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Special Theory of Relativity and Generalization of Relativistic Momentum

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Abstract

Lorentz invariance admits more general definitions for the relativistic momentum. In this paper, we study some consequences that follow one such generalization.

Keywords

Special Theory of Relativity, Relativistic Momentum, Lorentz Invariance

1. Introduction

The cardinal principle of momentum conservation (linear and additive) was at stake under Lorentz transformations (nonlinear) if momentum is defined as Newton did via:

$$\boldsymbol{P}_N = mc\boldsymbol{\beta}, \quad \boldsymbol{\beta} = \mathbf{v}/c, \tag{1}$$

where m is the mass, $\boldsymbol{v}$ is the velocity and c is the speed of light. Einstein was left with two options, either to change Newton's definition of momentum or to give up the conservation of momentum. He opted for the first to maintain Lorentz invariance, the stroke of genius, Einstein defined the relativistic momentum as [1]:

$$\boldsymbol{P}_E = mc\gamma\boldsymbol{\beta}, \tag{2}$$

where γ is the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{3}$$

The rest of special theory of relativity is just a routine follow-up and standard kinetic energy turns out to be:

$$T_E = c\int \boldsymbol{\beta}\cdot \mathrm{d}\boldsymbol{p} = (\gamma - 1)mc^2. \tag{4}$$

The velocity independent term (the constant of integration) is mc^2 which is the rest energy E_o given by:

$$E_o = mc^2. \tag{5}$$

Einstein defined the total energy as:

$$E = E_o + T_E = mc^2 + T_E = \gamma mc^2. \tag{6}$$

This mass-energy equivalence is the crux and the most famous landmark of science almost synonymous with the name of Einstein [2]. Equations (2) and (6) lead to the fundamental equation:

$$\left(\frac{E}{E_o}\right)^2 - \left(\frac{c\boldsymbol{P}_E}{E_o}\right)^2 = 1. \tag{7}$$

Equation (7) is the result of the identity:

$$\gamma^2 - (\gamma\boldsymbol{\beta})^2 = 1. \tag{8}$$

2. New Relativistic Momentum

As has been remarked on by many [3] [4], Equation (7) was never used by Einstein in his papers or letters. In an earlier communication [5], we showed that the identity in Equation (8) admits a more general definition of relativistic momentum given by:

$$\boldsymbol{P}_r = mc\tanh^{-1}(\boldsymbol{\beta}), \tag{9}$$

which satisfies the Legendre equation:

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{(1-\beta^2)\frac{\mathrm{d}\boldsymbol{P}_r}{\mathrm{d}\beta}\right\} = 0, \tag{10}$$

which has singular points at $\beta = \pm 1$. If the main motive is to preserve the linear (and additive) momentum conservation under Lorentz transformation (non-linear), a natural choice will be to use the inverse hyperbolic definition for the relativistic momentum. In fact, the hyperbolic transformation [6] [7]:

$$\gamma = \cosh\phi$$

$$\gamma\boldsymbol{\beta} = \sinh\phi$$

has been used in special theory for the invariant space-time $(\boldsymbol{x}, ct)$ distance:

$$d^2 = \boldsymbol{x}^2 - c^2t^2.$$

We have chosen the hyperbolic angle ϕ as:

$$\phi = \frac{P_r}{mc}.$$

It follows that:

$$\mathrm{d}\boldsymbol{P}_r = \frac{mc\mathrm{d}\boldsymbol{\beta}}{1-\beta^2}. \tag{11}$$

The expression for kinetic energy then becomes:

$$T = c\int \boldsymbol{\beta} \cdot \mathrm{d}\boldsymbol{P}_r = E_o \int \frac{\boldsymbol{\beta} \cdot \mathrm{d}\boldsymbol{\beta}}{1-\beta^2} = E_o \ln \gamma, \tag{12}$$

where we have chosen the constant of integration to be mc. Notice that the expression for the Lorentz factor becomes:

$$\gamma = \frac{E}{E_o} = 1 + \frac{T_E}{E_o} = \sqrt{1 + \left(\frac{\boldsymbol{P}_E}{mc}\right)^2} = \cosh\left(\frac{\boldsymbol{P}_r}{mc}\right). \tag{13}$$

Therefore,

$$\begin{aligned} \frac{E}{E_o} &= 1 + \frac{1}{2}\left(\frac{\boldsymbol{P}_E}{mc}\right)^2 - \frac{1}{8}\left(\frac{\boldsymbol{P}_E}{mc}\right)^4 + \cdots \\ &= 1 + \frac{1}{2}\left(\frac{\boldsymbol{P}_r}{mc}\right)^2 + \frac{1}{24}\left(\frac{\boldsymbol{P}_r}{mc}\right)^4 + \cdots \end{aligned} \tag{14}$$

The existing body of experimental evidence overwhelmingly supporting Einstein's formulation of special relativity rests on the mass-energy equivalence in Equation (7) and this still remains unchanged with the proposed generalization in Equation (9). The real test will be the relation between relativistic momentum and velocity depicted in **Figure 1**.

We remark that there is no change in Einstein's Equation (6) and thus all results related to the Lorentz factor γ (since they are based on $\boldsymbol{v}$). While the velocity dependence on E (the measured ones) remains the same, the critical difference is in the dependence of E on the momentum and the dependence of relativistic momentum on velocity:

$$E - E_o = (\gamma - 1)E_o = T_E = E_o\left(\mathrm{e}^{\frac{T}{E_o}} - 1\right) \tag{15}$$

The change from $\boldsymbol{P}_E$ to $\boldsymbol{P}_r$ then leads to the expression for the force:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{P}_r}{\mathrm{d}t} = \frac{m\boldsymbol{a}}{1-\beta^2} = \gamma^2 m\boldsymbol{a}, \tag{16}$$

where $\boldsymbol{a}$ is the acceleration. Notice that for small momentum (non-relativistic): $P_E \approx P_r$. Equation (7) now becomes only approximate Dirac's equation [8] which is based on the quadratic equation in Equation (7) now becomes only approximate in terms of P_r but is still quadratic in P_E, but $P_E = mc\sinh\left(\frac{P_r}{mc}\right)$ The relativistic correction to the kinetic energy which is (negative) is given by [9]:

$$\Delta T_E = -\frac{1}{8}\frac{P_E^4}{m^3c^2}, \tag{17}$$

and according to the usual description, now becomes (positive):

$$\Delta T_r = \frac{1}{24}\frac{P_r^4}{m^3c^2}. \tag{18}$$

This change has a profound impact on the atomic spectrum. The relativistic

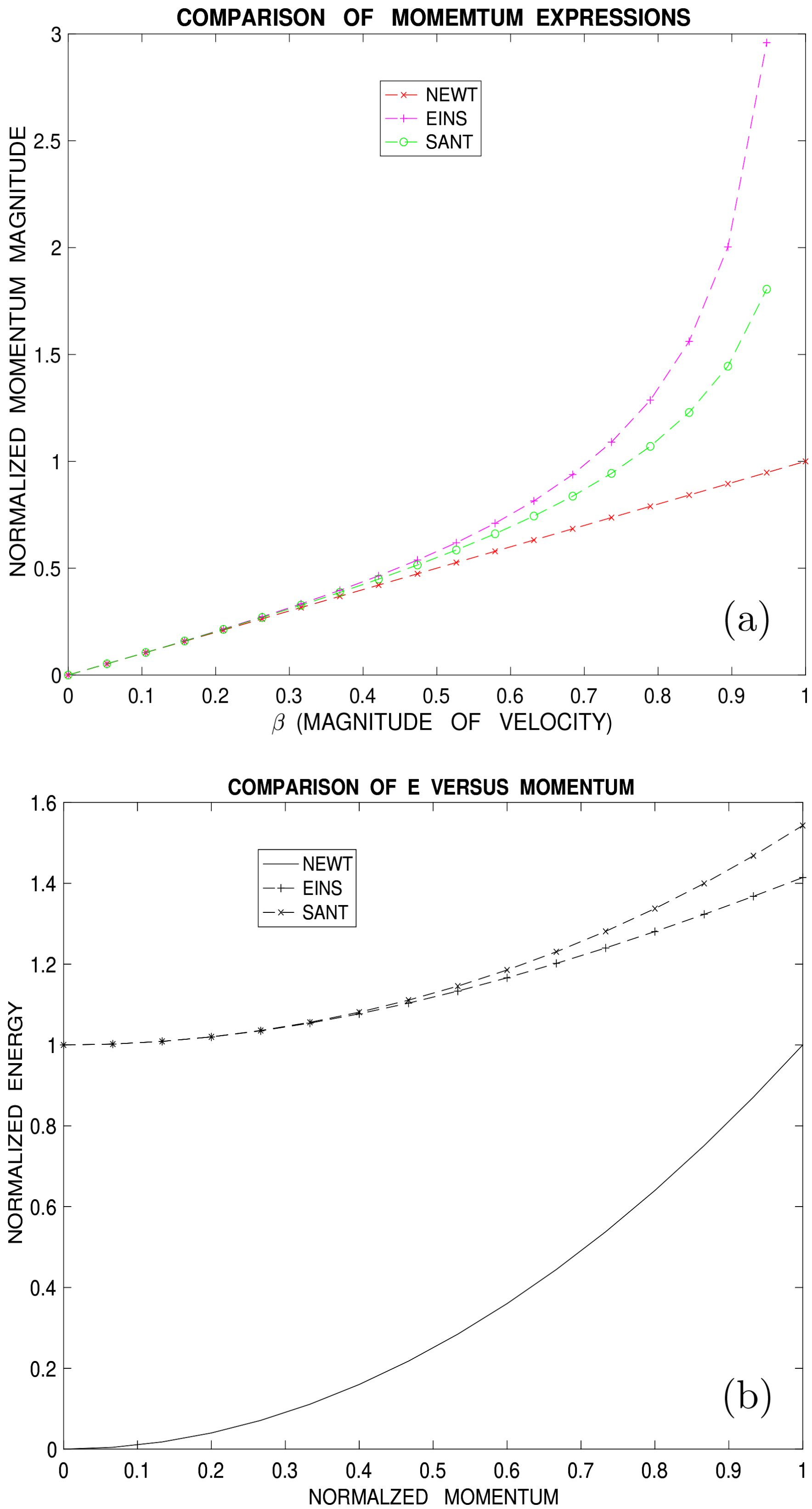

Figure 1. (a) comparison of different relativistic momentum expressions versus velocity and (b) comparison of different energy expressions versus momentum.

correction Equation (17) along with the correction due to spin-orbit interaction, in standard treatments, would imply that the fine-structure spectrum rests only on the total spin and not individually on orbital angular momentum or the spin. For instance, this would imply that the $2p_{1/2}$ and $2s_{1/2}$ states of the hydrogen atom are degenerate (apart from Lamb's shift) [10]. The relativistic correction in Equation (18) along with the correction due to spin-orbit interaction will lift this observed degeneracy of these states. All this means is that in addition to relativistic and spin-orbit corrections there must be some other sources of correction in order to restore this observed degeneracy. We cannot use Dirac's equation for this purpose since it is based on Equation (7). Dirac matrices are indeed built from Pauli spin matrices.

3. Summary

We have shown that Lorentz invariance admits a more general definition of relativistic momentum that alters the relationship between energy and momentum. Since most of the measurements are based on the measurement of velocity, they still hold with the new definition of relativistic momentum. Since $\frac{E}{mc^2}$ is still γ, there is no change in the kinematics. The main change is in the dynamics.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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