Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2025, 13(4), 1258-1269
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Existence and Stability of Standing Waves for the Inhomogeneous Schrödinger Equations with Mixed Fractional Laplacians

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How to cite this paper: Du, W.L. (2025) Existence and Stability of Standing Waves for the Inhomogeneous Schrödinger Equations with Mixed Fractional Laplacians. *Journal of Applied Mathematics and Physics*, **13**, 1258-1269.
https://doi.org/10.4236/jamp.2025.134067

Received: March 13, 2025
Accepted: April 14, 2025
Published: April 17, 2025

Abstract

In this paper, we consider the existence and orbital stability of standing waves for the inhomogeneous Schrödinger equations with mixed fractional Laplacians $i\partial_t\psi-(-\Delta)^{s_1}\psi-(-\Delta)^{s_2}\psi+|x|^{-\gamma}|\psi|^{2\sigma}\psi=0,(t,x)\in[0,T)\times\mathbb{R}^N$, where $N\geq 2$, $0<s_2<s_1<1$, and $0<\gamma<2s_1$. In the L^2-subcritical case, i.e., $0<\sigma<\frac{2s_1-\gamma}{N}$, we establish the existence and orbital stability of standing waves. Our approach is based on the concentration-compactness principle in the fractional Sobolev spaces H^{s_1} for $s_1\in(0,1)$.

Keywords

Laplace Operator, Existence, Stability

1. Introduction

This article focuses on studying the existence and orbital stability of standing waves for the inhomogeneous Schrödinger equations with mixed fractional Laplacians

$$\begin{cases} i\partial_t\psi-(-\Delta)^{s_1}\psi-(-\Delta)^{s_2}\psi+|x|^{-\gamma}|\psi|^{2\sigma}\psi=0,(t,x)\in[0,T)\times\mathbb{R}^N, \\ \psi(0,x)=\psi_0(x), \end{cases} \tag{1.1}$$

where $\psi:[0,T)\times\mathbb{R}^N\to\mathbb{C}$ is the complex valued function with $0<T\leq\infty$, $N\geq 2$, $0<s_2<s_1<1$, $0<\sigma<\frac{2s_1-\gamma}{N-2s_1}$, and $0<\gamma<2s_1$. The fractional Laplacian $(-\Delta)^s$ is characterized as $\mathcal{F}\left((-\Delta)^s\psi\right)(\xi)=|\xi|^{2s}\mathcal{F}(\psi)(\xi)$ for $\xi\in\mathbb{R}^N$, where $\mathcal{F}$ is the Fourier transform.

In recent years, the fractional Schrödinger equation has attracted widespread attention. Motivated by a variety of applications, remarkable progress has been achieved, as can be seen in [1]-[14]. Laskin initially introduced the fractional Schrödinger equation in [15] [16]. During the early research on this equation, the focus was mainly on the Hartree-type nonlinearity $\left(|x|^{-\gamma} * |\psi|^2\right)\psi$, as shown in [17]-[20]. However, regarding the local nonlinearity $|\psi|^{2\sigma}\psi$, the authors in [21] [22] investigated the well-posedness and ill-posedness in Sobolev space H^s. Recently, Boulenger et al. established the sufficient conditions for the blow-up in finite time of radial solutions in $\mathbb{R}^N$ in [23]. Nevertheless, for the mixed fractional inhomogeneous Schrödinger equation, as mentioned in [24] [25], there are currently very limited known results.

Equation (1.1) significantly broadens the application scope compared to equations with a single fractional Laplacian operator. It can describe a wider spectrum of physical scenarios. For example, the combination of fractional Laplacian operators of different orders can be used to depict the motion and interactions of particles in non-uniform media within special quantum-mechanical systems. Moreover, from a practical application perspective, Equation (1.1) has potential value in various fields. In population dynamics, for instance, the nonlinearity $|x|^{-\gamma}|\psi|^{2\sigma}\psi$ in Equation (1.1), analogous to the interaction patterns among population individuals, is vital for studying population distribution and evolution, as reported in references [26] [27]. For Equation (1.1) with $\gamma = 0$, the authors derived the existence and dynamics of solutions in mass-critical and supercritical cases by using the mountain pass lemma in [24]. In addition, the authors employed the constrained variational approaches to give a complete description of the existence of the normalized solution in [25], and this study addressed the mass-subcritical, critical and supercritical cases. However, there had been very little research on the situation when $\gamma \neq 0$ before. In this paper, we consider the case of $\gamma > 0$ for Equation (1.1). Moreover, in terms of research methods, this paper makes improvements on the deficiencies of predecessors in dealing with the non-locality of the fractional Laplacian operator and the nonlinearity $|x|^{-\gamma}|\psi|^{2\sigma}\psi$. In the proof of the existence of solutions, the application mode of the concentration-compactness principle is optimized to make it more suitable for this equation. In addition, based on the applicability of the current theory and in order to obtain clearer research results, this paper only studies the properties in the mass-subcritical case. Equation (1.1) admits a class of special solutions, which are called standing waves, namely solutions of the form $e^{i\omega t}u(x)$, where $\omega \in \mathbb{R}$ is a frequency and $u \in H^{s_1}\left(\mathbb{R}^N\right)$ is a nontrivial solution to the elliptic equation

$$(-\Delta)^{s_1} u + (-\Delta)^{s_2} u + \omega u = |x|^{-\gamma}|u|^{2\sigma} u. \tag{1.2}$$

Equation (1.2) is variational, whose action functional is defined by

$$S_\omega(u) = E(u) + \frac{\omega}{2}\|u\|_{L^2}^2, \tag{1.3}$$

where the corresponding energy $E(u)$ is defined by

$$E(u) := \frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}} u\right\|_2^2 + \frac{1}{2}\left\|(-\Delta)^{\frac{s_2}{2}} u\right\|_2^2 - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |x|^{-\gamma} |u|^{2\sigma+2} \mathrm{d}x. \tag{1.4}$$

It is interesting to study solutions of (1.2) having prescribed L^2-norm. That is, for any given constant $c>0$, consider solutions of (1.2) with the L^2-norm constraint

$$S(c) := \left\{u \in H^{s_1}\left(\mathbb{R}^N\right); \|u\|_2^2 = c\right\}. \tag{1.5}$$

Physically, such solution is called normalized solution to (1.2). In this case, the frequency $\omega \in \mathbb{R}$ is determined as Lagrange multiplier associated with $S(c)$ and is unknown.

For Equation (1.1), one of important problems is to consider the stability of standing waves, which is defined as follows.

Definition 1.1. The set $\mathcal{M}$ is orbitally stable if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any data $\psi_0 \in H^{s_1}$ satisfying

$$\inf_{u\in\mathcal{M}} \|\psi_0 - u\|_{H^{s_1}} < \delta,$$

the corresponding solution $\psi(t)$ of (1.2) satisfies

$$\inf_{u\in\mathcal{M}} \|\psi(t) - u\|_{H^{s_1}} < \varepsilon,$$

for any $t > 0$.

In the L^2-subcritical case, i.e., $0 < \sigma < \frac{2s_1 - \gamma}{N}$, by using the Gagliardo-Nirenberg inequality (2.3), we find that $E(u)$ restricted to $S(c)$ is bounded from below for any $c > 0$. Therefore, we consider the following constrained minimization problem

$$m(c) := \inf_{u\in S(c)} E(u). \tag{1.6}$$

In this case, we prove the existence of the solution to variational problem (1.6) by using the concentration compactness principle. However, compared with the work in [24] [25], one of the main difficulties is that due to the inhomogeneous nonlinearity $|x|^{-\gamma}|u|^{2\sigma}u$, Equation (1.1) does not possess translational invariance. In order to overcome the difficulties, we prove the boundedness of any translation sequence by using a similar argument in [19]. Denote the set of all minimizers of (1.6) by

$$\mathcal{M}_c = \left\{u \in H^{s_1}; u \text{ is a minimizer of the variational problem } (1.6)\right\}. \tag{1.7}$$

It is standard that for any $u_c \in \mathcal{M}_c$, there exists a $\omega_c \in \mathbb{R}$ such that (u_c, ω_c) solves Equation (1.2), and $\mathrm{e}^{i\omega_c t}u_c(x)$ is a standing wave solution of (1.1) with the initial data $\psi_0 = u_c$.

Theorem 1.2. Let $N \geq 2$, $0 < s_2 < s_1 < 1$, $0 < \gamma < 2s_1$, $0 < \sigma < \frac{2s_1 - \gamma}{N}$, and $c > 0$. Then the minimizing problem (1.6) has a positive normalized solution $u \in H^{s_1}$, and it satisfies (1.2) for some $\omega > 0$.

In view of Definition 1.1, in order to study the stability, we require that the solution of (1.1) exists globally, at least for initial data ψ_0 sufficiently close to $\mathcal{M}_c$. In fact, in the L^2-subcritical case, all solutions for the nonlinear Schrödinger Equation (1.1) exist globally. Therefore, we can obtain that if the initial value is close to an orbit in the set $\mathcal{M}_c$, then the solution of (1.1) remains close to the orbit in the set $\mathcal{M}_c$. Our main results are as follows:

Theorem 1.3. Let $N \geq 2$, $\frac{N}{2N-1} < s_2 < s_1 < 1$, $0 < \gamma < 2s_1$, and $0 < \sigma < \frac{2s_1 - \gamma}{N}$. Then, the $\mathcal{M}_c$ is orbitally stable.

Notation. For any $s \in (0,1)$, the definition of the fractional order Sobolev space $H^s(\mathbb{R}^N)$ is as follows

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} \left(1 + |\xi|^{2s}\right) |\hat{u}(\xi)|^2 \mathrm{d}\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|u\|_{\dot{H}^s(\mathbb{R}^N)},$$

where,

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_{L^2} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of u. We use $H^s_{rad}(\mathbb{R}^N)$ to denote the subspace of $H^s(\mathbb{R}^N)$, consisting of radially symmetric functions in $H^s(\mathbb{R}^N)$. In this paper, we often use the abbreviation $L^2 = L^2(\mathbb{R}^N)$, $H^s = H^s(\mathbb{R}^N)$.

2. Preliminaries

In this section, we recall some preliminary results that will be used later. Firstly, we recall the local well-posedness for the Cauchy problem (1.1), which can be proved by using the methods in [24], and the process is standard.

Lemma 2.1. Let $N \geq 2$, $\frac{N}{2N-1} < s_2 < s_1 < 1$, $0 < \gamma < 2s_1$, and $0 < \sigma < \frac{2s_1 - \gamma}{N - 2s_1}$. Then, for any $\psi_0 \in H^{s_1}_{rad}$, there exists a constant $T := T\left(\|\psi_0\|_{H^{s_1}}\right)$ and a unique maximal solution $\psi(t) \in C\left([0,T), H^{s_1}_{rad}\right)$ to the problem (1.1) which satisfies the alternative: either $T = \infty$ or $T < \infty$ and $\|\psi(t)\|_{H^{s_1}} \to \infty$ as $t \to T^-$. In addition, the solution $\psi(t)$ satisfies the following conservations of mass and energy

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \tag{2.1}$$

and

$$E(\psi(t)) = E(\psi_0), \tag{2.2}$$

for all $0 \leq t < T$, where $E(\psi(t))$ is defined by (1.4).

Next, we display the Gagliardo-Nirenberg inequality in $H^{s_1}(\mathbb{R}^{\mathbb{N}})$.

Lemma 2.2. ([28]) Let $N \geq 2$, $0 < s < 1$, $0 < \sigma < \dfrac{2s-\gamma}{N-2s}$, and $0 < \gamma < 2s$, then

$$\int_{\mathbb{R}^N} |x|^{-\gamma} |u|^{2\sigma+2} \mathrm{d}x \leq C_{N,\sigma,\gamma,s} \left(\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 \mathrm{d}x \right)^{\frac{N\sigma+\gamma}{2s}} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\sigma+1-\frac{N\sigma+\gamma}{2s}}, \quad (2.3)$$

where $C_{N,\sigma,\gamma,s} > 0$ denotes the optimal constant, and

$$C_{N,\sigma,\gamma,s} = \left(\frac{2s(\sigma+1)}{N\sigma+\gamma} - 1 \right)^{\frac{N\sigma+\gamma}{2s}} \frac{2s(\sigma+1)}{2s(\sigma+1)-(N\sigma+\gamma)} \|\Phi\|_2^{-2\sigma},$$

moreover, Φ is the solution of the following equation:

$$(-\Delta)^s \Phi + \Phi = |x|^{-\gamma} |\Phi|^{2\sigma} \Phi.$$

Furthermore, we present an estimate of the nonlinear term.

Lemma 2.3. Let $0 < \gamma < 2s_1$, $s_1 \in (0,1)$. if $u \in H^{s_1}(\mathbb{R}^{\mathbb{N}})$, then

$$\left\| |x|^{-\gamma} |u|^{2\sigma+2} \right\|_{L^1} \leq C \|u\|_{L^{m(2\sigma+2)}}^{2\sigma+2} + \|u\|_{L^{2\sigma+2}}^{2\sigma+2}, \quad (2.4)$$

and

$$\left\| |x|^{-\gamma} |u|^{2\sigma+2} \right\|_{L^1} \leq C \|u\|_{H^{s_1}}^{2\sigma+2}, \quad (2.5)$$

where $\dfrac{N}{N-\gamma} < m < \dfrac{2N}{N-2}\dfrac{1}{2\sigma+2}$ and $0 < \sigma < \dfrac{2s_1-\gamma}{N-2s_1}$.

Proof. We deduce from Hölder's inequality that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-\gamma} |u|^{2\sigma+2} \mathrm{d}x &= \int_{|x|\leq 1} |x|^{-\gamma} |u|^{2\sigma+2} \mathrm{d}x + \int_{|x|\geq 1} |x|^{-\gamma} |u|^{2\sigma+2} \mathrm{d}x \\ &\leq \left(\int_{|x|\leq 1} |x|^{-\gamma q} \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_{|x|\leq 1} |u|^{(2\sigma+2)m} \mathrm{d}x \right)^{\frac{1}{m}} + \|u\|_{L^{2\sigma+2}}^{2\sigma+2}. \end{aligned}$$

where $\dfrac{1}{q} + \dfrac{1}{m} = 1$. When $\gamma q < N$, we choose $m \in \left(\dfrac{N}{N-\gamma}, \dfrac{2N}{N-2}\dfrac{1}{2\sigma+2} \right)$, then (2.4) holds. In addition, by the Sobolev inequality, we have $H^{s_1}(\mathbb{R}^N) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N)$, where $2 < 2\sigma+2 < \dfrac{2N-2\gamma}{N-2s_1} < 2^*$, thus (2.5) holds.

□

We cite the concentration-compactness principle of [29], but we need to operate some modifications due to the difference of the parameters.

Lemma 2.4. ([29]) Let $N \geq 2$. Suppose $\{u_n\}_{n>0} \subset H^{s_1}$ and satisfy

$$\int_{\mathbb{R}^N} |u_n(x)|^2 \mathrm{d}x = \mu > 0, \quad (2.6)$$

$$\sup_{n>0} \|u_n\|_{H^{s_1}} < \infty. \quad (2.7)$$

Then there exists a subsequence $\{u_{n_k}\}_{k>0}$, for which one of the following properties holds.

i) Compactness: There exists a sequence $\{y_{n_k}\}_{k>0}$ in $\mathbb{R}^{\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exist $0 < r < \infty$ with

$$\int_{|x-y_{n_k}|\le r}|u_{n_k}(x)|^2\,\mathrm{d}x\ge \mu-\varepsilon. \tag{2.8}$$

ii) Vanishing: For all $r<\infty$, it follows that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le r}|u_{n_k}(x)|^2\,\mathrm{d}x=0. \tag{2.9}$$

iii) Dichotomy: There exists a constant $\beta\in(0,\mu)$ and two bounded sequences $\{v_k\}_{k>0},\{w_k\}_{k>0}\subset H^{s_1}$ such that

$$supp v_k\cap supp w_k=\varnothing, \tag{2.10}$$

$$|v_k|+|w_k|\le|u_{n_k}|, \tag{2.11}$$

$$\|v_k\|_{L^2}^2\to\beta,\|w_k\|_{L^2}^2\to(\mu-\beta)\text{ as }k\to\infty, \tag{2.12}$$

$$\|u_{n_k}-v_k-w_k\|_{L^{2\sigma+2}}\to 0\text{ for }0\le\sigma<\frac{2s_1}{N-2s_1}, \tag{2.13}$$

$$\liminf_{k\to\infty}\left\{\left\langle(-\Delta)^{s_1}u_{n_k},u_{n_k}\right\rangle-\left\langle(-\Delta)^{s_1}v_k,v_k\right\rangle-\left\langle(-\Delta)^{s_1}w_k,w_k\right\rangle\right\}\ge 0. \tag{2.14}$$

Finally, we present another version of the vanishing proved in [30].

Lemma 2.5. ([30]) Let $N\ge 2$. Assume that $\{u_k\}$ is bounded in $H^{s_1}(\mathbb{R}^N)$, and that it satisfies

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le r}|u_k(x)|^2\,\mathrm{d}x=0,$$

where $r>0$. Then $u_k\to 0$ in $L^{2\sigma+2}(\mathbb{R}^N)$ for $0<\sigma<\frac{2s_1}{N-2s_1}$.

3. Orbital Stability of Standing Waves

In this section, we study the existence and stability of standing waves to (1.1) in the mass-subcritical case.

Proof of Theorem 1.2. We proceed in four steps.

Step 1. Prove that the variational problem (1.6) is well-defined, and there exists a positive constant $c_0>0$ such that $m(c)\le -c_0<0$. We deduce from the Lemma 2.2 and Young's inequality that there exists a constant $C>0$ such that for any $0<\varepsilon<\frac{1}{2}$, the following inequalities hold:

$$\begin{aligned}E(u)&\ge\frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}}u\right\|_2^2+\frac{1}{2}\left\|(-\Delta)^{\frac{s_2}{2}}u\right\|_2^2-C\left\|(-\Delta)^{\frac{s_1}{2}}u\right\|_2^{\frac{N\sigma+\gamma}{s_1}}\|u\|_2^{2\sigma+2-\frac{N\sigma-\gamma}{s_1}}\\&\ge\left(\frac{1}{2}-\varepsilon\right)\left(\left\|(-\Delta)^{\frac{s_1}{2}}u\right\|_2^2+\left\|(-\Delta)^{\frac{s_2}{2}}u\right\|_2^2\right)-K,\end{aligned} \tag{3.1}$$

where $0<K<\infty$. This result implies that $-m(c)<\infty$, which indicates that the variational problem (1.6) is well-defined.

On the other hand, it is evident that $S(c)\ne\varnothing$. Let $u\in S(c)$ and $\lambda>0$. Define $u_\lambda(x):=\lambda^{\frac{N}{2}}u(\lambda x)$ for $x\in\mathbb{R}^N$. Through straightforward calculations, we obtain $\|u_\lambda\|_2=\|u\|_2$ and

$$E(u_\lambda):=\frac{1}{2}\lambda^{2s_1}\left\|(-\Delta)^{\frac{s_1}{2}}u\right\|_2^2+\frac{1}{2}\lambda^{2s_2}\left\|(-\Delta)^{\frac{s_2}{2}}u\right\|_2^2-\frac{1}{2\sigma+2}\lambda^{\gamma+N\sigma}\int_{\mathbb{R}^N}|x|^{-\gamma}|u|^{2\sigma+2}\,\mathrm{d}x. \quad (3.2)$$

Since $\gamma+N\sigma<2s_1$, when λ is small enough, we can deduce that $E(u_\lambda)<0$. Thus, there exists a positive constant $c_0>0$ such that $m(c)\le -c_0<0$.

Step 2. Estimates of the minimizing sequence $\{u_n\}_{n>0}$ for (1.6). Since $\{u_n\}_{n>0}\in S(c)$, the sequence $\{u_n\}_{n>0}$ is bounded in L^2. As can be observed from (3.1), $\left\|(-\Delta)^{\frac{s_1}{2}}u_n\right\|_2^2<\infty$. Therefore, $\{u_n\}_{n>0}$ is bounded in H^{s_1}. Moreover, given that $m(c)\le -c_0<0$, we obtain $E(u_n)\le -\frac{c_0}{2}$ for n large enough. From this, we can further deduce that

$$\int_{\mathbb{R}^N}|x|^{-\gamma}|u_n|^{2\sigma+2}\,\mathrm{d}x\ge(\sigma+1)c_0. \quad (3.3)$$

Step 3. We show that the vanishing and dichotomy cases do not occur. Let $\{u_n\}_{n>0}$ be a minimizing sequence of (1.6). Note that through scaling operations, we can assume $\mu=1$. Obviously, $\{|u_n|\}_{n>0}$ is also a minimizing sequence of (1.6). Thus, without loss of generality, we may suppose that $\{u_n\}_{n>0}$ is nonnegative. We now apply Lemma 2.4 to the minimizing sequence $\{u_n\}_{n>0}$.

Firstly, we claim that vanishing cannot occur. Indeed, if not, applying Lemma 2.5, we have $u_{n_k}\to 0$ in $L^{2\sigma+2}$, and combined with Lemma 2.3, we can get that

$$\int_{\mathbb{R}^N}|x|^{-\gamma}\left|u_{n_k}\right|^{2\sigma+2}\,\mathrm{d}x\to 0 \text{ as } k\to\infty,$$

which is a contradiction with (3.3).

Next, we show dichotomy cannot occur. If not, there exist a constant $\beta\in(0,1)$ and two sequences $\{v_k\}_{k>0}$ and $\{w_k\}_{k>0}$ which are introduced in Lemma 2.4. It follows from (2.13) and (2.14) that

$$\liminf_{k\to\infty}\left(E(u_{n_k})-E(v_k)-E(w_k)\right)\ge 0.$$

Hence,

$$\liminf_{k\to\infty}\left(E(v_k)+E(w_k)\right)\le\liminf_{k\to\infty}E(u_{n_k})=m(c). \quad (3.4)$$

On the other hand, given $u\in H^{s_1}$ and $a>0$, we have

$$E(u)=\frac{1}{a^2}E(au)+\frac{a^{2\sigma}-1}{2\sigma+2}\int_{\mathbb{R}^N}|x|^{-\gamma}|u|^{2\sigma+2}\,\mathrm{d}x.$$

Applying the above inequality with v_k and $a_k=\frac{1}{\|v_k\|_{L^2}}$, and since $a_kv_k\in S(c)$, we obtain that

$$E(v_k)\ge\frac{m(c)}{a_k^2}+\frac{a_k^{2\sigma}-1}{2\sigma+2}\int_{\mathbb{R}^N}|x|^{-\gamma}|v_k|^{2\sigma+2}\,\mathrm{d}x. \quad (3.5)$$

Similarly,

$$E(w_k) \geq \frac{m(c)}{b_k^2} + \frac{b_k^{2\sigma}-1}{2\sigma+2}\int_{\mathbb{R}^N}|x|^{-\gamma}|w_k|^{2\sigma+2}\,\mathrm{d}x, \tag{3.6}$$

with $b_k = \frac{1}{\|w_k\|_{L^2}}$, and so

$$E(v_k)+E(w_k) \geq m(c)\left(a_k^{-2}+b_k^{-2}\right)+\frac{a_k^{2\sigma}-1}{2\sigma+2}\int_{\mathbb{R}^N}|x|^{-\gamma}|v_k|^{2\sigma+2}\,\mathrm{d}x$$
$$+\frac{b_k^{2\sigma}-1}{2\sigma+2}\int_{\mathbb{R}^N}|x|^{-\gamma}|w_k|^{2\sigma+2}\,\mathrm{d}x.$$

Note that $a_k^{-2}\to\beta$ and $b_k^{-2}\to 1-\beta$ by (2.12). It follows from $0<\beta<1$ that

$$\theta := \min\left\{\beta^{-\sigma},(1-\beta)^{-\sigma}\right\} > 1.$$

Therefore, using (2.11) and (3.3) we deduce that

$$\liminf_{k\to\infty}\left(E(v_k)+E(w_k)\right) \geq m(c)+\frac{\theta-1}{2\sigma+2}\liminf_{k\to\infty}\int_{\mathbb{R}^N}|x|^{-\gamma}\left|u_{n_k}\right|^{2\sigma+2}\mathrm{d}x$$
$$\geq m(c)+\frac{\theta-1}{2}c_0 > m(c),$$

which contradicts (3.4).

Finally, since we have ruled out both vanishing and dichotomy, we conclude that indeed compactness occurs. Applying Lemma 2.4, we deduce that there exists a subsequence $\{u_{n_k}\}_{k>0}$ and a sequence $\{y_{n_k}\}_{k>0}\subset\mathbb{R}^N$ such that, for any $\varepsilon>0$, there exist $0<r<\infty$ with

$$\int_{|x-y_{n_k}|\leq r}\left|u_{n_k}(x)\right|^2\mathrm{d}x \geq c-\varepsilon.$$

Let $\tilde{u}_{n_k}(\cdot)=u_{n_k}(\cdot+y_{n_k})$. Since $\{u_n\}_{n>0}$ is a radially bounded sequence in $H_{rad}^{s_1}(\mathbb{R}^N)$, the embedding $H_{rad}^{s_1}(\mathbb{R}^N)\hookrightarrow L^{2\sigma+2}$ is compact for any $\sigma\in\left[0,\frac{2s_1}{N-2s_1}\right)$ and $N\geq 2$. Hence, there exists $\tilde{u}\in H_{rad}^{s_1}(\mathbb{R}^N)$ such that $u_{n_k}\rightharpoonup\tilde{u}$ in $H_{rad}^{s_1}(\mathbb{R}^N)$, thus, $u_{n_k}\rightharpoonup\tilde{u}$ in $L^{2\sigma+2}$. Therefore, we obtain

$$\int_{|x|\leq r}\left|\tilde{u}_{n_k}(x)\right|^2\mathrm{d}x \geq c-\varepsilon,$$

this implies $\int_{\mathbb{R}^N}|\tilde{u}(x)|^2\mathrm{d}x = c$, i.e., $\tilde{u}_{n_k}\to\tilde{u}$ in L^2. By Sobolev Embedding Theorem,

$$\tilde{u}_{n_k}\to\tilde{u} \text{ in } L^{2\sigma+2} \text{ for all } \sigma\in\left[0,\frac{2s_1}{N-2s_1}\right). \tag{3.7}$$

Step 4. Conclusion. We first prove that the sequence $\{y_{n_k}\}_{k>0}$ is bounded. Indeed, for the sake of contradiction, that it is unbounded. Then, passing to a subsequence if necessary, we can assume that $\left|y_{n_k}\right|\to\infty$ as $k\to\infty$. Consequently, from (3.7), we deduce that

$$\int_{\mathbb{R}^N} |x|^{-\gamma} \left|u_{n_k}(x)\right|^{2\sigma+2} \mathrm{d}x = \int_{\mathbb{R}^N} \left|x + y_{n_k}\right|^{-\gamma} \left|\tilde{u}_{n_k}(x)\right|^{2\sigma+2} \mathrm{d}x \to 0 \text{ as } k \to \infty.$$

This leads to the following inequality

$$\begin{aligned}\frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}} \tilde{u}\right\|_2^2 + \frac{1}{2}\left\|(-\Delta)^{\frac{s_2}{2}} \tilde{u}\right\|_2^2 &\le \lim_{k\to\infty}\left(\frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}} \tilde{u}_{n_k}\right\|_2^2 + \frac{1}{2}\left\|(-\Delta)^{\frac{s_2}{2}} \tilde{u}_{n_k}\right\|_2^2\right) \\ &= \lim_{k\to\infty} E\left(u_{n_k}\right) \\ &= m(c).\end{aligned}$$

By the definition of $E(\tilde{u})$, we obtain

$$E(\tilde{u}) + \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |x|^{-\gamma} |\tilde{u}|^{2\sigma+2} \mathrm{d}x \le m(c),$$

which implies $E(\tilde{u}) \le m(c)$. However, since $\|\tilde{u}\|_{L^2}^2 = c$, we know that $E(\tilde{u}) \ge m(c)$. This is a contradiction. Thus, the sequence $\left\{y_{n_k}\right\}_{k>0}$ is bounded. Then, there exists some $y_0 \in \mathbb{R}^N$ such that $y_{n_k} \to y_0$ as $k \to \infty$. We consequently deduce from (3.7) that for all $\sigma \in \left[0, \frac{2s_1}{N-2s_1}\right)$

$$\left\|u_{n_k} - \tilde{u}(x - y_0)\right\|_{L^{2\sigma+2}} \le \left\|u_{n_k} - \tilde{u}\left(x - y_{n_k}\right)\right\|_{L^{2\sigma+2}} + \left\|\tilde{u}\left(x - y_{n_k}\right) - \tilde{u}(x - y_0)\right\|_{L^{2\sigma+2}} \to 0$$

as $k \to \infty$. Let $u(x) = \tilde{u}(x - y_0)$, then $u \in S(c)$. Combining this with the week lower semicontinuity of the H^{s_1}-norm, this implies

$$E(u) \le \lim_{k\to\infty} E\left(u_{n_k}\right) = m(c).$$

According to the definition of $m(c)$, we conclude that $E(u) = m(c)$. In particular, since $E\left(u_{n_k}\right) \to E(u)$, it follows that

$$\left\|(-\Delta)^{\frac{s_1}{2}} u_{n_k}\right\|_2 \to \left\|(-\Delta)^{\frac{s_1}{2}} u\right\|_2 \text{ as } k \to \infty.$$

This implies that $u_{n_k} \to u$ in $H^{s_1}\left(\mathbb{R}^N\right)$ as $k \to \infty$, and thus the proof is complete.

□

Remark 3.1. When $0 < \sigma < \frac{2s_1 - \gamma}{N}$, we can obtain the global existence of (1.1) by using (2.1), (2.2) and (2.3). Indeed, we have that

$$\begin{aligned}E(\psi_0) = E(\psi(t)) &= \frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}} \psi\right\|_2^2 + \frac{1}{2}\left\|(-\Delta)^{\frac{s_2}{2}} \psi\right\|_2^2 - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |x|^{-\gamma} |\psi|^{2\sigma+2} \mathrm{d}x \\ &\ge \frac{1}{2}\left\|(-\Delta)^{\frac{s_1}{2}} \psi\right\|_2^2 - \frac{C_{N,\sigma,\gamma,s_1}}{2\sigma+2}\left\|(-\Delta)^{\frac{s_1}{2}} \psi\right\|_2^{\frac{N\sigma+\gamma}{s_1}} \|\psi\|_2^{2\sigma+2-\frac{N\sigma+\gamma}{s_1}} \\ &= \left\|(-\Delta)^{\frac{s_1}{2}} \psi\right\|_2^2 \left(\frac{1}{2} - \frac{C_{N,\sigma,\gamma,s_1}}{2\sigma+2}\left\|(-\Delta)^{\frac{s_1}{2}} \psi\right\|_2^{\frac{N\sigma+\gamma}{s_1}-2} \|\psi_0\|_2^{2\sigma+2-\frac{N\sigma+\gamma}{s_1}}\right).\end{aligned}$$

It follows that $\sup_{t\in[0,T)}\left\|(-\Delta)^{\frac{s_1}{2}}\psi\right\|_2^2<\infty$ if $\sigma<\frac{2s_1-\gamma}{N}$, which implies that $\psi\in H^{s_1}$.

Proof of Theorem 1.3.

We prove this theorem by contradiction. According to Remark 3.1, if $\psi_0\in H^{s_1}$, $\frac{N}{2N-1}<s_2<s_1<1$, and $0<\sigma<\frac{2s_1-\gamma}{N}$, then the solution $\psi(t,x)$ of (1.1) exists globally and $\|\psi(t,x)\|_{H^{s_1}}$ is bounded. Assume that exist $\varepsilon_0>0$ and a sequence $\{\psi_{0,n}\}_{n=1}^{\infty}$ such that

$$\inf_{u\in\mathcal{M}}\|\psi_{0,n}-u\|_{H^{s_1}}<\frac{1}{n}, \tag{3.8}$$

and there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that the relevant solution sequence $\{\psi_n(t_n)\}_{n=1}^{\infty}$ of (1.2) satisfies

$$\inf_{u\in\mathcal{M}}\|\psi_n(t_n)-u\|_{H^{s_1}}\geq\varepsilon_0. \tag{3.9}$$

It follows from (3.8) and the conservation laws that as $n\to\infty$

$$\|\psi_n(t_n,x)\|_{L^2}^2=\|\psi_{0,n}(x)\|_{L^2}^2\to\|\psi(x)\|_{L^2}^2=c,$$

and

$$E(\psi_n(t_n))=E(\psi_{0,n})\to E(\psi)=m(c).$$

Therefore, $\{\psi_n(t_n)\}_{n=1}^{\infty}$ is a minimizing sequence of the variational problem (1.6). According to Theorem 1.2, we deduce that there exists a minimizer $\omega\in\mathcal{M}_c$ such that as $n\to\infty$

$$\|\psi_n(t_n)-\omega\|_{H^{s_1}}\to 0, \tag{3.10}$$

which contradicts with (3.9). This completes the proof.

□

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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