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Asymptotic Behavior of the Solution of the Initial Boundary Value Problem for a Boussinesq Analog Equation

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Abstract

In this paper, the asymptotics of the solution of the initial boundary value problem for a sixth-order nonlinear partial differential equation of Boussinesq type is studied. First, the energy function is obtained. Then, apriori evaluations for this function are obtained. Then, by imposing some conditions on the function on the right-hand side of the equation and using appropriate inequalities, it is shown that the solution is asymptotically damped.

Keywords

Initial Boundary Value Problem, Boussinesq Equation, Asymptotic Behavior of the Solution

1. Introduction

[1] As I mentioned in my paper, the following equation:

$$\left(\sigma^2\Delta-1\right)\frac{\partial u^2}{\partial t^2}+\gamma^2\Delta u=F$$

It is called the Boussinesq equation. Δ where is the Laplace operator, σ,γ fixed numbers F if it is u is a linear or nonlinear function depending on derivatives of order less than the left-hand side derivatives of the solution function. This equation and its analogs appear in the study of long-wave motions in plasmas and liquids [1]-[35] They examined the solvability of initial boundary value problems and the stability of the solutions.

In this paper we investigate the asymptotic stability of the solution of the initial and boundary value problem for an equation which is an analog of the sixth-order

Boussinesq equation with respect to space variables.

2. Discussion

In this paper, we have investigated the asymptotic behavior of the strong solution of the initial boundary-value problem for the sixth-order nonlinear Boussinsq type equation with a dispersion term.

3. Methods

By using the multiplier method and the integral estimate methods, we prove that the strong solutions of the problem decay to zero exponentially as the time tends to infinite, under weaker conditions regarding the nonlinear term. And Gromwells lemma.

4. Results

Now let's move on to the content of the article.

In this study $x \in (0,1), t \geq 0$ of the initial boundary value problem for a sixth-order nonlinear partial differential equation of Boussinesq type given below

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} - \Delta^3 u = f(u), \ x \in (0,1), \ t \geq 0 \tag{1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ x \in [0,1] \tag{2}$$

$$u(0,t) = u(1,t) = 0, \ t \geq 0 \tag{3}$$

We will study the asymptotic behavior of the solution satisfying the initial conditions (2) and boundary conditions (3). Here $u(x,t)$ is the sought solution function, $f(s)$ is a nonlinear function and $u_0(x), u_1(x)$ are the initial data.

Theorem 1. Suppose,

$$0 \leq -F(u) \leq -f(u)u \quad \forall u \in R, \quad F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

Let u be the solution of the problem (1) - (3). Then for positive constants C and λ

$$E(t) \leq C \cdot E(0) \mathrm{e}^{-\lambda t}, \ 0 \leq t < +\infty \tag{4}$$

inequality is true.

Here

$$E(t) = \frac{\|u_t\|^2}{2} + \frac{\|\nabla u\|^2}{2} + \frac{\|\nabla u_t\|^2}{2} + \frac{\|\nabla^3 u\|^2}{2} - \int_0^1 F(u)\,\mathrm{d}x \tag{5}$$

It is a function of energy.

Proof: Let us scalar multiply Equation (1) by u_t and integrate it from 0 to 1:

$$\int_0^1 u_{tt} u_t \mathrm{d}x - \int_0^1 u_{xx} u_t \mathrm{d}x - \int_0^1 u_{xxt} u_t \mathrm{d}x - \int_0^1 u_{xxtt} u_t \mathrm{d}x - \int_0^1 u_{xxxxxx} u_t \mathrm{d}x = \int_0^1 f(u) u_t \mathrm{d}x$$

Now if we use the partial integration formula

$$\frac{\mathrm{d}}{2\mathrm{d}t}\|u_t\|^2 + \frac{\mathrm{d}}{2\mathrm{d}t}\|u_x\|^2 + \|u_{xt}\|^2 + \frac{\mathrm{d}}{2\mathrm{d}t}\|u_{xt}\|^2 + \frac{\mathrm{d}}{2\mathrm{d}t}\|\nabla^3 u\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 F(u)\,\mathrm{d}x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|u_t\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_t\|^2}{2}+\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,\mathrm{d}x\right)+\|u_{xt}\|^2=0 \tag{6}$$

we obtain equality.

If we substitute (6) in (5):

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\|\nabla u_t\|^2=0 \tag{7}$$

we get.

Let us multiply the inequality (7) by $\mathrm{e}^{\delta t}$ provided that $\delta>0$:

$$\mathrm{e}^{\delta t}\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\mathrm{e}^{\delta t}\|\nabla u_t\|^2=0$$

We can write this equation as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\delta t}E(t)\right)+\mathrm{e}^{\delta t}\|\nabla u_t\|^2=\delta\mathrm{e}^{\delta t}E(t) \tag{8}$$

If we integrate Equation (8) from 0 to t:

$$\int_0^t\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{\delta\tau}E(\tau)\right)\mathrm{d}\tau+\int_0^t\mathrm{e}^{\delta\tau}\|\nabla u_\tau\|^2\,\mathrm{d}\tau=\delta\int_0^t\mathrm{e}^{\delta\tau}E(\tau)\,\mathrm{d}\tau$$

$$\mathrm{e}^{\delta t}E(t)-E(0)+\int_0^t\mathrm{e}^{\delta\tau}\|\nabla u_\tau\|^2\,\mathrm{d}\tau=\delta\int_0^t\mathrm{e}^{\delta\tau}E(\tau)\,\mathrm{d}\tau$$

$$\mathrm{e}^{\delta t}E(t)+\int_0^t\mathrm{e}^{\delta\tau}\|\nabla u_\tau\|^2\,\mathrm{d}\tau=E(0)+\delta\int_0^t\mathrm{e}^{\delta\tau}E(\tau)\,\mathrm{d}\tau$$

we obtain. In the integral on the right side of the equation $E(t)$ if we use (5) instead:

$$\begin{aligned}&\mathrm{e}^{\delta t}E(t)+\int_0^t\mathrm{e}^{\delta\tau}\|\nabla u_\tau\|^2\,\mathrm{d}\tau\\&=E(0)+\delta\int_0^t\mathrm{e}^{\delta\tau}\left(\frac{\|u_\tau\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_\tau\|^2}{2}+\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,\mathrm{d}x\right)\mathrm{d}\tau\\&=E(0)+\delta\int_0^t\mathrm{e}^{\delta\tau}\left(\frac{\|u_\tau\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_\tau\|^2}{2}\right)\mathrm{d}\tau+\delta\int_0^t\mathrm{e}^{\delta\tau}\left(\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,\mathrm{d}x\right)\mathrm{d}\tau\end{aligned} \tag{9}$$

we get.

Considering the conditions of Theorem 1, let us evaluate the last integral in (9):

$$\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,\mathrm{d}x\le\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 f(u)u\,\mathrm{d}x$$

$$\delta\int_0^t\mathrm{e}^{\delta\tau}\left(\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,\mathrm{d}x\right)\mathrm{d}\tau\le\delta\int_0^t\mathrm{e}^{\delta\tau}\left(\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 f(u)u\,\mathrm{d}x\right)\mathrm{d}\tau \tag{10}$$

As seen in this inequality, if we substitute the left side of Equation (1) for $f(u)$ in the expression $\int_0^1 f(u)u\,\mathrm{d}x$ in the integral on the right side of the inequality

$$\int_0^t e^{\delta\tau}\left(\frac{\left\|\nabla^3 u\right\|^2}{2}-\int_0^1 F(u)\,dx\right)d\tau$$
$$\le\int_0^t e^{\delta\tau}\left(\frac{\left\|\nabla^3 u\right\|^2}{2}-\left\|\nabla^3 u\right\|^2-(u_{\tau\tau}u)-\left\|\nabla u\right\|^2-\frac{d}{2d\tau}\left\|\nabla u\right\|^2-(\nabla u_{\tau\tau}\nabla u)\right)d\tau \quad (11)$$
$$=\int_0^t e^{\delta\tau}\left(-\frac{\left\|\nabla^3 u\right\|^2}{2}-(u_{\tau\tau}u)-\left\|\nabla u\right\|^2-\frac{d}{2d\tau}\left\|\nabla u\right\|^2-(\nabla u_{\tau\tau}\nabla u)\right)d\tau$$

we get.

From the right hand side of (11)

$$-\frac{\left\|\nabla^3 u\right\|^2}{2} \text{ and } -\left\|\nabla u\right\|^2$$

If we drop terms that are strictly negative, such as (11), then (11) becomes the following inequality.

$$\delta\int_0^t e^{\delta\tau}\left(\frac{\left\|\nabla^3 u\right\|^2}{2}-\int_0^1 F(u)\,dx\right)d\tau$$
$$\le\delta\int_0^t e^{\delta\tau}\left(-(u_{\tau\tau}u)-\frac{d}{2d\tau}\left\|\nabla u\right\|^2-(\nabla u_{\tau\tau}u)\right)d\tau \quad (12)$$

Let us consider the terms on the right hand side of inequality (12) respectively: before

$$-\delta\int_0^t e^{\delta\tau}(u_{\tau\tau}u)\,d\tau$$

Let's look at the integral:

For this, let's examine the following expression

$$-\frac{d}{dt}\left(\int_0^t e^{\delta\tau}(u_\tau u)\,d\tau\right)=-\delta\int_0^t e^{\delta\tau}(u_\tau u)\,d\tau-\int_0^t e^{\delta\tau}(u_{\tau\tau}u)\,d\tau-\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau$$
$$-e^{\delta\tau}(u_t u)+(u_1 u_0)=-\delta\int_0^t e^{\delta\tau}(u_\tau u)\,d\tau-\int_0^t e^{\delta\tau}(u_{\tau\tau}u)\,d\tau-\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau$$

Now multiply both sides of the equation by δ, $(\delta>0)$ and leave the desired integral alone

$$-\delta\int_0^t e^{\delta\tau}(u_{\tau\tau}u)\,d\tau=-\delta e^{\delta\tau}(u_t u)+\delta(u_1 u_0)+\delta^2\int_0^t e^{\delta\tau}(u_\tau u)\,d\tau+\delta\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau$$

we obtain equality.

If we apply Young's inequality to the remaining terms in this equation except the last term, we obtain the following inequality.

$$-\delta\int_0^t e^{\delta\tau}(u_{\tau\tau}u)\,d\tau\le\frac{\delta}{2}e^{\delta\tau}\left(\left\|u_t\right\|^2+\left\|u\right\|^2\right)+\frac{\delta}{2}\left(\left\|u_1\right\|^2+\left\|u_0\right\|^2\right)$$
$$+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\left\|u_\tau\right\|^2+\left\|u\right\|^2\right)d\tau+\delta\int_0^t e^{\delta\tau}\left\|u_\tau\right\|^2 d\tau \quad (13)$$

Now let's look at the following integrand

$$-\delta\int_0^t e^{\delta\tau}\frac{d}{2d\tau}\|\nabla u\|^2\,d\tau$$

For this, let us examine the following expression:

$$-\int_0^t \frac{d}{2dt}\left(e^{\delta\tau}\|\nabla u\|^2\right)d\tau = -\frac{\delta}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau - \int_0^t e^{\delta\tau}\frac{d}{2dt}\|\nabla u\|^2\,d\tau$$

$$-\frac{1}{2}e^{\delta\tau}\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_0\|^2 = -\frac{\delta}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau - \int_0^t e^{\delta\tau}\frac{d}{2dt}\|\nabla u\|^2\,d\tau$$

Now multiply both sides of the equation by δ, $(\delta > 0)$ and leave the desired integral alone

$$-\delta\int_0^t e^{\delta\tau}\frac{d}{2d\tau}\|\nabla u\|^2\,d\tau = -\frac{\delta}{2}e^{\delta t}\|\nabla u\|^2 + \frac{\delta}{2}\|\nabla u_0\|^2 + \frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau$$

we obtain equality.

If we drop the strictly negative term in the above equation

$$-\delta\int_0^t e^{\delta\tau}\frac{d}{2d\tau}\|\nabla u\|^2\,d\tau \le \frac{\delta}{2}\|\nabla u_0\|^2 + \frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau \tag{14}$$

we obtain the inequality. Finally, let us look at the following integrand:

$$-\delta\int_0^t e^{\delta\tau}\left(\nabla u_{\tau\tau}\nabla u\right)d\tau$$

For this, let us examine the following expression:

$$-\frac{d}{dt}\left(\int_0^t e^{\delta\tau}\left(\nabla u_\tau\nabla u\right)d\tau\right) = -\delta\int_0^t e^{\delta\tau}\left(\nabla u_\tau\nabla u\right)d\tau - \int_0^t\left(\nabla u_{\tau\tau}\nabla u\right)d\tau - \int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau$$

$$-e^{\delta\tau}\left(\nabla u_\tau\nabla u\right) + \left(\nabla u_1\nabla u_0\right) = -\delta\int_0^t e^{\delta\tau}\left(\nabla u_\tau\nabla u\right)d\tau - \int_0^t\left(\nabla u_{\tau\tau}\nabla u\right)d\tau - \int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau$$

Now multiply both sides of the equation by δ, $(\delta > 0)$ and leave the desired integral alone

$$\begin{aligned}-\delta\int_0^t e^{\delta\tau}\left(\nabla u_{\tau\tau}\nabla u\right)d\tau &= -\delta e^{\delta\tau}\left(\nabla u_t\nabla u\right) + \delta\left(\nabla u_1\nabla u_0\right)\\ &\quad + \delta^2\int_0^t e^{\delta\tau}\left(\nabla u_\tau\nabla u\right)d\tau + \delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau\end{aligned}$$

we obtain equality.

If we apply Young's inequality to the remaining terms in this equation except the last term, we obtain the following inequality.

$$\begin{aligned}-\delta\int_0^t e^{\delta\tau}\left(\nabla u_{\tau\tau}\nabla u\right)d\tau &\le \frac{\delta}{2}e^{\delta t}\left(\|\nabla u_t\|^2 + \|\nabla u\|^2\right) + \frac{\delta}{2}\left(\|\nabla u_1\|^2 + \|\nabla u_0\|^2\right)\\ &\quad + \frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\|\nabla u_\tau\|^2 + \|\nabla u\|^2\right)d\tau + \delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau\end{aligned} \tag{15}$$

If we consider inequalities (13), (14) and (15) in (9) and (12)

$$
\begin{aligned}
& e^{\delta t}E(t)+\int_0^1 e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau \\
& \le E(0)+\delta\int_0^t e^{\delta\tau}\left(\frac{\|u_\tau\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_\tau\|^2}{2}\right)d\tau+\frac{\delta}{2}e^{\delta t}\left(\|\nabla u_t\|^2+\|\nabla u\|^2\right) \\
& \quad +\frac{\delta}{2}\left(\|\nabla u_1\|^2+\|\nabla u_0\|^2\right)+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\|\nabla u_\tau\|^2+\|\nabla u\|^2\right)d\tau+\delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau \qquad (16)\\
& \quad +\frac{\delta}{2}\|\nabla u_0\|^2+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau+\frac{\delta}{2}e^{\delta t}\left(\|\nabla u_t\|^2+\|\nabla u\|^2\right) \\
& \quad +\frac{\delta}{2}\left(\|\nabla u_1\|^2+\|\nabla u_0\|^2\right)+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\|\nabla u_\tau\|^2+\|\nabla u\|^2\right)d\tau+\delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau
\end{aligned}
$$

inequality is obtained.

(16) in the inequality

$$
\frac{\delta}{2}\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau,\ \frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau,\ \delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau,\ \frac{\delta}{2}e^{\delta t}\|\nabla u\|^2
$$

Using the Sobolev-Poincare inequality $\forall u\in H_0^1(\Omega)$, $\|u\|^2\le\lambda_0\|\nabla u\|^2$ we obtain the following inequalities.

$$
\delta\int_0^t e^{\delta\tau}\|u_\tau\|^2\,d\tau\le\delta\lambda_0\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau
$$

$$
\frac{\delta}{2}\int_0^t e^{\delta\tau}\|u_\tau\|^2\,d\tau\le\lambda_0\frac{\delta}{2}\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau
$$

$$
\frac{\delta}{2}e^{\delta t}\|u\|^2\le\lambda_0\frac{\delta}{2}e^{\delta t}\|\nabla u\|^2
$$

$$
\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|u\|^2\,d\tau\le\lambda_0\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau
$$

If we substitute these inequalities in (16)

$$
\begin{aligned}
& e^{\delta t}E(t)+\int_0^1 e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau \\
& \le E(0)+\delta\int_0^t e^{\delta\tau}\left(\frac{\lambda_0\|u_\tau\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_\tau\|^2}{2}\right)d\tau \\
& \quad +\frac{\delta}{2}\left(\|\nabla u_t\|^2+\lambda_0\|\nabla u\|^2\right)+\frac{\delta}{2}\left(\|\nabla u_1\|^2+\|\nabla u_0\|^2\right) \\
& \quad +\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\|\nabla u_\tau\|^2+\lambda_0\|\nabla u\|^2\right)d\tau+\delta\lambda_0\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau \qquad (17)\\
& \quad +\frac{\delta}{2}\|\nabla u_0\|^2+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\|\nabla u\|^2\,d\tau+\frac{\delta}{2}e^{\delta t}\left(\|\nabla u_t\|^2+\|\nabla u\|^2\right) \\
& \quad +\frac{\delta}{2}\left(\|\nabla u_1\|^2+\|\nabla u_0\|^2\right)+\frac{\delta^2}{2}\int_0^t e^{\delta\tau}\left(\|\nabla u_\tau\|^2+\|\nabla u\|^2\right)d\tau \\
& \quad +\delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau
\end{aligned}
$$

we obtain the inequality.

We can rearrange the terms on the right-hand side of inequality (17) as follows.

$$\frac{3}{2}\delta\lambda_0\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau+\frac{3}{2}\delta\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau=\frac{3}{2}\delta(1+\lambda_0)\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau$$

$$\delta e^{\delta t}\left(\frac{\|u_t\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_t\|^2}{2}+\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,dx\right)\le c_1\delta e^{\delta t}E(t)$$

$$\delta^2\int_0^t e^{\delta\tau}\left(\frac{\|u_\tau\|^2}{2}+\frac{\|\nabla u\|^2}{2}+\frac{\|\nabla u_\tau\|^2}{2}+\frac{\|\nabla^3 u\|^2}{2}-\int_0^1 F(u)\,dx\right)d\tau\le c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau$$

If we substitute these expressions in (17)

$$\begin{aligned}&e^{\delta t}E(t)+\int_0^1 e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau\\&\le E(0)+\frac{3}{2}\delta(1+\lambda_0)\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau+c_1\delta e^{\delta\tau}E(t)+c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau\end{aligned}\tag{18}$$

we get.

From (18), substitute δ for $0<\min\left\{\frac{2}{3(1+\lambda_0)},\frac{1}{2c_1}\right\}$ to get the following inequality

$$\begin{aligned}&e^{\delta t}E(t)+\int_0^1 e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau\\&\le E(0)+\int_0^t e^{\delta\tau}\|\nabla u_\tau\|^2\,d\tau+\frac{e^{\delta\tau}}{2}E(t)+c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau\end{aligned}$$

$$e^{\delta t}E(t)\le E(0)+\frac{e^{\delta t}}{2}E(t)+c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau$$

$$\frac{e^{\delta t}}{2}E(t)\le E(0)+c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau$$

$$e^{\delta t}E(t)\le 2E(0)+2c_2\delta^2\int_0^t e^{\delta\tau}E(\tau)\,d\tau$$

we get.

If we apply Gronwall's lemma here

$$e^{\delta t}E(t)\le 2E(0)e^{2c_2\delta^2 t},\ \ 0\le t<\infty$$

and

$$E(t)\le CE(0)e^{-\lambda t}$$

we get.

If we select δ from the following range

$$0<\delta<\min\left\{\frac{2}{3(1+\lambda_0)},\frac{1}{2c_1},\frac{1}{2c_2}\right\}$$

(4). That is, we show that the solution of the problem converges to zero of exponential order when $t \to \infty$ under the condition $c_1, c_2 > 1$.

5. Conclusion

This paper has investigated the asymptotic behavior of the strong solution to a class of sixth-order nonlinear evolutions equations with both dispersive terms.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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