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Controllability of Non-Linear Stochastic Systems with Bilinear Mode and Delays in Control

Murugesan Sathya[1*], Emmanuel Hagenimana[2*]

[1]Department of Mathematics, Muthayammal College of Arts and Science, Namakkal, Tamilnadu, India
[2]University of Applied Sciences, Institut d'Enseignement Supérieur de Ruhengeri (INES-Ruhengeri), Musanze, Rwanda
Email: msathyapu@gmail.com, ehagenimana@ines.ac.rw

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Abstract

This study examines the comparative controllability of bilinear stochastic systems. By applying the Banach fixed-point theorem, necessary conditions for the relative controllability of stochastic bilinear systems with delays are derived. A numerical example is presented to illustrate the applicability and effectiveness of the proposed results.

Keywords

Relative Controllability, Stochastic Bilinear Systems, Banach Fixed-Point Theorem

1. Introduction

Bilinear systems are a specific type of non-linear system that can model a wide range of significant physical processes. Bilinear models (BM) are effective in approximating various non-linear systems and are commonly employed to represent non-linear phenomena in fields suchas sign with image treating, as well as communication to this system of modeling. These models are particularly utilized in various domains such as channel equalization, echo cancellation, non-linear tracking, and the modeling of multiplicative disturbances and tracking. Additionally, refer to [1], they find applications in a wide range of fields including manufacturing, socio-economics, and biology.

Mathematically, bilinear models offer a more manageable structure compared to Volterra models for non-linear systems. Furthermore, bilinear models can capture Bilinear models capture the dynamics of non-linear systems more accurately than

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*Both authors contribute equally to this article.

linear models. Consequently, modeling and controlling non-linear systems within the bilinear framework present significant challenges in engineering. Over the past few decades, a considerable body of literature has emerged, focusing on the control issues related to these systems Many real-world systems can be adequately approximated by bilinear models rather than linear models [2] where non-linear systems of Stochastic bilinear systems (SBS) can be viewed as a generalization of bilinear systems or as a specific subset of stochastic non-linear systems, and have been extensively studied in the literature ([3]-[8]). While these works provide significant insights into SBS, several challenging issues remain unresolved. Controllability [9]-[12] is a critical issue in the context of stochastic non-linear systems. The stability and robust stabilization of non-linear singular systems were investigated by [13]-[15], who also addressed local asymptotic stabilization for general singular non-linear systems. [16] [17] explored stabilization of singular systems with non-linear perturbations. Furthermore, the global asymptotic stabilization of singular bilevel systems (SBS) has been studied by [18]. In their studies, they established a set of sufficient conditions where the focus is on achieving global asymptotic stabilization through continuous static state systems, this approach not only guarantees the existence of a solution but also ensures the global stabilization of the closed-loop system.

[19] employed the WF method for time-invariant stochastic bilinear systems (SBS), while [20], [21] utilized Haar wavelets to analyze time-varying SBS. In this paper, we propose a new numerical technique for solving time-varying SBS.

In the past two decades, considerable attention has been directed toward the analysis of linear systems subjected to multiplicative random disturbances, where the system states are influenced by a random sequence, commonly referred to as stochastic bilinear systems to the controllability of linear stochastic system as can be seen in [22]-[24]. This interest has been driven by various application areas, including population models, nuclear fission, heat transfer, and immunology, among others. Several aspects of the structural properties of such systems, both in discrete and continuous time, as well as in finite and infinite dimensions, have been explored in the existing literature. These investigations address fundamental questions and provide practical and theoretical motivations for considering this particular class of systems (for additional references, see...).

The focus or aim of this work is to establish the given necessary of conditions for full controllability of the stochastic linear system [25]-[28] of a finite-dimensional stochastically bilinear system. In the absence of multiplicative noise, which corresponds to the deterministic case, this problem reduces to the conventional control problem for continuous-time linear systems. The problem addressed in this study is categorized as a stochastic optimum control problem as can be seen in [6] due to the multiplicative noise influencing the system's state. This noise inherently makes the problem probabilistic, as the system's state becomes a stochastic process, independent of any additive disturbances.

In the deterministic case, where no multiplicative random disturbances are present, the problem simplifies to the conventional H control problem within a state-space framework [29].

This research is structured such as: Section 2 provides the fundamental construction of mixture functions, specifically block of pulse and Legendre of polynomials, which are crucial for the consequent analysis. Section 3, discusses the construction of time varying stochastic bilinear systems (SBS). In Section 4, the suggested methods are applied to estimate the SBS, while Section 5, presents the mathematical results and validates the exactitude of the proposed mathematical scheme through demonstrative examples.

The investigation of this problem is motivated by three primary factors: It is well-established that, in the linear case, solving the stochastic differential equations realization problem also provides solutions to linear filtering [30] and [31]. Thus, successfully addressing the bilinear stochastic control problem is expected to yield valuable insights into non-linear stochastic control issues. A notable gap exists within bilinear system theory, particularly regarding bilinear stochastic systems. From an engineering perspective, solving the bilinear stochastic control problem represents a crucial. This represents a critical step in the non-linear modeling of stochastic processes. The majority of research on the realization of bilinear systems has primarily focused on deterministic bilinear systems for instance refer to [32] and [33], the bilinear stochatic of controllable system is well defined. Assumed a mechanism system, approximately that a point y is accessible from the required point x if there is an acceptable control u and the finite state time T, in such that, the trajectory with initial conditions x of the trajectory field identified by u permits through y at time T. Indicate the set of the points accessible from x as $R(x)$. If $R(x)$ is equivalent to the state space for each point, x, in the given state space, then the system is absolutely well-regulated.

2. System Framework

Let us assume that the nonhomogeneous bilinear stochastic of the control system be modeled by Ito equation in the following form

$$\left.\begin{aligned} &\mathrm{d}x(t)=\left[A(t)+\sum_{i=1}^{m}A_i(t)u_i(t)\right]x(t)\mathrm{d}t+\sum_{i=0}^{M}B_i(t)u\left(\delta_i(t)\right)\mathrm{d}t+\tilde{\sigma}(t)\mathrm{d}w(t)\\ &x(0)=x_0,\quad t\in[t_0,T]\end{aligned}\right\}\tag{1}$$

and the equivalent non-linear stochastic control system

$$\left.\begin{aligned} &\mathrm{d}x(t)=\left[A(t)+\sum_{i=1}^{m}A_i(t)u_i(t)\right]x(t)\mathrm{d}t+\sum_{i=0}^{M}B_i(t)u\left(\delta_i(t)\right)\mathrm{d}t+\sigma\left(t,x(t)\right)\mathrm{d}w(t)\\ &x(0)=x_0,\quad t\in[t_0,T]\end{aligned}\right\}\tag{2}$$

where $x(t)\in\mathbb{R}^n$ It represents the system's state at a given moment in time, $u(t)$ is an $m\times 1$ input vector with components u_i and $A_i(t)$ and $B_i(t)(i=1,2,\cdots,m)$ are $n\times n$ and $n\times m$ matrix valued functions respectively. $\tilde{\sigma}:[t_0,T]\to\mathbb{R}^{n\times n}$ and $\sigma:[t_0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$. The given functions

$\delta_i : [t_0, T] \to \mathbb{R}, i = 0, 1, \cdots, M$ They are constantly which is differentiable to the second order and exhibit strict monotonicity $[t_0, T]$ and moreover

$$\delta_i(t) \le t \text{ for } t \in [t_0, T], i = 0, 1, \cdots, M$$

Here, the control function $u(t)$ regulates the system state by fusing the values of $u(t)$ at various time moments $\delta_i(t), i = 1, \cdots, n$, where $\delta_i(t)$ are time varying delays systems [34] and [35] as well at the current time t, which assumes that the current state of the systems depends not only on the current value of $u(t)$ but also on its values after certain lags $\delta_i(t), i = 1, \cdots, n$.

For a given initial condition (1) and any admissible control $u \in U_{ad}$, there exists a unique solution $x(t; x_0, u) \in L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ of the linear system (1) which can be represented in the following integral form:

$$\begin{aligned} x(t) = F(t, t_0)x_0 + \int_{t_0}^{t} F(t, s) \sum_{i=0}^{M} B_i(s) u(\delta_i(s)) \mathrm{d}s \\ + \int_{t_0}^{t} F(t, s) \tilde{\sigma}(s) \mathrm{d}w(s) \end{aligned} \tag{3}$$

where $F(t, t_0)$ It denotes the transition matrix of the linear system. $\left[A(t) + \sum_{i=1}^{m} A_i(t) u_i(t)\right] x(t)$ with $F(t_0, t_0) = I$, the identity matrix.

We now present the time-lead of the functions $r_i(t) : [\delta_i(t_0), \delta_i(T)] \to [t_0, T]$, so that

$$r_i(\delta_i(t)) = t, i = 0, 1, \cdots, M, t \in [t_0, T].$$

We also define what we called widespread state of the given system (1) at the given time t for the set $y(t) = \{x(t), u_t(s)\}$ for $u_t(s) = u(s)$ for $s \in \left[\min_i \delta_i(t), t\right)$.

Taking $\delta_i(s) = \tau$ in (3) and applying the time lead of the function $r_i(t)$, we get

$$s = r_i(\tau) \text{ and } \mathrm{d}s = \dot{r}_i(\tau) \mathrm{d}\tau.$$

Thus (3) can be taken as

$$\begin{aligned} x(t) = F(t, t_0)x_0 + \sum_{i=0}^{M} \int_{\delta_i(t_0)}^{\delta_i(t)} F(t, r_i(s)) B_i(r_i(s)) \dot{r}_i(s) u(s) \mathrm{d}s \\ + \int_{t_0}^{t} F(t, s) \tilde{\sigma}(s) \mathrm{d}w(s) \end{aligned} \tag{4}$$

By taking into consideration to the loss of generalization, it can be expected that

$$\delta_0(t) = t$$

and the next differences hold for $t = T$:

$$\begin{aligned} \delta_M(T) \le \delta_{M-1}(T) \le \cdots \le \delta_{m+1}(T) \le t_0 \\ = \delta_m(T) < \delta_{m-1}(T) = \cdots = \delta_1(T) = \delta_0(T) = T \end{aligned} \tag{5}$$

By applying (5), the equation (4) for $t = T$ will be stated as

$$\begin{aligned} x(T) &= F(T,t_0)x_0 + \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \sum_{i=0}^{m}\int_{t_0}^{\delta_i(T)} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s)\mathrm{d}s \\ &\quad + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(T)} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \int_{t_0}^{T} F(T,s)\tilde{\sigma}(s)\mathrm{d}w(s) \\ &= F(T,t_0)x_0 + \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \sum_{i=0}^{m}\int_{t_0}^{T} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u(s)\mathrm{d}s \\ &\quad + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(T)} F(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \int_{t_0}^{T} F(T,s)\tilde{\sigma}(s)\mathrm{d}w(s) \end{aligned}$$

It is significant to note that the preceding term of this third integral has to be equivalent to zero due to the given definition in the time-lead of the function. $r_m(t)$ that is taken as constant term $r_m(t_0)$ in that interval $[t_0,T]$.

For ease of reference, we present the subsequent notations:

$$\begin{aligned} H(t,t_0) &= \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F(t,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(t)} F(t,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ G_i(t,s) &= \sum_{j=0}^{i} F(t,r_j(s))B_j(r_j(s))\dot{r}_j(s), i=1,2,\cdots,M. \end{aligned}$$

We describe the linear function and the bounded of the control of the operator $\mathbb{L}: L_2^{\mathcal{F}}([t_0,T],\mathbb{R}^l) \to L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ as follows:

$$\mathbb{L}u = \int_{t_0}^{T} G_m(T,s)u(s)\mathrm{d}s$$

and it is the adjoint bounded linear of the operator $\mathbb{L}^*: L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n) \to L_2^{\mathcal{F}}([t_0,T],\mathbb{R}^l)$ is defined by

$$(\mathbb{L}^*z)(t) = G_m^*(T,t)\mathbb{E}\{z \mid \mathcal{F}_t\}, \quad t \in [t_0,T]$$

in such that, the equation star $(*)$ means the adjoint of the matrix. From the given above symbolization, it is given that the set of given all states system that can be reached from the initial state is given by $x(t_0) = x_0 \in L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n)$ in time $T > 0$, The system, governed by admissible controls, takes the form of

$$\begin{aligned} \mathcal{R}_T(\mathcal{U}_{ad}) &= \left\{x(T;x_0,u) \in L_2(\Omega,\mathcal{F}_T,\mathbb{R}^n) : u(\cdot) \in \mathcal{U}_{ad}\right\} \\ &= \Phi(T,t_0)x_0 + \operatorname{Im}\mathbb{L} + \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} \Phi(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(T)} \Phi(T,r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\mathrm{d}s \\ &\quad + \int_{t_0}^{T} \Phi(T,s)\tilde{\sigma}(s)\mathrm{d}w(s) \end{aligned}$$

The given linear controllability of operator $\mathcal{W}: L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right) \to L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ is associated with the system (1) is defined by

$$\mathcal{W} = \mathbb{L}\mathbb{L}^*\{\cdot\} = \int_{t_0}^{T} G_m(T,s) G_m^*(T,s) \mathbb{E}\{\cdot \mid \mathcal{F}_t\} \mathrm{d}s$$

and the deterministic to controllability matrix $\Gamma_s^T \in \mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^n\right)$ is

$$\Gamma_s^T = \int_s^T G_m(T,s) G_m^*(T,s) \mathrm{d}s, \ s \in [t_0, T].$$

Definition 2.1 *The given stochastic method* (1) *is said to be comparatively controllable on* $[t_0, T]$ *when, for every whole state system* $y(t_0)$ *and every* $x_1 \in \mathbb{R}^n$, *there exists or we may find a control* $u(t)$ *defined by* $[t_0, T]$ *In such a way that is the trajectory to the given stochastic system linked with it* (1) *satisfies thefollowing condition* $x(T) = x_1$.

Definition 2.2 *The stated stochastic method* (1) *is supposed to be relatively accurate controllable on* $[t_0, T]$ *if*

$$\mathcal{R}_T(\mathcal{U}_{ad}) = L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right),$$

that is, if every points in $L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ can be accurately reached at the given time T from any given arbitrary to the initial point $x_0 \in L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ at time $T > 0$.

Definition 2.3 *The stochastic method* (1) *is taken to be relative estimated controllable on* $[t_0, T]$ *if*

$$\overline{\mathcal{R}_T(\mathcal{U}_{ad})} = L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right),$$

that means, if all the given points in $L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ be able to be approximately extended at time T from any kind of arbitrary of the initial point which is $x_0 \in L_2\left(\Omega, \mathcal{F}_T, \mathbb{R}^n\right)$ at time $T > 0$.

3. Bilinear Systems

From this unit, we review key outcomes necessary for establishing the qualified controllability of the given linear stochastic method (1).

Let us take the corresponding of the deterministic system of this kind

$$z'(t) = \left[A(t) + \sum_{i=1}^{m} A_i(t) v_i(t)\right] z(t)\mathrm{d}t + \sum_{i=0}^{M} B_i(t) v(\delta_i(t))\mathrm{d}t \tag{6}$$

In which the acceptable controls $v \in L_2\left([t_0, T], \mathbb{R}^l\right)$.

For that deterministic of the system (6) let us represent by R_T the set of all of states system reachable to the initial state $z(t_0) = z_0$ in time $T > 0$ using the given admissible of controls system.

Lemma 3.1 *The deterministic of the state system* (6) *is said or taken to be reasonably controllability on* $[t_0, T]$ *if* $R_T = \mathbb{R}^n$

Lemma 3.2 *The given situations are taken to be equally*:

i) Deterministic of the system (6) is relatively implies the controllability on $[t_0, T]$.

ii) The controllability of the matrix defined by $\mathcal{W}$ is non-singular.

The following given lemma establish the relativity of the controllable of the corresponding deterministic of the linear state system [36]-[44] where these are corresponding to both relatively exact controllability which is relatively approximate of the controllability of the linearity of the stochastic method.

Lemma 3.3 *The given circumstances are corresponding*:

i) The deterministic of the state system is relatively said controllable at $[t_0, T]$.

ii) The Stochastic of the state system is also said relatively precise controllable at $[t_0, T]$.

iii) The Stochastic linear system is closely relatively approximated to the controllability of $[t_0, T]$.

We notice that from the given above mentioned research, it observed that, if the given linear stochastic of the system is closely relatively totally controllability as can be seen in [45]-[48], therefore, the given operator defined by $\mathcal{W}$ is strictly positive to the definite integral and that is, the inverse of the linear Operator defined $\mathcal{W}^{-1}$ is totally bounded, say

$$\left\|\mathcal{W}^{-1}\right\| \leq k_3 \tag{7}$$

where k_3 is constant.

Now, shall express and resolve the minimum of the given energyof the control of problem for relative precise controllable of stochastic dynamic organization. By the fact that if the given operator defined $\mathcal{W}^{-1}$ is limited we may build the control given by $u^0(t), t \in [t_0, T]$ which is that navigates the system which is coming from the given initial of state-run x_0 to a favorite of the final of the given state x_1 at time T.

Lemma 3.4 *Accept that the given stochastic of the state system* (1) *is closely relatively precisely controllable to the* $[t_0, T]$. *Then, for the given arbitrary condition or target of* $x_1 \in L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ *and* $\tilde{\sigma}(\cdot) \in L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^{n\times n})$, *the control*

$$\begin{aligned} u^0(t) = G_m^*(t,T)\mathbb{E}\Big\{\mathcal{W}^{-1}\Big(x_1 - F(T,t_0)x_0 \\ -\sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\ -\sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(T)} F(T, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\ -\int_{t_0}^{T} F(T,s)\tilde{\sigma}(s)\,\mathrm{d}w(s)\Big) \mid \mathcal{F}_t\Big\} \end{aligned} \tag{8}$$

transfers the given state system

$$\begin{aligned} x(t) = F(t,t_0)x_0 + \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\ + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(t)} F(t, r_i(s))B_i(r_i(s))\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\ + \int_{t_0}^{t} G_m(t,s)u(s)\,\mathrm{d}s + \int_{t_0}^{t} F(t,s)\tilde{\sigma}(s)\,\mathrm{d}w(s) \end{aligned}$$

from the $x_0 \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}^n$ at time T. Furthermore, among all of the admissible of the controls $u(t)$ transmitting the initial of the state x_0 to the final of the given state system x_1 at the given time $T > 0$, the controllable system $u^0(t)$ reduces the integrity of the performance of the index

$$\mathcal{J}(u) = \mathbb{E}\int_{t_0}^{T} \left\| u(t) \right\|^2 \mathrm{d}t.$$

Proof. Since the given stochastic dynamic method (1) is totally relatively strict controllable on $[t_0, T]$, the given controllability to the operator $\mathcal{W}$ is taken as invertible, and it is opposite $\mathcal{W}^{-1}$ which is the linear and bounded of that given operator, that is $\mathcal{W}^{-1} \in \mathcal{L}\left(L_2\left(\Omega, \mathcal{F}_t, \mathbb{R}^n\right), L_2\left(\Omega, \mathcal{F}_t, \mathbb{R}^n\right)\right)$. Replacing the control system $u^0(t)$ into the given solution to the given formula of the differential equation and replacing $t = T$, everyone can easily prove that the given control system (8) navigates the linear state system from x_0 to x_1. The next part of this proof is totally similar to that given in Theorem 2.

4. Non-Linear Systems

Pleasing into consideration or account that, the given notations and the given results, we may develop necessary controllability conditions of the state system for the semi state linear system of stochastic system with bilinear mode and stays in control of the form

$$\left.\begin{aligned} &\mathrm{d}x(t) = \left[A(t) + \sum_{i=1}^{m} A_i(t) u_i(t) \right] x(t)\,\mathrm{d}t + \sum_{i=0}^{M} B_i(t) u\left(\delta_i(t)\right)\mathrm{d}t + \sigma\left(t, x(t)\right)\mathrm{d}w(t) \\ &x(0) = x_0,\ t \in [t_0, T] \end{aligned}\right\} \tag{9}$$

Then the given solution to the linear system (9) can be stated in the given form

$$\begin{aligned} x(t) = {} & F(t, t_0) x_0 + \int_{t_0}^{t} F(t, s) \sum_{i=0}^{M} B_i(s) u\left(\delta_i(s)\right)\mathrm{d}s \\ & + \int_{t_0}^{t} F(t, s) \sigma\left(s, x(s)\right)\mathrm{d}w(s). \end{aligned}$$

Now applying the time principal function and we take

$$\begin{aligned} x(t) = {} & F(t, t_0) x_0 + \sum_{i=0}^{M} \int_{\delta_i(t_0)}^{\delta_i(t)} F\left(t, r_i(s)\right) B_i\left(r_i(s)\right) \dot{r}_i(s) u(s)\,\mathrm{d}s \\ & + \int_{t_0}^{t} F(t, s) \sigma\left(s, x(s)\right)\mathrm{d}w(s). \end{aligned} \tag{10}$$

and inequalities (5), the given above equation when $t = T$ could be stated as

$$\begin{aligned} x(T) = {} & F(T, t_0) x_0 + \sum_{i=0}^{m} \int_{\delta_i(t_0)}^{t_0} F\left(T, r_i(s)\right) B_i\left(r_i(s)\right) \dot{r}_i(s) u_{t_0}(s)\,\mathrm{d}s \\ & + \sum_{i=0}^{m} \int_{t_0}^{T} F\left(T, r_i(s)\right) B_i\left(r_i(s)\right) \dot{r}_i(s) u(s)\,\mathrm{d}s \\ & + \sum_{i=m+1}^{M} \int_{\delta_i(t_0)}^{\delta_i(T)} F\left(T, r_i(s)\right) B_i\left(r_i(s)\right) \dot{r}_i(s) u_{t_0}(s)\,\mathrm{d}s \\ & + \int_{t_0}^{t} F(T, s) \sigma\left(s, x(s)\right)\mathrm{d}w(s). \end{aligned}$$

Now consider or let us express the controllable of the operator and the controllability of function connected with the given system (9) as follows:

$$\mathcal{W} = \mathcal{W}(t_0, T) = \int_{t_0}^{T} G_m(T,s) G_m^*(T,s) \mathbb{E}\{\cdot \mid \mathcal{F}_t\} \mathrm{d}s$$

$$\begin{aligned} u(t) = G_m^*(T,t) \mathbb{E}\Big\{ \mathcal{W}^{-1}\Big(& x_1 - \Phi(T,t_0) x_0 \\ & - \sum_{i=0}^{m} \int_{\delta_i(t_0)}^{t_0} F(T, r_i(s)) B_i(r_i(s)) \dot{r}_i(s) u_{t_0}(s) \mathrm{d}s \\ & - \sum_{i=m+1}^{M} \int_{\delta_i(t_0)}^{\delta_i(T)} F(T, r_i(s)) B_i(r_i(s)) \dot{r}_i(s) u_{t_0}(s) \mathrm{d}s \\ & - \int_{t_0}^{T} F(T,s) \sigma(s, x(s)) \mathrm{d}w(s) \Big) \mid \mathcal{F}_t \Big\} \end{aligned} \tag{11}$$

for G_m is taken as in the state linear system.

Introducing (11) in (10), it is very easy to prove that the control of the system $u(t)$ transfers to the x_0 for the desired of vector x_1 at any given time T. For the demonstration of the given main result, we execute the following given assumptions of the data to the problem:

(H1) is the function σ is Lipschitz continuously, that is, for $x, y \in \mathbb{R}^n$ and $t_0 \le t \le T$ there exists a constant $L_1 > 0$ in such that

$$\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_1 \|x - y\|^2 .$$

(H2) the given function σ fulfills the usual of the linear growth to the condition, that, there exists a constant $L_2 > 0$ in such that for all $t \in [t_0, T]$ and all $x \in \mathbb{R}^n$

$$\|\sigma(t,x)\|^2 \le L_2 \left(1 + \|x\|^2\right).$$

Assume that $\mathcal{B}_2$ indicates that the Banach space to all square integral and $\mathcal{F}_t$ -adapted process $\varphi(t)$ with norm

$$\|x\|^2 := \sup_{t \in [t_0, T]} \mathbb{E}\|x(t)\|^2 .$$

Express the non-linear operator $\mathcal{P}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ by

$$\begin{aligned} (\mathcal{P}x)(t) = {} & F(t,t_0) x_0 + \sum_{i=0}^{m} \int_{\delta_i(t_0)}^{t_0} F(t, r_i(s)) B_i(r_i(s)) \dot{r}_i(s) u_{t_0}(s) \mathrm{d}s \\ & + \sum_{i=m+1}^{M} \int_{\delta_i(t_0)}^{\delta_i(t)} F(t, r_i(s)) B_i(r_i(s)) \dot{r}_i(s) u_{t_0}(s) \mathrm{d}s \\ & + \int_{t_0}^{t} G_m(t,s) u(s) \mathrm{d}s + \int_{t_0}^{t} F(t,s) \sigma(s, x(s)) \mathrm{d}w(s), \end{aligned} \tag{12}$$

Helped by the Lemma 3, it is revealed that, if the given operator $\mathcal{P}$ is well defined in the equation (12) has one fixed point and then the state system [9] have a solution given by $x(t)$ well-defined in the equatin Eq. (10) with respectivly to $u(\cdot)$, with $(\mathcal{P}x)(T) = x(T) = x_1$, which suggests that, for the system of the equation (9) is relatively totally controllable. Hence, the given problem to the controllability of the given semi-linear state system (9) can be condensed into the existence of the unique fixed points of the given operator $\mathcal{P}$.

Therefore, for our suitability, let us present the following notations:

$$M = \max\left\{\|F(t,s)\|^2 : t_0 \le s < t \le T\right\},$$

$$k_1 = \max\left\{\left\|\Gamma_s^T\right\|^2 : s \in \left[t_0, T\right]\right\},$$

$$k_2 = \max\left\{\left\|H\left(t, t_0\right)\right\|^2 : t_0 \le t \le T\right\}$$

and we have

$$\left\|\mathcal{W}^{-1}\right\| \le k_3.$$

Theorem 4.1 *Accept that, the given conditions* $(H1)$-$(H2)$ *holds and assume that, the given linear state of stochastic system* (1) *is closely relatively totally controllable. Further, if the following inequality*

$$2ML_1\left(1 + k_1 k_3\right)T < 1 \tag{13}$$

is fulfilled, then the given semi-linear of the stochastic system of equation (9) is relatively totally controllable.

Proof. To demonstrate that, the system is relative controllable of the system of Equation (9), it is sufficient to express that the linear operator $\mathcal{P}$ has a given fixed points in $\mathcal{B}_2$. To do so, we can work the contraction of the mapping of principle. To smear the principle, first we demonstrate that $\mathcal{P}$ applies $\mathcal{B}_2$ with itself. Therefore, by using Lemma 3, we get

$$\begin{aligned}
&\mathbb{E}\left\|(\mathcal{P}x)(t)\right\|^2 \\
&= \mathbb{E}\Bigg\| F\left(t, t_0\right)x_0 + \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F\left(t, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\
&\quad + \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(t)} F\left(t, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t} G_m(t,s)u(s)\,\mathrm{d}s + \int_{t_0}^{t} F(t,s)\sigma\left(s, x(s)\right)\mathrm{d}w(s)\Bigg\|^2 \qquad (14)\\
&\le 4\mathbb{E}\left\|F\left(t, t_0\right)\right\|^2\left\|x_0\right\|^2 + 4\mathbb{E}\left\|\sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F\left(t, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(t)} F\left(t, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s\right\|^2 \\
&\quad + 4\mathbb{E}\left\|\int_{t_0}^{t} G_m(t,s)u(s)\,\mathrm{d}s\right\|^2 + 4\mathbb{E}\left\|\int_{t_0}^{t} F(t,s)\sigma\left(s, x(s)\right)\mathrm{d}w(s)\right\|^2.
\end{aligned}$$

To simplify this, first let us consider the fourth term in the above inequalities

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_0}^{t} G_m(t,\tau)u(\tau)\,\mathrm{d}\tau\right\|^2 \\
&= \mathbb{E}\Bigg\|\int_{t_0}^{t} G_m(t,\tau)G_m^*(T,\tau)\mathbb{E}\Big\{\mathcal{W}^{-1}\Big(x_1 - F\left(T, t_0\right)x_0 \\
&\quad - \sum_{i=0}^{m}\int_{\delta_i(t_0)}^{t_0} F\left(T, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \\
&\quad - \sum_{i=m+1}^{M}\int_{\delta_i(t_0)}^{\delta_i(T)} F\left(T, r_i(s)\right)B_i\left(r_i(s)\right)\dot{r}_i(s)u_{t_0}(s)\,\mathrm{d}s \qquad (15)\\
&\quad - \int_{t_0}^{T} F(T,s)\sigma\left(s, x(s)\right)\mathrm{d}w(s)\Big) \mid \mathcal{F}_t\Big\}\,\mathrm{d}\tau\Bigg\|^2 \\
&\le 4k_1k_3\left[\left\|x_1\right\|^2 + M\left\|x_0\right\|^2 + k_2 + ML_2\int_{t_0}^{t}\left(1 + \mathbb{E}\left\|x(s)\right\|^2\right)\mathrm{d}s\right].
\end{aligned}$$

Substituting the value of u(t) in the L.H.S of Equation (15) and then we completed a simplification [using M, k_1, k_2 and k_3] to get this answer.

Using (15) in (14), we have

$$\begin{aligned}\mathbb{E}\left\|(\mathcal{P}x)(t)\right\|^2 &\le 4M\left\|x_0\right\|^2+4k_2+16k_1k_3\Big[\left\|x_1\right\|^2+M\left\|x_0\right\|^2+k_2\\ &\quad +ML_2\int_{t_0}^{t}\left(1+\mathbb{E}\left\|x(s)\right\|^2\right)\mathrm{d}s\Big]+4ML_2\int_{t_0}^{t}\left(1+\mathbb{E}\left\|x(s)\right\|^2\right)\mathrm{d}s\\ &\le 4M\left\|x_0\right\|^2+4k_2+16k_1k_3\left(\left\|x_1\right\|^2+M\left\|x_0\right\|^2+k_2\right)\\ &\quad +\left(4M+16Mk_3k_1\right)L_2\int_{t_0}^{T}\left(1+\mathbb{E}\left\|x(s)\right\|^2\right)\mathrm{d}s.\end{aligned} \tag{16}$$

It is given that, from (16) and thegiven condition of (H2) that there is $C>0$ depending on x_0, T, L, M, k_1, k_2 and k_3 in such that

$$\mathbb{E}\left\|(\mathcal{P}x)(t)\right\|^2 \le C\left(1+\int_{t_0}^{T}\mathbb{E}\left\|x(r)\right\|^2\mathrm{d}r\right).$$

Thus we have

$$\mathbb{E}\left\|(\mathcal{P}x)(t)\right\|^2 \le C\left(1+T\sup_{r\in[t_0,T]}\mathbb{E}\left\|x(r)\right\|^2\right).$$

Consequently $\mathcal{P}$ maps $\mathcal{B}_2$ with itself. Secondly, we suppose that that, $\mathcal{P}$ is taken as a contraction mapping to the $\mathcal{B}_2$. Given $x, y\in\mathcal{B}_2$,

$$\begin{aligned}&\mathbb{E}\left\|(\mathcal{P}x_1)(t)-(\mathcal{P}x_2)(t)\right\|^2\\ &\le \mathbb{E}\Big\|\int_{t_0}^{t}F(t,s)\left[\sigma(s,x_1(s))-\sigma(s,x_2(s))\right]\mathrm{d}w(s)\\ &\qquad +\Gamma_{t_0}^{T}\mathcal{W}^{-1}\left(\int_{t_0}^{T}F(T,\tau)\left[\sigma(\tau,x_2(\tau))-\sigma(\tau,x_1(\tau))\right]\mathrm{d}w(\tau)\right)\Big\|^2\\ &\le 2ML_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s+2Mk_1k_3L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s\\ &\le 2M\left(1+k_1k_3\right)L_1\int_{t_0}^{T}\mathbb{E}\left\|x_1(s)-x_2(s)\right\|^2\mathrm{d}s.\end{aligned}$$

It results that

$$\sup_{t\in[t_0,T]}\mathbb{E}\left\|(\mathcal{P}x_1)(t)-(\mathcal{P}x_2)(t)\right\|^2 \le 2ML_1\left(1+k_1k_3\right)T\sup_{t\in[t_0,T]}\mathbb{E}\left\|x_1(t)-x_2(t)\right\|^2.$$

Therefore, we accomplish from (13) that $\mathcal{P}$ is the contraction of mapping on $\mathcal{B}_2$. Then the given mapping $\mathcal{P}$ has the unique of the fixed point $x(\cdot)\in\mathcal{B}_2$, which is the required solution of Equation (10). Thus the system is relatively totally controllable to this $[t_0, T]$.

5. Example

To prove the following applicability to the given above-mentioned results, this section examines the following given semi-linear stochastic state system, which is well-defined by

$$\left.\begin{aligned}&\mathrm{d}x(t)=\left[A(t)+\sum_{i=1}^{m}A_i(t)u_i(t)\right]x(t)\mathrm{d}t+\sum_{i=0}^{M}B_i(t)u(\delta_i(t))\mathrm{d}t+\sigma(t,x(t))\mathrm{d}w(t)\\ &x(0)=x_0,\ t\in[t_0,T]\end{aligned}\right\} \tag{17}$$

The above system can be formulated in the form, with $M = 3$:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ u_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$u_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ u_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_0(t) = \begin{bmatrix} 1 & \mathrm{e}^{-0.3t} \\ t & 0.2 \end{bmatrix},\ B_1(t) = \begin{bmatrix} 0.02t & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_2(t) = \begin{bmatrix} \mathrm{e}^{-0.6t} & 0.05t^2 \\ 0 & 0 \end{bmatrix},\ B_3(t) = \begin{bmatrix} 0 & \mathrm{e}^{-0.1t} \\ 0 & 0 \end{bmatrix},$$

$$\sigma(t, x(t)) = \begin{bmatrix} \dfrac{|t|\mathrm{e}^t x_1(t) + t\sin(t + x_1(t))}{58} & 0 \\ 0 & \dfrac{|t|\mathrm{e}^t x_2(t) + t\sin(t + x_2(t))}{58} \end{bmatrix}.$$

More over,

$$\delta_0(t) = t,\ \delta_1(t) = 0.75t,\ \delta_2(t) = 0.5t,\ \delta_3(t) = 0.25t,$$

for $t \in [0, 2]$ and

$$\delta_m(t) < \delta_{m-1}(t) < \cdots < \delta_k(t) < \cdots < \delta_1(t) < \delta_0(t) = t$$

for $t \in [t_0, t_1]$. Consider the lead functions as follows,

$$r_0(t) = t,\ r_1(t) = \frac{4}{3}t,\ r_2(t) = 2t,\ r_3(t) = 4t,$$

More over, for $T = 2$, we have

$$\delta_3(2) < \delta_2(2) < \delta_1(2) < \delta_0(2) = 2$$

By taking into consideration to the given form of the matrices $A_0(t), A_1(t), A_2(t), u_0(t), u_1(t), u_2(t)$ Using the given formula for calculating the exponent of the matrix of the function, we get the transition of the matrix

$$F(t, t_0) = \begin{bmatrix} 2 + \mathrm{e}^{0.4t} & 0 \\ 0 & 1 + \mathrm{e}^{0.3t} + \mathrm{e}^t \end{bmatrix},$$

and the given controllability of the Grammian

$$\mathcal{W}(0, 2) = \int_0^2 G_m(t, s) G_m^*(t, s)\,\mathrm{d}s$$
$$= \begin{bmatrix} 68.82 & 101.516 \\ 101.516 & 176.732 \end{bmatrix}$$

Hence, we will have rank given by $W(0, 2) = 2$. Taking the final of the point as $x_T \in \mathbb{R}^2$. Moreover, it can be readily verified that for all value $x \in \mathbb{R}^2$,

$$\left\|\sigma\left(t,x\left(t\right)\right)\right\|^{2}\le\frac{1}{29}\left|x_{2}-x_{1}\right|.$$

everyone can understood that, the inequality (13) is fulfilled, and all others conditions drawn in Theorem 1 are met. Therefore, the given state system (17) is relatively totally controllable to the $\left[0,2\right]$, that is we the system (17) can be directed from x_0 to x_1.

6. Conclusion

In this research paper, controllability of bilinear stochastic systems by applying the Banach fixed point theorem, necessary and sufficient conditions for the relative controllability of stochastic bilinear systems with delays are satisfied and the numerical examples are demonstrated to emphisize the applicability and the applicability of the proposed research work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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