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The Existence of Global Attractor for Kirchhoff-Type Strongly Damped Wave Equation with Nonlinear Memory

Jixin Xu*, **Xinghui Wang**

College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China
Email: *xjx915326@163.com

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Abstract

This paper addresses the existence of a global attractor for Kirchhoff-type strongly damped wave equation with nonlinear memory effects. The key innovation of our work lies in reformulating the historical memory term as a convolution integral involving the memory kernel μ and a nonlinear power-law function $|u(s)|^{\beta} u(s)$. First, we rigorously establish the existence, uniqueness, and regularity of solutions for Equation (1.2) through a systematic application of a priori energy estimates and the Faedo-Galerkin approximation method, while simultaneously demonstrating the presence of a bounded absorbing set. To analyze the asymptotic dynamics, we decompose the solution semigroup $S(t)$ into two components: $S_1(t)$, governed by higher-order regularity, and $S_2(t)$, capturing dissipative effects. The compactness of $S_1(t)$ is established via operator regularity analysis combined with the compact sobolev embedding theorem, while the uniform exponential decay of $S_2(t)$ is proven through refined energy estimation techniques. By synthesizing these results, we conclusively demonstrate the existence of a global attractor for the system under study, thereby extending the theoretical framework for nonlinear wave equations with memory-driven dissipation.

Keywords

Wave Equation, Nonlinear Memory, Global Attractor

1. Introduction

The wave equation, as a cornerstone of continuum mechanics, plays a pivotal role in modeling dissipative wave propagation phenomena across diverse physical systems. These include but are not limited to: acoustic attenuation in viscoelastic media (e.g., polymer melts), stress relaxation in aging composite materials, and energy dissipation in nonlinear metamaterials. The mathematical formulation of dissipative wave equations traces back to foundational work by Hale *et al.* [1], with the general form:

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + f(u) = h(x), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

where the dissipative term $g(u_t)$ governs energy loss mechanisms. Physically, the strong damping term $g(u_t) = -\Delta u_t$ (as studied in [2]-[4]) models highly viscous systems where the damping force depends on the Laplacian of the velocity field—a feature critical for simulating wave attenuation in frequency-dependent media such as biological tissues or rheological fluids.

Recent advances in materials science have highlighted the necessity of incorporating memory effects to describe time-history-dependent responses. For instance, in viscoelastic polymers subjected to cyclic loading, the stress-strain relationship depends nonlinearly on both current deformations and accumulated microstructural damage—a phenomenon that cannot be captured by linear memory models. This has motivated studies on memory-driven wave equations of the form:

$$u_{tt} - \Delta u - \sigma\left(\|\nabla u\|^2\right)\Delta u_t + \rho|u(t)|^\gamma u(t) - \int_0^\infty \mu(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s + f(u) = h(x)$$

where the nonlinear memory term $\int_0^\infty \mu(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s$ introduces a power-law dependence on historical states $\beta > 0$, enabling the modeling of fatigue damage accumulation in alloys or creep behavior in geomaterials. In particular, wave equations with linear memory terms have emerged as powerful tools for characterizing the time-dependent evolution of viscoelastic materials, where the current state depends intrinsically on its historical configurationsa framework rigorously explored in [5] [6], see related literature [7]-[9]. Such equations typically incorporate nonlocal operators expressed as convolution integrals between the unknown function $u(t)$ and a prescribed memory kernel μ. The kernel, often assumed to decay monotonically at infinity, encodes the systems fading memory properties and governs its asymptotic dissipation dynamics. These memory-driven models hold profound implications across multiple disciplines, including solid mechanics, fluid dynamics, and heat transfer theory, where they provide critical insights into energy dissipation mechanisms and long-term stability. Meanwhile, recent advances in dynamical systems theory and global attractor analysis have spurred significant interest in mathematical models incorporating memory effects, as evidenced by foundational studies in [2] [6] [10].

In their seminal 2003 work [11], Yang Meihua and Sun Chunyou established the existence of a global attractor for a strongly damped wave equation with critically growing nonlinearity in the phase space $H_0^1(\Omega) \times H_0^1(\Omega)$. They further demonstrated that this attractor is bounded in the refined space

$\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times\left(H^2(\Omega)\cap H_0^1(\Omega)\right)$, exhibiting attraction properties for bounded sets $H_0^1(\Omega)\times L^2(\Omega)$ under its natural norm. Subsequent advances include Pata's 2008 study [4], which derived conditions for the existence of global attractors in strongly damped wave equations with memory, requiring the memory kernel $\mu(s)$ to satisfy the exponential decay criterion $\mu(s+\sigma)\le K\mathrm{e}^{-\delta\sigma}\mu(s)$ ($K\ge 1,\delta>0$). Further contributions emerged in 2012 with Han Yinghao's analysis of nonlinear memory effects in damped wave equations [12], followed by D'Abbicco's 2014 investigations into weakly damped and structurally damped wave equations with nonlinear memory in [3] [13], respectively. Parallel developments in Kirchhoff-type damped wave equations have also been extensively explored, as evidenced by works such as [14]-[17]. Despite these advancements, the literature remains sparse regarding systems combining nonlinear memory with Kirchhoff-type damping. Motivated by the methodologies in [3] [12]-[14], this paper addresses this gap by investigating the existence of global attractors for a Kirchhoff-type strongly damped wave equation with nonlinear memory. The governing equation under consideration is formulated as follows:

$$\begin{cases} u_{tt}-\Delta u-\sigma\left(\|\nabla u\|^2\right)\Delta u_t+\rho|u(t)|^\gamma u(t) & \\ -\int_0^\infty \mu(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s+f(u)=h(x), & x\in\Omega,t>0, \\ u(x,t)=0, & x\in\partial\Omega,t\ge 0, \\ u(x,t)=u_0(x),u_t(x,t)=u_1(x), & x\in\Omega,t\le 0 \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and γ and β be positive constants, $\sigma\left(\|\nabla u\|^2\right)\Delta u_t$ is a Kirchhoff-type strong damping. The damping coefficient $\sigma(\cdot)\in C^1\left(\mathbb{R}^+\right)$ monotonically decreases, and for all $s\in\mathbb{R}^+$ there exists $\sigma(s)>0$. Moreover, $h(\cdot)\in L^2(\Omega)$.

This paper is organized as follows. Section 2 presents fundamental definitions and key theorems in dynamical systems theory. In Section 3, we rigorously establish the existence and uniqueness of solutions for system (1.2), thereby demonstrating that the system generates a well-defined dynamical system. Finally, Section 4 constructs a compact and connected global attractor for the dynamical system associated with Equation (1.2).

2. Preliminaries

In this Section, we introduce some notations and the useful theorem, which is the key technique to establish the well-posedness and existence of global attractors, which are from [1] [16] [18] [19].

Let $L^p(\Omega)$, $1\le p\le\infty$ is the usual Lebesgue spaces. The norm and inner-product of $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Let C be an arbitrary positive constant, which may be different from line to line and even in the same line. We set $H=V_0=L^2(\Omega)$, $V=V_1=H_0^1(\Omega)$. Let $V_s=D\left(\mathcal{A}^{\frac{s}{2}}\right)$, and $\mathcal{A}=-\Delta$ is a positive definite operator on H with discrete spectrum, with

$$\mathcal{A}e_i = \lambda_i e_i,\ 0 < \lambda_1 \le \lambda_2 \le \cdots,\ \lim_{k\to\infty}\lambda_k = \infty.$$

Therefore, $\{e_k\}$ forms a normalized orthogonal basis of space H and simultaneously serves as a normalized orthogonal basis in V. The inner product and norm for space V_s are defined as follow

$$(u,v)_{V_s} = \left(\mathcal{A}^{\frac{s}{2}}u, \mathcal{A}^{\frac{s}{2}}v\right),\ \|u\|_{V_s} = \|\mathcal{A}^s u\|.$$

Using H' and V' respectively represent the dual spaces of H and V, and in this paper, we use C to represent different constants. $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$.

We assume that the memory kernel μ is subject to the following hypotheses:

(h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$;

(h_2) $\mu(s) \ge 0$, $\mu'(s) \le 0$, $\forall\, s \in \mathbb{R}^+$;

(h_3) $\int_0^\infty \mu(s)\mathrm{d}s = 1$;

(h_4) $m_1\mu(s) \le -\mu'(s) \le m_2\mu(s)$, $\forall\, s \in \mathbb{R}^+$, where $m_1, m_2 > 0$.

In addition, the memory kernel μ satisfying assumptions (h_1)-(h_4) exists, such as $\mu(s) = m\mathrm{e}^{-ms}$.

We assume that the nonlinearity $f \in C(\mathbb{R})$ and $f(0) = 0$, is subject to the following hypotheses: there exists positive constants $\nu \ge 0$, $\gamma \ge \beta$ and $\rho > \frac{\gamma+2}{\beta+2}|\Omega|^{\frac{1}{\gamma}-\frac{1}{\beta}}$ such that

$$(f_1)\quad -bs^2 - A(x) \le F(s) \le sf(s) + \theta;$$

$$(f_2)\quad |f'(s)| \le c_1\left(1+|s|^\nu\right),\ \nu \ge 0,$$

where $0 < b < \frac{\lambda_1}{2}$, $A(\cdot) \in L^1(\Omega)$, and $A(\cdot) \ge -\theta$, $F(s) = \int_0^s f(y)\mathrm{d}y$.

Definition 2.1. Let $\mathcal{B}_0$ be an open subset of a matric space X, and $S(t)$ be a semigroup on X. We say that $\mathcal{B}_0$ is an absorbing set of $S(t)$ if the orbit of any bounded set $\mathcal{B}$ enters into $\mathcal{B}_0$ after a certain time (which may depend on the set $\mathcal{B} \subset X$). Namely, for any bounded set $\mathcal{B}$, there exists $t_1(\mathcal{B})$, such that $S(t)\mathcal{B} \subset \mathcal{B}_0, \forall t \ge t_1(\mathcal{B})$.

Definition 2.2. Let X be a metric space. The Kuratowski measure of a noncompact set $\mathcal{B}$ in X is defined by

$$\mathcal{K}_X(\mathcal{B}) = \inf\{d : \mathcal{B} \text{ has a finite cover of open balls of } X \text{ of diameter less than } d\}$$

Theorem 2.1. [18] Let X be a Banach space, and let $S(t)(t \ge 0)$ be a continuous operator semigroup on X. Suppose there exist an open set $\mathcal{U} \subset X$ and a bounded set $\mathcal{B} \subset \mathcal{U}$ that is absorbing in $\mathcal{U}$, and one of the following conditions holds:

i) Uniform Compactness for Large t: For every bounded set $\mathcal{B}$, there exists $t = t_0(\mathcal{B})$ such that

$$\bigcup_{t \ge t_0} S(t)\mathcal{B}$$

is relatively compact in X.

ii) Decomposition into Compact and Decaying Parts: $S(t)=S_1(t)+S_2(t)$, where

- $S_1(t)$: is uniformly compact for sufficiently large t,
- $S_2(t)$: $X\to X$ is a continuous mapping, and for every bounded set $\mathcal{B}\subset X$,

$$\sup_{\varphi\in\mathcal{B}}\left\|S_2(t)\varphi\right\|_X\to 0.$$

Then, the ω-limit set $\mathcal{A}=\omega(\mathcal{B})$ is a compact attractor that attracts all bounded sets in $\mathcal{U}$, is the maximal bounded attractor in $\mathcal{U}$, and is connected if $\mathcal{U}$ is both convex and connected.

Theorem 2.2. [18] Let X be a complete metric space, and $S(t)$ be a continuous semigroup on X. Suppose $S(t)$ admits a bounded absorbing set $\mathfrak{U}\subset X$ and satisfies that for every bounded subset $\mathcal{B}\subset X$, the Kuratowski measure of noncompactness $\alpha\left(S(t)\mathcal{B}\right)\to 0$ as $t\to\infty$. Then, the ω-limit set $\mathcal{A}=\omega(\mathfrak{U})$ is a compact maximal attractor for $S(t)$.

3. Well-Posedness of Solutions

We have the existence and uniqueness of solutions which are obtained by the Faedo-Galerkin approximation and a compactness method.

Theorem 3.1. If (h_1)-(h_4) and (f_1)-(f_2) hold, and γ,β,ν satisfy the following conditions

$$\begin{cases}0\le\nu,\beta,\gamma<\infty,\beta\le\gamma, & \text{when } n=1,2,\\ 0\le\nu,\beta,\gamma<2,\beta\le\gamma, & \text{when } n=3,\\ \nu=\beta=\gamma=0, & \text{when } n\ge 4,\end{cases}$$

then for any $u_0\in V,u_1\in H$ and $h\in H$, Equation (1.2) has a unique global solution u such that

$$u\in C\left(\left[0,\infty\right);V\right),\ u_t\in C\left(\left[0,\infty\right);H\right),\ u_{tt}\in C\left(\left[0,\infty\right);V'\right).$$

Proof. To prove the existence of solutions, we use the standard Galerkin method. Let $V_m=span\{e_1,e_2,e_3,\cdots,e_m\}$, be the subspace of H. We consider

$$u_m=\sum_{i=1}^{m}a_{im}(t)e_i,\ \forall\, t\ge 0, \tag{3.1}$$

which is the approximate solution of Faedo-Galerkin of order m, that is, we get a system of ODEs in the variables $a_{im}(t)$ $(i=1,2,\cdots,m)$ of the form

$$\begin{aligned}&\left(\partial_{tt}u_m,e_i\right)-\left(\Delta u_m,e_i\right)-\left(\sigma\left(\|\nabla u_m\|^2\right)\partial_t\Delta u_m,e_i\right)+\left(\rho\left|u_m(t)\right|^{\gamma}u_m(t),e_i\right)\\&-\left(\int_0^{\infty}\mu(t-s)\left|u_m(s)\right|^{\beta}u_m(s)\mathrm{d}s,e_i\right)+\left(f(u_m),e_i\right)=\left(h(x),e_i\right),\end{aligned} \tag{3.2}$$

subject to the initial conditions

$$u_m(0)=u_{m0}=\sum_{i=1}^{m}a_{im}(0)e_i\to u_0, \tag{3.3}$$

$$\partial_t u_m(0) = u_{m1} = \partial_t \sum_{i=1}^{n} a_{im}(0) e_i \to u_1. \tag{3.4}$$

It is well-known that the above finite-dimensional system of ordinary functional differential equations is well-posed at least locally (see for example [19]). Indeed, for fixed m, the system (3.2) defines a linear system of differential equations on $\Omega \subset \mathbb{R}^N$. Then we can apply differential equations theory for local existence and uniqueness of solutions to the system (3.2). Hence, the system of (3.2) possesses a unique local solution $a_{im}(t)$ defined in $[0,T]$, with $0 < T \le \infty$. Also, let us prove that a priori estimate for the Faedo-Galerkin approximate solutions $u_m(t)$. Multiplying Equation of (3.2) by $\partial_t a_{im}$ and εa_{im}, respectively, then summing over i and adding the results, we obtain

$$\begin{aligned}&\left(\partial_{tt} u_m, \partial_t u_m + \varepsilon u_m\right) - \left(\Delta u_m, \partial_t u_m + \varepsilon u_m\right) - \left(\sigma\left(\|\nabla u_m\|^2\right)\partial_t \Delta u_m, \partial_t u_m + \varepsilon u_m\right)\\&+\left(\rho\left|u_m(t)\right|^{\gamma} u_m(t), \partial_t u_m + \varepsilon u_m\right) - \left(\int_0^\infty \mu(t-s)\left|u_m(s)\right|^{\beta} u_m(s)\,\mathrm{d}s, \partial_t u_m + \varepsilon u_m\right)\\&+\left(f(u_m), \partial_t u_m + \varepsilon u_m\right) = \left(h(x), \partial_t u_m + \varepsilon u_m\right).\end{aligned} \tag{3.5}$$

Define also

$$(\mu \otimes u_m)(t) = \int_0^\infty \mu(t-s)\left\|u_m(s)\right|^{\frac{\beta}{2}} u_m(s) - \left|u_m(s)\right|^{\frac{\beta}{2}} u_m(t)\right|^2 \mathrm{d}s; \tag{3.6}$$

$$(\mu' \otimes u_m)(t) = \int_0^\infty \mu'(t-s)\left\|u_m(s)\right|^{\frac{\beta}{2}} u_m(s) - \left|u_m(s)\right|^{\frac{\beta}{2}} u_m(t)\right|^2 \mathrm{d}s; \tag{3.7}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \mu(t-s)\left\|u_m(s)\right\|_{\beta+2}^{\beta+2}\mathrm{d}s\right)\\&=\frac{1}{2}\mu(0)\left\|u_m(t)\right\|_{\beta+2}^{\beta+2} + \frac{1}{2}\int_0^\infty \mu'(t-s)\left\|u_m(s)\right\|_{\beta+2}^{\beta+2}\mathrm{d}s.\end{aligned} \tag{3.8}$$

Combining (3.6) - (3.8), we get

$$\begin{aligned}&\left(\int_0^\infty \mu(t-s)\left|u_m(s)\right|^{\beta} u_m(s)\,\mathrm{d}s, \partial_t u_m\right)\\&=\frac{1}{2}(\mu' \otimes u_m)(t) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mu \otimes u_m)(t) - \frac{1}{2}\mu(0)\left\|u_m(t)\right\|_{\beta+2}^{\beta+2}\\&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \mu(t-s)\left\|u_m(s)\right\|^{\frac{\beta}{2}} u_m(t)\right|^2 \mathrm{d}s\right)\\&\quad-\frac{1}{2}\int_0^\infty \mu'(t-s)\left\|u_m(s)\right\|^{\frac{\beta}{2}} u_m(t)\right|^2 \mathrm{d}s.\end{aligned} \tag{3.9}$$

Let $v = \partial_t u_m + \varepsilon u_m$, we substitute (3.8) and (3.9) in (3.5), by direct calculations, one gets

$$
\begin{aligned}
(h,v)=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\|v\|^2+\left(1+\varepsilon\sigma\left(\|\nabla u_m\|^2\right)\right)\|\nabla u_m\|^2+\frac{2\rho}{\gamma+2}\|u_m(t)\|^{\gamma+2}+(\mu\otimes u_m)(t)\\
&-\int_0^\infty\mu(t-s)\left\|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right\|^2\mathrm{d}s+\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\\
&+2\int_\Omega\left(F(u_m)+A(x)\right)\mathrm{d}x\Bigg]-\varepsilon\|v\|^2+\varepsilon^2(u_m,v)+\varepsilon\|\nabla u_m\|^2\\
&+\sigma\left(\|\nabla u_m\|^2\right)\|\nabla\partial_t u_m\|^2-\frac{1}{2}\|\nabla u_m\|^2\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma\left(\|\nabla u_m\|^2\right)\right]\\
&+\varepsilon\rho\|u_m(t)\|^{\gamma+2}-\frac{1}{2}(\mu'\otimes u_m)(t)-\frac{1}{2}\int_0^\infty\mu'(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\\
&+\frac{1}{2}\int_0^\infty\mu'(t-s)\left\|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right\|^2\mathrm{d}s+\varepsilon\int_\Omega f(u_m)u_m\mathrm{d}x\\
&-\varepsilon\left(\int_0^\infty\mu(t-s)|u_m(s)|^\beta u_m(s)\mathrm{d}s,u_m(t)\right).
\end{aligned}
\tag{3.10}
$$

By Hölder inequality and Young's inequality, we obtain

$$
\begin{aligned}
&\left(\int_0^\infty\mu(t-s)|u_m(s)|^\beta u_m(s)\mathrm{d}s,u_m(t)\right)\\
&=\int_\Omega\int_0^\infty\mu(t-s)|u_m(s)|^\beta u_m(s)u_m(t)\mathrm{d}s\mathrm{d}x\\
&\le\int_0^\infty\mu(t-s)\|u_m(s)\|^{\frac{\beta}{2}+1}|u_m(s)|^{\frac{\beta}{2}}u_m(t)\mathrm{d}s\\
&\le\frac{1}{2}\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{1}{2}\int_0^\infty\mu(t-s)\left\|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right\|^2\mathrm{d}s.
\end{aligned}
\tag{3.11}
$$

On the one hand, since $\sigma(\cdot)$ monotonically decreases and strictly positive in $\mathbb{R}^+$, there exists a constant $\sigma_0>0$ such that $\forall t\ge 0$, one gets

$$
\sigma\left(\|\nabla u\|^2\right)\ge\sigma_0,
\tag{3.12}
$$

and $\frac{\mathrm{d}}{\mathrm{d}t}\sigma\left(\|\nabla u\|^2\right)<0$. On the other hand, by the following inequality

$$
\|v\|^2=\|\partial_t u_m+\varepsilon u_m\|^2\le 2\left(\|\partial_t u_m\|^2+\varepsilon^2\|u_m\|^2\right),
$$

we have

$$
\begin{aligned}
&-\varepsilon\|v\|^2+\varepsilon^2(u_m,v)+\varepsilon\|\nabla u_m\|^2+\sigma\left(\|\nabla u_m\|^2\right)\|\nabla\partial_t u_m\|^2-\frac{1}{2}\|\nabla u_m\|^2\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma\left(\|\nabla u_m\|^2\right)\right]\\
&\ge-\varepsilon\|v\|^2+\varepsilon\|\nabla u_m\|^2+\sigma_0\lambda_1\left(\frac{1}{2}\|v\|^2-\frac{\varepsilon^2}{\lambda_1}\|\nabla u_m\|^2\right)-\frac{\varepsilon^2}{\sqrt{\lambda_1}}\left(\sqrt{\lambda_1}\|v\|^2+\frac{\|\nabla u_m\|^2}{4\sqrt{\lambda_1}}\right)\\
&\ge\left[\frac{\sigma_0\lambda_1}{2}-\varepsilon(1+\varepsilon)\right]\|v\|^2+\varepsilon\left(1-\varepsilon\left(\sigma_0+\frac{1}{4\lambda_1}\right)\right)\|\nabla u_m\|^2.
\end{aligned}
\tag{3.13}
$$

Meanwhile, by hypothesis (f_1), one has

$$
\begin{aligned}
\varepsilon\int_\Omega f(u_m)u_m&\ge\varepsilon\int_\Omega\left(F(u_m)-\theta\right)\mathrm{d}x\\
&=\varepsilon\int_\Omega\left(F(u_m)+A(x)\right)\mathrm{d}x-\varepsilon\int_\Omega\left(A(x)+\theta\right)\mathrm{d}x.
\end{aligned}
\tag{3.14}
$$

Then, taking into account hypothesis (f_1) and (3.11) - (3.14) in (3.10), we obtain

$$
\begin{aligned}
(h,v) \ge{}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+\frac{2\rho}{\gamma+2}\|u_m(t)\|^{\gamma+2}+(\mu\otimes u_m)(t) \\
&-\int_0^\infty \mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s+\int_0^\infty \mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s \\
&+2\int_\Omega\big(F(u_m)+A(x)\big)\mathrm{d}x\Bigg]+\left[\frac{\sigma_0\lambda_1}{2}-\varepsilon(1+\varepsilon)\right]\|v\|^2 \\
&+\varepsilon\left(1-\varepsilon\left(\sigma_0+\frac{1}{4\lambda_1}\right)\right)\|\nabla u_m\|^2+\varepsilon\rho\|u_m(t)\|^{\gamma+2} \\
&+\frac{m_1}{2}(\mu\otimes u_m)(t)-\frac{m_2+\varepsilon}{2}\int_0^\infty \mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s \\
&+\frac{m_1-\varepsilon}{2}\int_0^\infty \mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\varepsilon\int_\Omega\big(F(u_m)+A(x)\big)\mathrm{d}x \\
&-\varepsilon\int_\Omega\big(A(x)+\theta\big)\mathrm{d}x.
\end{aligned}
\tag{3.15}
$$

Besides, since

$$
(h,v)\le\|h\|\|v\|\le\frac{1}{\sigma_0\lambda_1}\|h\|^2+\frac{\sigma_0\lambda_1}{4}\|v\|^2,
$$

from (3.15), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+\frac{2\rho}{\gamma+2}\|u_m(t)\|^{\gamma+2}+(\mu\otimes u_m)(t) \\
&-\int_0^\infty \mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s+\int_0^\infty \mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s \\
&+2\int_\Omega F(u_m)+A(x)\mathrm{d}x\Bigg]+\left[\frac{\sigma_0\lambda_1}{4}-\varepsilon(1+\varepsilon)\right]\|v\|^2+\varepsilon\left[1-\varepsilon\left(\sigma_0+\frac{1}{4\lambda_1}\right)\right]\|\nabla u_m\|^2 \\
&+\varepsilon\rho\|u_m(t)\|^{\gamma+2}+\frac{m_1}{2}(\mu\otimes u_m)(t)-\frac{m_2+\varepsilon}{2}\int_0^\infty \mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s \\
&+\frac{m_1-\varepsilon}{2}\int_0^\infty \mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\varepsilon\int_\Omega\big(F(u_m)+A(x)\big)\mathrm{d}x \\
&\le\varepsilon\int_\Omega\big(A(x)+\theta\big)\mathrm{d}x+\frac{1}{\sigma_0\lambda_1}\|h\|^2.
\end{aligned}
\tag{3.16}
$$

Choosing ε small enough such that $\frac{\sigma_0\lambda_1}{4}-\varepsilon(1+\varepsilon)$, $\varepsilon\left(1-\varepsilon\left(\sigma_0+\frac{1}{4\lambda_1}\right)\right)$, $\frac{m_1-\varepsilon}{2}$, are positive constants, and let

$$
\delta=2\min\left\{\frac{\sigma_0\lambda_1}{4}-\varepsilon(1+\varepsilon),\frac{\varepsilon\left(1-\varepsilon\left(\sigma_0+\frac{1}{4\lambda_1}\right)\right)}{1+\varepsilon\sigma_0},\frac{\varepsilon}{2},\frac{\varepsilon(\gamma+2)}{2},\frac{m_1}{2},\frac{m_1-\varepsilon}{2},\frac{m_2+\varepsilon}{2}\right\},
$$

thus, (3.16) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t)+\delta E_m(t)\le 2\varepsilon\int_\Omega\left(A(x)+\theta\right)\mathrm{d}x+\frac{2}{\sigma_0\lambda_1}\|h\|^2, \tag{3.17}$$

where

$$\begin{aligned}E_m(t)=&\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+\frac{2\rho}{\gamma+2}\|u_m(t)\|^{\gamma+2}+(\mu\otimes u_m)(t)\\&-\int_0^\infty\mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s+\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\\&+2\int_\Omega\left(F(u_m)+A(x)\right)\mathrm{d}x.\end{aligned}$$

Using Young's inequality and hypothesis (h_3), we get that

$$\begin{aligned}&\int_0^\infty\mu(t-s)\left|\|u_m(s)\|^{\frac{\beta}{2}}u_m(t)\right|^2\mathrm{d}s\\&\le\frac{\beta}{\beta+2}\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{2}{\beta+2}\int_0^\infty\mu(t-s)\|u_m(t)\|_{\beta+2}^{\beta+2}\mathrm{d}s\\&\le\frac{\beta}{\beta+2}\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\frac{2}{\beta+2}|\Omega|^{\frac{1}{\gamma}-\frac{1}{\beta}}\|u_m(t)\|^{\beta+2}.\end{aligned} \tag{3.18}$$

Because of $\rho>\frac{\gamma+2}{\beta+2}|\Omega|^{\frac{1}{\gamma}-\frac{1}{\beta}}$, putting (3.18) in $E_m(t)$, and by means of the hypothesis (f_1), it immediately gives that

$$\begin{aligned}E_m(t)\ge&\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+(\mu\otimes u_m)(t)+2\int_\Omega\left(F(u_m)+A(x)\right)\mathrm{d}x\\&+\frac{2}{\beta+2}\int_0^\infty\mu(t-s)\|u_m(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s+\left(\frac{2\rho}{\gamma+2}-\frac{2}{\beta+2}|\Omega|^{\frac{1}{\gamma}-\frac{1}{\beta}}\right)\|u_m(t)\|^{\beta+2}\\\ge&\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+(\mu\otimes u_m)(t)+2\int_\Omega\left(F(u_m)+A(x)\right)\mathrm{d}x\\\ge&\|v\|^2+(1+\varepsilon\sigma_0)\|\nabla u_m\|^2+(\mu\otimes u_m)(t)-\frac{2b}{\lambda_1}\|\nabla u_m\|^2\\\ge&\|v\|^2+\varepsilon\sigma_0\|\nabla u_m\|^2.\end{aligned}$$

Applying Gronwall's inequality in (3.17), we deduce that

$$E_m(t)\le E_m(0)\mathrm{e}^{-\delta t}+\left(2\int_\Omega\left(A(x)+\theta\right)\mathrm{d}x+\frac{2}{\varepsilon\sigma_0\lambda_1}\|h\|^2\right)\left(1-\mathrm{e}^{-\delta t}\right). \tag{3.19}$$

According to the hypothesis (f_2), when $u(0)$ and $u_t(0)$ are bounded, $\int_\Omega F(u(0))\mathrm{d}x$ is also bounded, it is obvious that $E_m(0)$ is bounded, therefore, we claim that

$$R=\sup_{(u_0,u_1)\in B}E_m(0)<\infty,$$

if we set $\mu_0^2=2\int_\Omega\left(A(x)+\theta\right)\mathrm{d}x+\frac{2}{\varepsilon\sigma_0\lambda_1}\|h\|^2$, by using (3.19), for all $t\in[0,\infty)$, we have that

$$\|\partial_t u_m+\varepsilon u_m\|^2+\varepsilon\sigma_0\|\nabla u_m\|^2=\|v\|^2+\varepsilon\sigma_0\|\nabla u_m\|^2\le E_m(t)\le R\mathrm{e}^{-\delta t}+\mu_0^2. \tag{3.20}$$

Consequently, we can get from the above estimations that

$$u_m \text{ is bounded uniformly in } L^\infty\left([0,\infty);H_0^1\right);$$

$$\partial_t u_m \text{ is bounded uniformly in } L^\infty\left([0,\infty);L^2\right).$$

And because of the weak compactness, we can conclude that there exists a subsequence of solutions of the Galerkin approximations, denoted also by $\{u_m\}_{m\in\mathbb{N}}$, such that, for some u,

$$u_m \overset{*}{\rightharpoonup} u \text{ weakly star in } L^\infty\left([0,\infty);H_0^1\right);$$

$$\partial_t u_m \overset{*}{\rightharpoonup} \partial_t u \text{ weakly star in } L^\infty\left([0,\infty);L^2\right).$$

Meanwhile, since $H_0^1 \hookrightarrow\hookrightarrow L^{2\gamma+2} \hookrightarrow L^2$, by *Aubin-Lions* theorem, $u_m \to u$ is convergent strongly in $L^2\left([0,T];L^2\right)$. Therefore, $u_m \to u$ is true almost everywhere in $\Omega\times[0,T]$.

In the following, we will verify the convergence of the term $\rho|u_m|^\gamma u_m$. Since $u_m \to u$ is almost everywhere in $\Omega\times[0,T]$, and $|x|^a x\,(a>0)$ is a continuous function, we have

$$\rho|u_m|^\gamma u_m \to \rho|u|^\gamma u \text{ is convergent almost everywhere in } \Omega\times[0,T].$$

By the embedding $H_0^1 \hookrightarrow L^6(\Omega) \hookrightarrow L^{2\gamma+2}(\Omega)$ and (3.20), we obtain

$$\int_0^\infty\int_\Omega \left|\rho|u_m|^\gamma u_m\right| \mathrm{d}x\mathrm{d}t \le \rho\int_0^T \|u_m\|_{2\gamma+2}^{2\gamma+2}\,\mathrm{d}t \le C. \tag{3.21}$$

Using (3.20) and the Lions lemma we also have that

$$\rho|u_m|^\gamma u_m \to \rho|u|^\gamma u \text{ in } L^2\left(\Omega\times[0,T]\right).$$

Next we establish the convergence of the nonlinear term f. It follows from $u_m \to u$ is almost everywhere in $\Omega\times[0,T]$ and from $f(u_m)$ is continuous, one deduces that

$$f(u_m) \to f(u) \text{ is convergent almost everywhere in } \Omega\times[0,T],$$

we apply the hypothesis (f_2) and the Differential mean value theorem, there exists a constant $\theta\in(0,1)$, such that

$$\begin{aligned}\|f(u_m)\|^2 &= \|f(u_m)-f(0)\|^2 = \|f'(\theta u_m)u_m\|^2\\ &\le \int_\Omega \left|C\left(1+|u_m|^\nu\right)|u_m|\right|^2 \mathrm{d}x\\ &\le C\int_\Omega \left(|u_m|^2+|u_m|^{2\nu+2}\right)\mathrm{d}x\\ &\le C\left(\|u_m\|^2+\|u_m\|_{2\nu+2}^{2\nu+2}\right).\end{aligned}$$

Due to the embedding $H_0^1 \hookrightarrow L^6(\Omega) \hookrightarrow L^{2\nu+2}(\Omega)$, one implies that

$$\int_0^t \|f(u_m)\|^2\,\mathrm{d}t \le C\left(\|u_m\|^2+\|u_m\|_{2\nu+2}^{2\nu+2}\right) \le C. \tag{3.22}$$

Moreover, by the Lions lemma, it holds that

$$f(u_m) \rightharpoonup f(u) \text{ in } L^2(\Omega\times[0,T]).$$

By the above convergence results, integrating (3.2) over $[0,T]$ and taking the limit $m\to\infty$, it is clear that we obtain the solutions of (1.2) exist in $L^\infty(0,T;V')$.

Now, we will prove the continuous dependence of the solutions on the initial data. To do this, we consider u and v are two solutions of (1.2) corresponding to the initial conditions (u_0,u_1) and (v_0,v_1) respectively. Set $w=u-v$, therefore, it has $w(0)=u_0-v_0$, $w_t(0)=u_1-v_1$. Substituting u and v into (1.2) respectively, then subtracting the two formulas, we readily obtain that

$$\begin{aligned}&w_{tt}-\Delta w-\left(\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\sigma\left(\|\nabla v\|^2\right)\Delta v_t\right)+\left(\rho|u(t)|^\gamma u(t)-\rho|v(t)|^\gamma v(t)\right)\\&-\left(\int_0^\infty\mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s-\int_0^\infty\mu(t-s)|v(s)|^\beta v(s)\mathrm{d}s\right)+\left(f(u)-f(v)\right)=0.\end{aligned}\tag{3.23}$$

Multiplying this Equation (3.23) by w_t and integrating over Ω, we derive that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2-\left(\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\sigma\left(\|\nabla v\|^2\right)\Delta v_t,w_t\right)\\&=\left(\rho|v(t)|^\gamma v(t)-\rho|u(t)|^\gamma u(t),w_t\right)+\left(\int_0^\infty\mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s\right.\\&\quad\left.-\int_0^\infty\mu(t-s)|v(s)|^\beta v(s)\mathrm{d}s,w_t\right)+\left(f(v)-f(u),w_t\right).\end{aligned}\tag{3.24}$$

In the following, we will estimate the third item on the left and the items on the right of (3.24), respectively, where

$$\begin{aligned}&-\left(\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\sigma\left(\|\nabla v\|^2\right)\Delta v_t,w_t\right)\\&=\left(\sigma\left(\|\nabla u\|^2\right)\nabla u_t-\sigma\left(\|\nabla v\|^2\right)\nabla v_t,\nabla w_t\right)\\&\ge\sigma_0\left(\nabla u_t-\nabla v_t,\nabla w_t\right)=\sigma_0\|\nabla w_t\|^2\ge\sigma_0\lambda_1.\end{aligned}\tag{3.25}$$

By applying the Mazur inequality, the Young's inequality, the Hölder inequality, and the embedding $H_0^1\hookrightarrow L^6(\Omega)\hookrightarrow L^{\gamma+2}(\Omega)$, we get that

$$\begin{aligned}&\left(\rho|v(t)|^\gamma v(t)-\rho|u(t)|^\gamma u(t),w_t\right)\\&\le\rho\int_\Omega\left||v(t)|^\gamma v(t)-|u(t)|^\gamma u(t)\right||w_t|\mathrm{d}x\\&\le\rho(\gamma+1)\int_\Omega\left(|v(t)|^\gamma+|u(t)|^\gamma\right)|w||w_t|\mathrm{d}x\\&\le\rho(\gamma+1)\left(\|v(t)\|^\gamma+\|u(t)\|^\gamma\right)\|w\|\|w_t\|\\&\le C\rho(\gamma+1)\left(\|\nabla v(t)\|^\gamma+\|\nabla u(t)\|^\gamma\right)\|w\|\|w_t\|\\&\le C\rho(\gamma+1)\left(\|w\|^2+\|w_t\|^2\right).\end{aligned}\tag{3.26}$$

Similarly, we conclude that

$$\begin{aligned}
&\left(\int_0^\infty \mu(t-s)|u(s)|^\beta u(s)\mathrm{d}s-\int_0^\infty \mu(t-s)|v(s)|^\beta v(s)\mathrm{d}s, w_t\right)\\
&\le(\beta+1)\int_\Omega\int_0^\infty \mu(t-s)\left\||u(s)|^\beta+|v(s)|^\beta\|w\|w_t\right|\mathrm{d}s\mathrm{d}x\\
&\le C(\beta+1)\left\||u(s)|^\beta+|v(s)|^\beta\right\|_{\beta+2}\|w\|\|w_t\| \qquad (3.27)\\
&\le C(\beta+1)\left(\|\nabla u(s)\|^\beta+\|v(s)\|^\beta\right)\|w\|\|w_t\|\\
&\le C(\beta+1)\left(\|w\|^2+\|w_t\|^2\right).
\end{aligned}$$

From the hypothesis (f_2) and the Differential mean value theorem, combining the Hölder inequality, the Young inequality and the embedding $H_0^1\hookrightarrow L^6(\Omega)\hookrightarrow L^{\nu+2}(\Omega)$, yield that

$$\begin{aligned}
&\left(f(u)-f(v),w_t\right)\\
&=\left(f\left(\theta' u+(1-\theta)v\right)(-w),w_t\right)\\
&\le C\int_\Omega\left(1+|u|^\nu+|v|^\nu\right)|w||w_t|\mathrm{d}x\\
&\le C\left(1+\|u\|_{\nu+2}^\nu+\|v\|_{\nu+2}^\nu\right)\|w\|\|w_t\| \qquad (3.28)\\
&\le C\left(1+\|\nabla u\|_{\nu+2}^\nu+\|\nabla v\|_{\nu+2}^\nu\right)\left(\frac{1}{2}\|w\|^2+\frac{1}{2}\|w_t\|^2\right)\\
&\le C\left(\|w\|^2+\|w_t\|^2\right).
\end{aligned}$$

Plugging (2.25) - (2.28) into (2.24), and combining the Poincaré inequality, we show that

$$\begin{aligned}
C'\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\|w\|^2\right)\le{}& C\left[\rho(\gamma+1)+(\beta+1)+1\right]\|w\|^2\\
&+\left[C\left(\rho(\gamma+1)+(\beta+1)+1\right)-\sigma_0\lambda_1\right]\|w_t\|^2,
\end{aligned} \qquad (3.29)$$

where $C'\le\min\left\{\frac{1}{2},\frac{\lambda_1}{2}\right\}$. Consequently, thanks to the Gronwall lemma, we deduce

$$\|w_t\|^2+\|w\|^2\le C\left(\|w_t(0)\|^2+\|w(0)\|^2\right),\ \forall t\ge 0. \qquad (3.30)$$

hence, the continuous dependence of the solutions on the initial data is turned out. Meanwhile, the formula (3.30) suggests that the solutions of (1.2) are unique.

□

4. Existence of Global Attractor

By Theorem 3.1 problem (1.2) generates an evolution semigroup $S(t)$ in the space $V\times H$ by the formula $S(t)y=(u(t);u_t(t))$ where $y=(u_0;u_1)\in V\times H$ and $u(t)$ is a weak solution to (1.2).

In this Section, we first show the existence of a bounded absorbing set for $S(t)$ in $V\times H$, then it follows from Theorem 2.1 that we prove the ω-limit set of $\mathcal{B}_0$ is the connected and compact global attractor of $S(t)$.

As in the proof of the priori estimates of Theorem 3.1 using the formula (3.20) one can see that there exists a constant $t_0=t_0(\mathcal{B})$ such that for all $t\ge t_0$

$$\|\partial_t u(t_0)\|^2 + \|\nabla u(t_0)\|^2 \le R\mathrm{e}^{-\delta t} + \mu_0^2.$$

Let $\mathcal{B}_0 = \bigcup_{t\ge 0} S(t)\mathcal{B}_0'$, where

$$\mathcal{B}_0' = \left\{(u_0,u_1)\in V\times H : \|\nabla u_0\|^2 + \|u_1\|^2 \le 2\mu_0^2\right\},$$

this implies that $\mathcal{B}_0$ is the bounded absorbing set of semigroup $S(t)$ in $V\times H$. Therefore, the bounded dissipativity of the semigroup $S(t)$ can be obtained as follows:

Theorem 4.1. Let the forcing term $h\in H$. Assume that the nonlinear term $f(\cdot)\in C^1(\mathbb{R})$ satisfies conditions (f_1) - (f_2), and further assume that conditions (h_1) - (h_4) hold. Let (u,u_t) denote the solution to equation (1.2). corresponding to the initial data (u_0,u_1). Then there exists a constant $R_0>0$ such that the closed ball $\mathcal{B}_0 = B(0,R_0)$ is a bounded absorbing set in $V\times H$ for the solution semigroup $S(t)$ generated by Problem (1.2). Specifically, for every bounded subset $\mathcal{B}\subset V\times H$, there exists a critical time $t_0>0$ ensuring the inclusion $S(t)\mathcal{B}\subseteq\mathcal{B}_0$ holds for all $t\ge t_0$.

Next, we will decompose the semigroup $S(t)$ to demonstrate its asymptotic compactness using the Kuratowski measure and $\alpha-$noncompactness measure. Specifically, the semigroup $S(t)$ can be decomposed into two components $S_1(t)$ and $S_2(t)$, where:

- $S_1(t)$: Uniformly compact in $V\times H$,
- $S_2(t)$: Exponential decay.

Consequently, for any bounded set $\mathcal{B}\subset V\times H$, as $t\to\infty$,

$$\alpha(S(t)\mathcal{B}) \le \alpha(S_1(t)\mathcal{B}) + \alpha(S_2(t)\mathcal{B}) = \alpha(S_2(t)\mathcal{B}) \to 0.$$

Decompose u into $\bar{u}+\tilde{u}$, where $\bar{u}$ is the unique solution to the following problem:

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} - \sigma\left(\|\nabla u\|^2\right)\Delta\bar{u}_t & \\ = \int_0^\infty \mu(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s - \rho|u(t)|^\gamma u(t) - f(u) + h(x), & x\in\Omega, t>0, \\ \bar{u}(x,t) = 0, & x\in\partial\Omega, t\ge 0, \\ \bar{u}(x,0) = 0, \bar{u}_t(x,0) = 0, & x\in\Omega, t\le 0 \end{cases} \tag{4.1}$$

and, $\tilde{u}$ is the unique solution to the complementary problem:

$$\begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} - \sigma\left(\|\nabla u\|^2\right)\Delta\tilde{u}_t = 0, & x\in\Omega, t>0, \\ \tilde{u}(x,t) = 0, & x\in\partial\Omega, t\ge 0, \\ \tilde{u}(x,0) = u_0, \tilde{u}_t(x,0) = u_1, & x\in\Omega, t\le 0. \end{cases} \tag{4.2}$$

Lemma 4.1. Define the abstract equation:

$$\begin{cases} u_{tt} + Au + \sigma\left(\|\nabla u\|^2\right)Au_t = h(x), & x\in\Omega, t>0, \\ u(x,t) = 0, & x\in\partial\Omega, t\ge 0, \\ u(x,0) = u_0, u_t(x,0) = u_1, & x\in\Omega, t\le 0. \end{cases} \tag{4.3}$$

If $h\in H$, $(u_0,u_1)\in V\times H$, u is the solution to the equation (4.3), then

$(u,u_t)\in C_b(\mathbb{R}^+;V\times H)$ such that the solution u satisfies the relation

$$\|u_t\|^2+\|\nabla u\|^2\le\left(\|u_1\|^2+\|\nabla u_0\|^2\right)\mathrm{e}^{-\alpha_1 t}+\frac{C}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right),$$

where $\alpha_1>0$.

Proof. Taking the inner product in H of both sides of equation (4.3) with u_t, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2+\|\nabla u\|^2\right)+\sigma\left(\|\nabla u\|^2\right)\|\nabla u_t\|^2=(h,u_t),$$

since $\sigma(\cdot)\in C^1(\mathbb{R}^+)$ is monotonically decreasing and satisfies $\sigma(s)>0$ for all $s\in\mathbb{R}^+$, then there exists a constant $\alpha_1>0$ such that $\sigma\left(\|\nabla u\|^2\right)>\alpha_1>0$. By applying the Cauchy-Schwarz inequality and Young's inequality, we deduce that

$$(h,u_t)\le\|h\|\|u_t\|\le\frac{1}{2\alpha_1}\|h\|^2+\frac{\alpha_1}{2}\|u_t\|. \tag{4.4}$$

Consequently, we derive the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2+\|\nabla u\|^2\right)+\alpha_1\left(\|u_t\|^2+\|\nabla u\|^2\right)\le C, \tag{4.5}$$

by applying Gronwall's inequality, we obtain the following estimate:

$$\|u_t\|^2+\|\nabla u\|^2\le\left(\|u_1\|^2+\|\nabla u_0\|^2\right)\mathrm{e}^{-\alpha_1 t}+\frac{C}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right). \tag{4.6}$$

□

We now proceed to demonstrate the uniform compactness of the semigroup $S_1(t)$ in the product space $V\times H$.

Lemma 4.2. Under the assumptions of Theorem 3.1 on parameters γ, β, ν, there exists $\sigma'>0$ such that for any $(u,u_t)\in C_b(\mathbb{R}^+,V\times H)$, the following regularity holds:

$$\partial_t\mathcal{F}(x,t)\in L^\infty\left(\mathbb{R}^+,V_{\sigma'-1}\right). \tag{4.7}$$

Furthermore, for any bounded set $\mathcal{B}$ in $V\times H$, there exists a constant $C(\mathcal{B})>0$ such that whenever $(u_0,u_1)\in\mathcal{B}$, the following uniform bound is satisfied:

$$\sup_{u\in V}\|\partial_t\mathcal{F}(x,t)\|_{L^\infty\left(\mathbb{R}^+,V_{\sigma'-1}\right)}\le C(\mathcal{B}), \tag{4.8}$$

where

$$\mathcal{F}(x,t)=\int_0^\infty\mu(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s-\rho|u(t)|^\gamma u(t)-f(u)+h(x). \tag{4.9}$$

Proof. By direct computation, we obtain the following expression for the time derivative of $\mathcal{F}(x,t)$:

$$\begin{aligned}\partial_t\mathcal{F}(x,t)=&\int_0^\infty\mu'(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s+\mu(0)|u(t)|^\beta u(t)\\&-\rho(\gamma+1)|u(t)|^\gamma u_t(t)-f'(u)u_t.\end{aligned} \tag{4.10}$$

Under the assumptions of Theorem 3.1, we analyze the problem by considering three cases for the dimension n.

Case 1: For $n=1,2$, we choose $1-\sigma'>0$ such that the following inequality holds:

$$\frac{n\tau}{2\tau+4}<1-\sigma'<\frac{n}{2}, \tag{4.11}$$

where $\tau=\max\{\gamma,\nu\}$. Consequently, we obtain

$$\beta+2\le\gamma+2\le\tau+2\le\frac{2n}{n-2(1-\sigma')},$$

which implies the following chain of continuous embeddings:

$$L^{\frac{2n}{n-2(1-\sigma')}}(\Omega)\subset L^{\tau+2}(\Omega)\subset L^{\gamma+2}(\Omega)\subset L^{\beta+2}(\Omega). \tag{4.12}$$

Moreover, by the Sobolev embedding theorem and interpolation theory, we have

$$V_{1-\sigma'}\subset H^{1-\sigma'}(\Omega)\subset L^{\frac{2n}{n-2(1-\sigma')}}(\Omega). \tag{4.13}$$

Since all embeddings are continuous, it follows from (4.12) and (4.13) that

$$V_{1-\sigma'}\subset L^{\tau+2}(\Omega)\subset L^{\gamma+2}(\Omega)\subset L^{\beta+2}(\Omega). \tag{4.14}$$

Clearly, for any $1\le q<\infty$, $H_0^1(\Omega)\subset L^q(\Omega)$. If we set $q=\frac{n\nu}{1-\sigma'}$, then for all $u\in H_0^1(\Omega)$ with $\|u\|\le M$, the following estimate holds:

$$\|u(t)\|_{\frac{\gamma n}{1-\sigma'}}\le C\|u(t)\|, \tag{4.15}$$

finally, by assumption (f_2), we conclude that

$$\|f'(u)\|_{L^{\frac{q}{\nu}}}\le C(M). \tag{4.16}$$

Case 2: For $n=3$, the Sobolev embedding theorem yields

$$H_0^1(\Omega)\subset L^6(\Omega);\ H^s(\Omega)\subset L^{\frac{6}{3-2s}}(\Omega);\ 0\le s<\frac{3}{2}, \tag{4.17}$$

choosing $1-\sigma'=\frac{\tau}{2}$, we have

$$0<2(1-\sigma')=\tau<2<\frac{6}{2}, \tag{4.18}$$

and

$$\tau+2\le\frac{2n}{n-2(1-\sigma')}. \tag{4.19}$$

Thus, we recover (4.13) and (4.14). Since

$$\frac{\gamma n}{1-\sigma'}=\frac{2\gamma n}{\tau}<2n, \tag{4.20}$$

we can deduce (4.15). This completes the proof that (4.13) - (4.14) hold for $1\le n\le 3$.

First, let $w(t)\in V_{1-\sigma'}$. Taking the inner product of $w(t)$ with (4.10) in

$L^2(\Omega)$, we obtain

$$\begin{aligned}\left(\partial_t\mathcal{F}(x,t),w(t)\right)=&\left(\int_0^\infty \mu'(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s,w(t)\right)\\&+\left(\mu(0)|u(t)|^\beta u(t),w(t)\right)\\&-\left(\rho(\gamma+1)|u(t)|^\gamma u_t(t),w(t)\right)-\left(f'(u)u_t,w(t)\right).\end{aligned}\tag{4.21}$$

We now estimate each term on the right-hand side of (4.21). By Hölder inequality and Young's inequality, we get

$$\begin{aligned}&\left(\mu(0)|u(t)|^\beta u(t),w(t)\right)\\&\le\mu(0)\left(\int_\Omega\left(|u(t)|^{\beta+1}\right)^{\frac{\beta+2}{\beta+1}}\mathrm{d}x\right)^{\frac{\beta+1}{\beta+2}}\|w(t)\|_{\beta+2}\\&\le\mu(0)\|u(t)\|_{\beta+2}^{\beta+1}\|w(t)\|_{\beta+2}\\&\le\frac{\mu(0)(\beta+1)}{\beta+2}\|u(t)\|_{\beta+2}^{\beta+2}\|w(t)\|_{\beta+2}+\frac{\mu(0)}{\beta+2}\|w(t)\|_{\beta+2}.\end{aligned}\tag{4.22}$$

Similarly, applying Hölder inequality, Young's inequality, and (h_4), we derive

$$\begin{aligned}&\left(\int_0^\infty\mu'(t-s)|u(s)|^\beta u(s)\,\mathrm{d}s,w(t)\right)\\&\le m_1\int_0^\infty\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+1}\|w(t)\|_{\beta+2}\,\mathrm{d}s\\&\le\frac{m_1(\beta+1)}{\beta+2}\int_0^\infty\mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\|w(t)\|_{\beta+2}\,\mathrm{d}s+\frac{m_1}{\beta+2}\|w(t)\|_{\beta+2}.\end{aligned}\tag{4.23}$$

Using Hölder inequality $\left(\frac{2(1-\sigma')}{2n}+\frac{n}{2n}+\frac{n-2(1-\sigma')}{2n}=1\right)$ and (4.15), we obtain

$$\begin{aligned}&\left(\rho(\gamma+1)|u(t)|^\gamma u_t(t),w(t)\right)\\&\le\rho(\gamma+1)\|u(t)\|_{\gamma n/(1-\sigma')}^\gamma\|u_t(t)\|_2\|w(t)\|_{2n/(n-2(1-\sigma'))}\\&\le C\rho(\gamma+1)\|u(t)\|^\gamma\|u_t(t)\|\|w(t)\|_{2n/(n-2(1-\sigma'))}.\end{aligned}\tag{4.24}$$

Similarly, using Hölder inequality $\left(\frac{2(1-\sigma')}{2n}+\frac{n}{2n}+\frac{n-2(1-\sigma')}{2n}=1\right)$ and (4.16), we derive

$$\begin{aligned}\left(f'(u)u_t,w(t)\right)&\le\|f'(u)\|_{n/(1-\sigma')}\|u_t(t)\|_2\|w(t)\|_{2n/(n-2(1-\sigma'))}\\&\le C(M)\|u_t(t)\|\|w(t)\|_{2n/(n-2(1-\sigma'))}.\end{aligned}\tag{4.25}$$

Substituting (4.22) - (4.25) into (4.21) and combining with (4.12) and (4.13), we obtain

$$
\begin{aligned}
&\left(\partial_t \mathcal{F}(x,t), w(t)\right) \\
&\le \Bigg(\frac{\mu(0)(\beta+1)C}{\beta+2} \|u(t)\|_{\beta+2}^{\beta+2} + C\rho(\gamma+1)\|u(t)\|^{\gamma}\|u_t(t)\| + \frac{\mu(0)C}{\beta+2} + \frac{m_1 C}{\beta+2} \\
&\quad + \frac{m_1(\beta+1)C}{\beta+2}\int_0^\infty \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\,\mathrm{d}s + C(M)\|u_t(t)\| \Bigg) \|w(t)\|_{V_{1-\sigma'}} \\
&\le C(\mathcal{B})\|w(t)\|_{V_{1-\sigma'}},
\end{aligned}
\tag{4.26}
$$

where

$$
\begin{aligned}
C(\mathcal{B}) = \sup_{(u_0,u_1)\in\mathcal{B}, t\in\mathbb{R}^+} \Bigg\{ & \frac{\mu(0)(\beta+1)C}{\beta+2}\|u(t)\|_{\beta+2}^{\beta+2} + C\rho(\gamma+1)\|u(t)\|^{\gamma}\|u_t(t)\| + \frac{\mu(0)C}{\beta+2} \\
& + \frac{m_1 C}{\beta+2} + \frac{m_1(\beta+1)C}{\beta+2}\int_0^\infty \mu(t-s)\|u(s)\|_{\beta+2}^{\beta+2}\,\mathrm{d}s + C(M)\|u_t(t)\| \Bigg\}.
\end{aligned}
$$

In summary, combining (4.19) and (4.16), we conclude that $C(\mathcal{B})$ is bounded. This implies that $\partial_t \mathcal{F}(x,t)$ $V_{1-\sigma'}$ belongs to the dual space $V_{\sigma'-1}$ of $V_{1-\sigma'}$, and its norm is bounded in $V_{\sigma'-1}$. Consequently, (4.7) and (4.8) hold.

Case 3: For $n \ge 4$, it follows from (4.9) and (4.10) that

$$
\mathcal{F}(x,t) = -f(u),\ \partial_t \mathcal{F}(x,t) = -f'(u)u_t.
$$

Under the assumptions of Theorem 3.1, for all $u \in H$, we have $f'(u) \in L^\infty(\Omega)$. Thus, the conclusion holds.

Therefore, for all $n \ge 1$, Lemma 4.3 holds.

□

Lemma 4.3. For sufficiently large t, the operator $S_1(t)$ is uniformly compact in $V \times H$.

Proof. Let $\bar{u}$ be a solution to equation (4.1). By differentiating Equation (4.1) with respect to t, we conclude that $w = \bar{u}_t$ satisfies the following equation:

$$
\begin{cases}
w_{tt} - \Delta w - \sigma\left(\|\nabla u\|^2\right)\Delta w_t + 2\sigma'\left(\|\nabla u\|^2\right)(\Delta u, u_t)\Delta w \\
= \int_0^\infty \mu'(t-s)|u(s)|^{\beta} u(s)\,\mathrm{d}s + \mu(0)|u(t)|^{\beta} u(t) \\
\quad -\rho(\gamma+1)|u(t)|^{\gamma} u_t(t) - f'(u)u_t, & x\in\Omega, t>0, \\
w(0) = 0, w_t(0) = h - f(u_0), & x\in\Omega, t\le 0.
\end{cases}
\tag{4.27}
$$

From Lemma 4.2, we obtain

$$
\partial_t \mathcal{F}(x,t) \in C_b\left(\mathbb{R}^+; V_{\sigma'-1}\right),
$$

and thus

$$
w = \bar{u}_t \in C_b\left(\mathbb{R}^+; V_{\sigma'}\right),\ w_t = \bar{u}_{tt} \in C_b\left(\mathbb{R}^+; V_{\sigma'-1}\right),\ w_{tt} = \bar{u}_{ttt} \in C_b\left(\mathbb{R}^+; V'_{\sigma'}\right).
$$

From (4.1), it follows that

$$
\int_0^\infty \mu(t-s)|u(s)|^{\beta} u(s)\,\mathrm{d}s - \rho|u(t)|^{\gamma} u(t) - f(u) + h(x) \in C_b\left(\mathbb{R}^+, L^2(\Omega)\right),
$$

and

$$
\mathcal{A}\bar{u} \in C_b\left(\mathbb{R}^+, V_{\sigma'-1}\right).
$$

Thus, we have $\bar{u} \in C_b\left(\mathbb{R}^+, V_{\sigma'+1}\right)$. Furthermore, by Lemma 4.1, $\bar{u}_t$ is bounded, and $\bar{u}_t \in C\left(\mathbb{R}^+, V_{\sigma'}\right)$, which implies $\Delta w = \Delta \bar{u}_t \in C_b\left(\mathbb{R}^+; V_{\sigma'}\right)$.

Consequently,

$$\left(\bar{u}, \bar{u}_t\right) \in C\left(\mathbb{R}^+; V_{\sigma'+1} \times V_{\sigma'}\right),$$

and

$$\bigcup_{t \geq t_0} S(t)\mathcal{B} \subset V_{\sigma'+1} \times V_{\sigma'}.$$

By the embedding theorem, for any $\sigma_1' > \sigma_2'$, we have $V_{\sigma_1'} \hookrightarrow V_{\sigma_2'}$. Since $V_{\sigma'+1} \times V_{\sigma'} \hookrightarrow V \times H$, it follows that $\bigcup_{t \geq t_0} S(t)\mathcal{B}$ is compact in $V \times H$. Therefore, $S_1(t)$ is uniformly compact in $V \times H$, and the conclusion is proved.

□

Theorem 4.2. For any bounded set $\mathcal{B}$, there exists $t_0 > 0$ such that $\bigcup_{t \geq t_0} S(t)\mathcal{B}$ is relatively compact in $V \times H$. Therefore, $S(t)$ is asymptotically compact in $V \times H$.

Proof. We decompose $S(t)$ into $S_1(t)$ and $S_2(t)$, i.e., $S(t) = S_1(t) + S_2(t)$, where $S_1(t) = \Sigma(t)$ and $S_2(t) = S(t) - \Sigma(t)$. For any solution $\tilde{u}$ to equation (4.2) with initial condition $u_0 \in \mathcal{B}$, the following inequality holds:

$$\left\|\tilde{u}_t\right\|^2 + \left\|\nabla \tilde{u}\right\|^2 \leq C(\mathcal{B})\exp\left(-\alpha_1 t\right).$$

Thus, as $t \to \infty$, we have

$$\alpha\left(S_2(t)(\mathcal{B})\right) \to 0.$$

By Lemma 4.3, the operator $S_1(t)$ is uniformly compact in $V \times H$. Therefore, as $t \to \infty$,

$$\alpha\left(S(t)\mathcal{B}\right) \leq \alpha\left(S_1(t)\mathcal{B}\right) + \alpha\left(S_2(t)\mathcal{B}\right) = \alpha\left(S_2(t)\mathcal{B}\right) \to 0.$$

This completes the proof.

Theorem 4.3. If the external force term $h \in L^2(\Omega)$, the nonlinear term $f \in C^1(\mathbb{R})$ satisfies (f_1) - (f_2), and (h_1) - (h_4) hold, then the solution semigroup $S(t)$ associated with Equation (1.2) possesses a compact and connected global attractor in the space $V \times H$.

Proof. Following the same reasoning, Theorem 4.3 can be derived by applying Theorem 4.1 and Theorem 4.2 within the framework of Theorem 2.1.

□

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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