatorname{cfd}_{\mathcal{A}}(N)$, namely $r > n$. Applying $A \otimes_R -$ to the sequence $0 \to N \to E \to M \to 0$ for any $A \in \mathcal{A}$, we have $0 = \operatorname{Tor}_{r+i}^R(A, E) \to \operatorname{Tor}_{r+i}^R(A, M) \to \operatorname{Tor}_{r+i-1}^R(A, N) \to \operatorname{Tor}_{r+i-1}^R(A, E) = 0$ for $i \geq 2$, it follows that $\operatorname{Tor}_{r+i}^R(A, M) = 0$ for $i \geq 2$. By definition, $\operatorname{cfd}_{\mathcal{A}}(M) \leq 1 + r$. If $n > r$, the proof is the same, so we get $\operatorname{cfd}_{\mathcal{A}}(M) \leq 1 + n$. Consequently, we have $\operatorname{cfd}_{\mathcal{A}}(M) \leq 1 + \max\left\{\operatorname{cfd}_{\mathcal{A}}(E), \operatorname{cfd}_{\mathcal{A}}(N)\right\}$.

c) can be proved by using the same argument in (b).

3) By summing and shifting, we get the follwing exact sequence

$$\cdots \xrightarrow{\oplus f_i} \underset{i\in\mathbb{Z}}{\oplus} E_i \xrightarrow{\oplus f_i} \underset{i\in\mathbb{Z}}{\oplus} E_i \xrightarrow{\oplus f_i} \underset{i\in\mathbb{Z}}{\oplus} E_i \xrightarrow{\oplus f_i} \cdots$$

with $\operatorname{Im}(\oplus f_i) = \oplus_i M_i$. Consider the exact sequence

$$\alpha : 0 \to \underset{i}{\oplus} M_i \to \underset{i}{\oplus} E_i \to \underset{i}{\oplus} M_i \to 0$$

and by the 2(c), we have $\operatorname{cfd}_{\mathcal{A}}(\oplus_i E_i) \le \operatorname{cfd}_{\mathcal{A}}(\oplus_i M_i)$, it follows that $\sup\{\operatorname{cfd}_{\mathcal{A}}(E_i) : i \in \mathbb{Z}\} \le \sup\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\}$. Assume that $\sup\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\}$ is finite, then $\operatorname{cfd}_{\mathcal{A}}(\oplus_i M_i)$ is finite. Let $\operatorname{cfd}_{\mathcal{A}}(\oplus_i M_i) = n < \infty$ and applying $A \otimes_R -$ to the sequence α for any $A \in \mathcal{A}$, we obtain the exact sequence

$$\operatorname{Tor}_{n+1}^R(A, \oplus_i E_i) \to \operatorname{Tor}_{n+1}^R(A, \oplus_i M_i) \to \operatorname{Tor}_n^R(A, \oplus_i M_i) \to \operatorname{Tor}_n^R(A, \oplus_i E_i).$$

If $\operatorname{Tor}_n^R(A, \oplus_i E_i) = 0$, then $\operatorname{Tor}_n^R(A, \oplus_i M_i) = 0$. This contracdicts the assumption, it follows that $\operatorname{Tor}_n^R(A, \oplus_i E_i) \neq 0$. Since $\operatorname{Tor}_k^R(A, \oplus_i E_i) = 0\,(k > n)$, this equation $\operatorname{cfd}_{\mathcal{A}}(\oplus_i E_i) = \operatorname{cfd}_{\mathcal{A}}(\oplus_i M_i)$ holds true.

□

Definition 4. Let $\mathcal{A} \subseteq \operatorname{Mod}(R^{op})$.

1) An exact sequence in $\operatorname{Mod}(R)$

$$\cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots$$

is called *generalized* $\mathcal{A}$*-complete flat resolution*, if $\sup\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\}$ and $\sup\{\operatorname{fd}_R(E_i) : i \in \mathbb{Z}\}$ are bounded sets, for each $M_i := \operatorname{Im}(f_i)$ and each integer $i \in \mathbb{Z}$.

2) The module $M \in \operatorname{Mod}(R)$ is called *generalized* $(\mathcal{F}, \mathcal{A})$*-Gorenstein flat module* if M is the kernel or the image of a generalized $\mathcal{A}$-complete flat resolution.

3) Let $\mathbf{E} = \cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \to \cdots$ be a generalized $\mathcal{A}$-complete flat resolution and let $M_i := \operatorname{Im}(f_i)$ and each integer i. The common value $\sup\{\operatorname{cfd}_{\mathcal{A}}(M_i) : i \in \mathbb{Z}\} = \sup\{\operatorname{fd}_R(E_i) : i \in \mathbb{Z}\}$ is called the *degree* of $\mathbf{E}$.

4) An exact sequence in $\operatorname{Mod}(R)$

$$\cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots$$

is called $\mathcal{A}$*-complete* n*-flat resolution* if it is a generalized $\mathcal{A}$-complete flat resolution of degree n.

5) For any $M \in \operatorname{Mod}(R)$. The *generalized* $(\mathcal{F}, \mathcal{A})$*-Gorenstein flat dimension of* M is defined as $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M)$

$$= \begin{cases} \operatorname{cfd}_{\mathcal{A}}(M) & \text{if } M \text{ is a generalized } (\mathcal{F}, \mathcal{A})\text{-Gorenstein flat module,} \\ \infty & \text{otherwise.} \end{cases}$$

An R-module M is called G-$(\mathcal{F}, \mathcal{A})$*-Gorenstein flat* if $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = 0$. We denote by $\mathcal{GGF}_{(\mathcal{F},\mathcal{A})}$ the category of G-$(\mathcal{F}, \mathcal{A})$-Gorenstein flat modules.

Proposition 4. The following assertions hold.

1) $\mathcal{GF}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GGF}_{(\mathcal{F},\mathcal{A})}$.

2) If $\mathcal{I}\left(R^{\mathrm{op}}\right) \subseteq \mathcal{A}$, then $\mathcal{GGF}^{+}_{(\mathcal{F},\mathcal{A})} \subseteq \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(\mathrm{R}^{\mathrm{op}}\right)\right)}$.

Proof. 1) It is obvious by definition.

2) Let M be a G-$(\mathcal{F},\mathcal{A})$-Gorenstein flat module, there is a generalized $\mathcal{A}$-complete flat resolution with degree n (for fixed integer n)

$$\mathbf{E} = \cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots$$

with $M = \mathrm{Im}\left(f_0\right)$. Since $(-)^{+}$ is an exact functor, the dual sequence

$$\mathbf{E}^{+} = \cdots \to E_{-2}^{+} \to E_{-1}^{+} \to E_0^{+} \to E_1^{+} \to \cdots$$

is exact. Then $\mathrm{id}_R\left(E_i^{+}\right) = \mathrm{fd}_R\left(E_i\right) \le n$ and, by ([12], Lemma 3.4) $\mathrm{cid}_{\mathcal{A}}\left(M_i^{+}\right) = \mathrm{cfd}_{\mathcal{A}}\left(M_i\right) \le n$ for each integer i and by Proposition 2, $\mathrm{Gid}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}\left(M^{+}\right) < \infty$. When $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) = 0$, we have $\mathrm{cfd}_{\mathcal{A}}\left(M\right) = \mathrm{cid}_{\mathcal{A}}\left(M^{+}\right) = 0$. Note that $\mathrm{cid}_{\mathcal{A}}\left(M^{+}\right) = \mathrm{Gid}_{\left(\mathcal{A},\mathcal{I}\left(R^{\mathrm{op}}\right)\right)}\left(M^{+}\right) = 0$ by Proposition 2. Consequently, $M^{+} \in \mathcal{GI}_{\left(\mathcal{A},\mathcal{I}\left(\mathrm{R}^{\mathrm{op}}\right)\right)}\left(R\right)$.

□

Lemma 5. Let $0 \to N \to E \xrightarrow{f} M \to 0$ be an exact sequence such that $\mathrm{fd}_R\left(E\right) < \infty$ and M is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module. Then N is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module.

Proof. Since M is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module, we may assume that $\cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ is an $\mathcal{A}$-complete n-flat resolution. Let $M := \mathrm{Im}\left(E_0 \to E_{-1}\right)$ and $m := \max\left\{\mathrm{fd}_R\left(E\right), n\right\}$. Note that $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) = \mathrm{cfd}_{\mathcal{A}}\left(M\right) \le n$, and by the long exact theorem, we have $\mathrm{cfd}_{\mathcal{A}}\left(N\right) \le m$. Consider a flat resolution $\cdots \to F_1 \to F_0 \to N \to 0$ of N. Then the sequence $\cdots \to F_1 \to F_0 \to E \to E_{-1} \to E_{-2} \to \cdots$ is a generalized $\mathcal{A}$-complete flat resolution of degree $r := \sup\left\{\mathrm{fd}_R\left(E\right), \mathrm{fd}_R\left(E_i\right) : i \le -1\right\} \le m$. Hence N is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module.

□

Theorem 6. Let $\mathcal{A} \subseteq \mathrm{Mod}\left(R^{\mathrm{op}}\right)$ and M be a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat R-module issued from a $\mathcal{A}$-complete flat resolution of degree m. Assume $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. Then:

1) $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) < \infty$, and more precisely, $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) \le \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) \le m$.

2) Let $n > 0$ be an integer. The following statements are equivalent:

a) $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) \le n$.

b) $\mathrm{Tor}_{k+1}^{R}\left(A, M\right) = 0$ for any $A \in \mathcal{A}$ and each integer $k \ge n$.

c) For each exact sequence $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that the F_i are flat modules, the nth yoke K is a G-$(\mathcal{F},\mathcal{A})$-Gorestein flat module.

Proof. 1) Since $\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) < \infty$, by ([13], Theorem 3.12), we have

$$\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}\left(M\right) = \mathrm{cfd}_{\mathcal{A}}\left(M\right) \le \mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}\left(M\right).$$

There is an $\mathcal{A}$-complete m-flat resolution:

$$\cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots.$$

Let $M_i := \operatorname{Im}(d_i)$ (with $M = M_0$) for each integer i. Let i be a fixed integer and consider the exact sequence $0 \to M_{i+1} \to E_i \to M_i \to 0$ and the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'_{i+1} \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d] & M'_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_{i+1,m-1} \arrow[r] \arrow[d] & P_{i+1,m-1} \oplus P_{i,m-1} \arrow[r] \arrow[d] & P_{i,m-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_{i+1,0} \arrow[r] \arrow[d] & P_{i+1,0} \oplus P_{i,0} \arrow[r] \arrow[d] & P_{i,0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{i+1} \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where $P_{i,j}, P_{i+1,j} \in \mathcal{P}(R)$. Observe that $\operatorname{fd}_R(E_i) \le m$ and $F_i \in \mathcal{F}(R)$. As $\operatorname{cfd}_{\mathcal{A}}(M_i) \le m$ and $\operatorname{cfd}_{\mathcal{A}}(M_{i+1}) \le m$, by Proposition 3(1), we have $\operatorname{cfd}_{\mathcal{A}}(M'_i) = \operatorname{cfd}_{\mathcal{A}}(M'_{i+1}) = 0$. So we can derive the following exact sequence

$$\mathbf{F}: \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \to \cdots$$

is a complete flat resolution by ([7], Remark 2), and thus every M'_i is $(\mathcal{F},\mathcal{A})$-Gorenstein flat. Since there is an exact sequence $0 \to M'_0 \to P_{0,m-1} \to P_{0,m-2} \to \cdots \to P_{0,0} \to M_0 = M \to 0$ with $P_{0,j} \in \mathcal{P}(R)$ and $M'_0 \in \mathcal{GF}_{(\mathcal{F},\mathcal{A})}$, we have $\operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) \le m$.

2) (a) ⇔ (b) by definition.

(c) ⇒ (a). We consider a flat resolution $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ of M with nth yoke K. Because $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{cfd}_{\mathcal{A}}(M)$ and $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(K) = \operatorname{cfd}_{\mathcal{A}}(K) = 0$ by Proposition 3(1), we get $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le n$.

(a) ⇒ (c). Assume that $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le n$. Let $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence of modules such that the F_i are flat. Continuous application of Lemma 5 shows that K is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module. Then $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(K) = \operatorname{cfd}_{\mathcal{A}}(K)$. Since $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{cfd}_{\mathcal{A}}(M) \le n$, we get $\operatorname{cfd}_{\mathcal{A}}(K) = 0$ by Proposition 3(1). It follows that $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(K) = 0$, by definition, K is a G-$(\mathcal{F},\mathcal{A})$-Gorestein flat module.

□

Corollary 1. Let $\mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ such that $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$. Then for any $M \in \operatorname{Mod}(R)$:

1) $\operatorname{cfd}_{\mathcal{A}}(M) \le \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) \le \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le \operatorname{fd}_R(M)$.

2) If $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) < \infty$, then

$$\operatorname{cfd}_{\mathcal{A}}(M) = \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) = \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M).$$

3) If $({}^{\perp_1}\mathcal{A},\mathcal{A})$ is a complete cotorsion pair and R is a right coherent ring, then, for all M with $\operatorname{fd}_R(M) < \infty$,

$$\operatorname{cfd}_{\mathcal{A}}(M) = \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) = \operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{fd}_R(M).$$

Proof. 1) By using the following adjointness isomorphism

$$\operatorname{Tor}_i^R(A,M)^+ \cong \operatorname{Ext}_R^i(A,M^+)$$

for any $A \in \mathcal{A}$, we have

$$\begin{aligned}\operatorname{cfd}_{\mathcal{A}}(M) &= \sup\left\{ j \in \mathbb{N} : \operatorname{Ext}_R^j(A,M^+) \ne 0 \mid \exists A \in \mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}}) \right\} \\ &\le \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+).\end{aligned}$$

Assume that $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le n$ for some positive integer n. Let $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be an exact sequence with F_i flat. Applying $(-)^+$ to the sequence, we get that $0 \to M^+ \to F_0^+ \to F_1^+ \to \cdots \to F_{n-1}^+ \to K^+ \to 0$ is exact with F_i^+ injective. By Theorem 6(2), $K \in \mathcal{GGF}_{(\mathcal{F},\mathcal{A})}$, and thus, by Proposition 4(2), $K^+ \in \mathcal{GI}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}$. So $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) \le n$. It can be concluded that $\operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) \le \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M)$. Next, we prove $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le \operatorname{fd}_R(M)$. Assume that $\operatorname{fd}_R(M) = n < \infty$. We consider the sequence $0 \to M \xrightarrow{f} M \to 0$. Since $\operatorname{cfd}_{\mathcal{A}}(M) \le \operatorname{fd}_R(M) < \infty$, by definition $\operatorname{Im}(f) = M$ is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module. Hence, we have that $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le \operatorname{fd}_R(M)$.

2) If $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) < \infty$, then by definition, we have $\operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{cfd}_{\mathcal{A}}(M)$. Hence, by (1), the equation is as follows $\operatorname{cfd}_{\mathcal{A}}(M) = \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) = \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M)$.

3) Since R is a right coherent ring, the class $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is projectively resolving by ([4], Proposition 4.5), $\mathcal{GF}_{(\mathcal{F},\mathcal{A})}$ is closed under extensions. We have that $\operatorname{cfd}_{\mathcal{A}}(M) = \operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{fd}_R(M)$ by ([13], Lemma 4.4). So by (1), we have that

$$\operatorname{cfd}_{\mathcal{A}}(M) = \operatorname{Gid}_{(\mathcal{A},\mathcal{I}(R^{\mathrm{op}}))}(M^+) = \operatorname{Gfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \operatorname{fd}_R(M).$$

□

Theorem 7. Let $\mathcal{A} \subseteq \operatorname{Mod}(R^{\mathrm{op}})$ such that $\mathcal{I}(R^{\mathrm{op}}) \subseteq \mathcal{A}$. If $({}^{\perp_1}\mathcal{A},\mathcal{A})$ is a complete cotorsion pair and R is a right coherent ring, then the following assertions hold:

1) Assume that $0 \to N \to E \to M \to 0$ is an exact sequence such that $\mathrm{fd}_R(E) < \infty$ and M is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module.

a) If $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le \mathrm{fd}_R(E)$, then $\max\left\{\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N), \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M)\right\} = \mathrm{fd}_R(E)$.

b) If $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) > \mathrm{fd}_R(E)$, then $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = 1 + \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)$.

2) Let

$$\mathbf{E} = \cdots \to E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} E_{-1} \xrightarrow{f_{-1}} E_{-2} \to \cdots$$

be a generalized complete $\mathcal{A}$-flat resolution. Let $M_i := \mathrm{Im}(d_i)$ for each integer i. Then:

$$\begin{aligned}\sup\left\{\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M_i) : i \in \mathbb{Z}\right\} &= \sup\left\{\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M_i) : i \in \mathbb{Z}\right\} \\ &= \sup\left\{\mathrm{fd}_R(E_i) : i \in \mathbb{Z}\right\}.\end{aligned}$$

Proof. First, by Lemma 5, N is a generalized $(\mathcal{F},\mathcal{A})$-Gorenstein flat module.

1) a) Suppose that $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le \mathrm{fd}_R(E) = n < \infty$. Since $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{cfd}_{\mathcal{A}}(M)$, applying $A \otimes_R -$ to the sequence $0 \to N \to E \to M \to 0$ for any $A \in \mathcal{A}$, we have $0 = \mathrm{Tor}_{i+1}^R(A,M) \to \mathrm{Tor}_i^R(A,N) \to \mathrm{Tor}_i^R(A,E) \to \mathrm{Tor}_i^R(A,M) = 0$ for $i \ge n+1$. We can easily get $\mathrm{Tor}_i^R(A,N) = 0$ for $i \ge n+1$, by definition, $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N) = \mathrm{cfd}_{\mathcal{A}}(N) \le n = \mathrm{fd}_R(E)$. Hence,

$$\max\left\{\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M), \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)\right\} \le \mathrm{fd}_R(E).$$

Next, we prove the other side of the equation. Note that, by Corollary 1(3)

$$\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(E) = \mathrm{cfd}_{\mathcal{A}}(E) = \mathrm{fd}_R(E).$$

Also, $\mathrm{cfd}_{\mathcal{A}}(E) \le \max\left\{\mathrm{cfd}_{\mathcal{A}}(M), \mathrm{cfd}_{\mathcal{A}}(N)\right\}$ by Proposition 3(2)c, it follows that $\mathrm{cfd}_{\mathcal{A}}(E) = \mathrm{fd}_R(E) \le \max\left\{\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M), \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)\right\}$. In summary, the desired equation is established.

b) First of all, we already know $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N) = \mathrm{cfd}_{\mathcal{A}}(N)$ and $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) = \mathrm{cfd}_{\mathcal{A}}(M)$. Assume that $m = \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) > \mathrm{fd}_R(E) = r < \infty$. By Corollary 1(3), we know that $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(E) = \mathrm{cfd}_{\mathcal{A}}(E) = \mathrm{fd}_R(E)$. Suppose $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N) = n$ is finite, if $n < r$, there are some modules $A \in \mathcal{A}$, applying the long exact sequence theorem to the exact sequence $0 \to N \to E \to M \to 0$, we have $0 = \mathrm{Tor}_{r+1}^R(A,N) \to \mathrm{Tor}_{r+1}^R(A,E) \to \mathrm{Tor}_{r+1}^R(A,M) \to \mathrm{Tor}_r^R(A,N) = 0$ for $r \ge 1$, so $\mathrm{Tor}_{r+1}^R(A,M) = 0$. This contradicts Proposition 3(2)(b) and the assumption. It follows that $n \ge r$, so by Proposition 3(2)(b) $m \le n+1$, namely $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \le 1 + \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)$. Now let's prove that $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \ge 1 + \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)$. Applying $A \otimes_R -$ to the sequence $0 \to N \to E \to M \to 0$ for any $A \in \mathcal{A}$, we have $0 = \mathrm{Tor}_{m+i}^R(A,E) \to \mathrm{Tor}_{m+i}^R(A,M) \to \mathrm{Tor}_{m+i-1}^R(A,N) \to \mathrm{Tor}_{m+i-1}^R(A,E) = 0$ for $i \ge 1$. It follows that $\mathrm{Tor}_{m+i-1}^R(A,N) = 0$ for $i \ge 1$, so $\mathrm{cfd}_{\mathcal{A}}(N) \le m-1$, namely $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M) \ge 1 + \mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)$. Consequently,

$\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M)=1+\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(N)$.

2) By definition, $\mathrm{fd}_R(E_i)$ and $\mathrm{cfd}_{\mathcal{A}}(M_i)$ are finite for each i. So we can get $\sup\{\mathrm{cfd}_{\mathcal{A}}(E_i):i\in\mathbb{Z}\}=\sup\{\mathrm{cfd}_{\mathcal{A}}(M_i):i\in\mathbb{Z}\}$ by Proposition 3(3). By Corollary 1(3), we have

$$\begin{aligned}\sup\left\{\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M_i):i\in\mathbb{Z}\right\}&=\sup\{\mathrm{cfd}_{\mathcal{A}}(M_i):i\in\mathbb{Z}\}\\&=\sup\{\mathrm{cfd}_{\mathcal{A}}(E_i):i\in\mathbb{Z}\}=\sup\{\mathrm{fd}_R(E_i):i\in\mathbb{Z}\}\end{aligned}.$$

Also, by Theorem 6(1), $\mathrm{GGfd}_{(\mathcal{F},\mathcal{A})}(M_i)\le\mathrm{Gfd}_{(\mathcal{F},\mathcal{A})}(M_i)\le\sup\left\{\mathrm{fd}_R\left(E_j\right):j\in\mathbb{Z}\right\}$ for each integer i. Thus the equation follows.

□

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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