Journal of Applied Mathematics and Physics, 2025, 13(3), 797-820
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Bending Analysis of Different Types of Non-Homogenous Viscoelastic Sandwich Plates

Hamada Fawzy El-Mekawy[1*], Montaser Othman Sadiq Hilal[1], Ashraf M. Zenkour[2,3]

[1]Department of Management Information Systems, College of Business and Economics, Qassim University, Buraidah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[3]Department of Mathematics, Faculty of Science, Kafrelsheikh University, Kafrelsheikh, Egypt
Email: *hamf.halawa@qu.edu.sa

How to cite this paper: El-Mekawy, H.F., Hilal, M.O. and Zenkour, A.M. (2025) Bending Analysis of Different Types of NonHomogenous Viscoelastic Sandwich Plates. *Journal of Applied Mathematics and Physics*, **13**, 797-820.
https://doi.org/10.4236/jamp.2025.133041

Received: February 14, 2025
Accepted: March 18, 2025
Published: March 21, 2025

Abstract

The bending analysis of homogenous and non-homogenous sandwich plates is investigated using classical thin plate theory. Two types of homogenous and non-homogenous sandwich plates are considered. The case of the first type is made of viscoelastic material while the faces have elastic properties and vice versa for the second type. Young's modulus is assumed to be a function of the thickness while Poisson's ratio is assumed to be a constant. The method of effective moduli and Illyushin's approximation method are used to solve the governing equations for bending of simply supported viscoelastic sandwich plates. Numerical computations were carried out and the results show how the stresses change with time: Comparison between the behavior of stresses with various parameters for homogenous and non-homogenous viscoelastic sandwich plates are presented.

Keywords

Thin Plate Theory, Non-Homogenous, Sandwich Plate

1. Introduction

Applications of composite materials have grown rapidly over the last three decades because of their high strength and lightweight. Composite materials configured in a sandwich topology are finding increased usage in a variety of applications, such as ships and aircraft. Composite materials consist of two or more materials, which together produce desirable properties that cannot be achieved with any of the constituents alone. The feasibility of having different lamination schemes and different material properties for different lamina provides added flexibility to designers to tailor the stiffness and strength of the laminate to match the structural requirements of specific applications.

A sandwich structure consists of three distinct layers (*i.e.*, two faces and one core), which are bonded together to form an efficient load-carrying assembly. The greatest advantage of sandwich construction compared to solid laminates is that the strength and stiffness are increased without a corresponding increase in weight. It is well known that the dynamic characteristics of sandwich structures are altered by the existence of static sandwich plates are widely used in modern engineering applications, especially in aviation, marine, civil, and mechanical industries. This is because they have a combination of features like lightweight, high stiffness, high structural efficiency, and durability. Sandwich plates have been the subject of many investigations; an extensive list of references up to 1965 can be found in the monograph by Plantema [1]. Originally, most authors dealt with sandwiches in which the facings were thin, stiff, and heavy as compared with the core, henceforth, this configuration will be referred to as "classical". This has been done to relax or even eliminate the restriction of dealing only with classical sandwiches. However, the governing equations of motion of these more complicated strength-of-materials theories have necessarily become more involved as well. Several studies have been performed to analyze the structures of sandwich plates [2]-[9]. Zenkour [10]-[12] has discussed many kinds of sandwich plates using different plate theories.

The pioneering work on the analysis of sandwich beams with viscoelastic core was done in 1965 by DiTaranto [13] and in 1969 by Mead and Markus [14] for the axial and bending vibration of the beam. Since then, formulations and techniques have been reported in this area for various structural elements, e.g., beams, plates, and shells. The viscoelastic heterogeneous media of several discrete linear viscoelastic phases with known stress-strain relations has shown that the effective relaxation and creep functions can be obtained by the corresponding principle of the theory of linear viscoelasticity. In some cases, explicit results in terms of general linear viscoelastic matrix properties have been given, thus permitting direct use of experimental information [15]. Some adopted their model to study the damping mechanism of the viscoelastic layer, see e.g., Douglas and Yang [16]. In a review by Ahmed and Jones [17] of particulate reinforcement theories for polymer composites, it was concluded that the size, shape, distribution, and interfacial adhesion of the inclusions affected macroscopic behavior. The stability of rectangular, viscoelastic, orthotropic plates subjected to biaxial compression was analyzed by Wilson and Vinson [18]. In their analysis, the equations governing stability were obtained by using the quasi-elastic approximation, which overlooks the hereditary material behavior. Kim and Hong [19] have examined the viscoelastic-buckling load of sandwich plates with cross-ply faces. Huang [20] has studied the viscoelastic buckling and post-buckling of circular cylindrical laminated shells. These works, as in [18], were conducted within the framework of the quasi-elastic analysis, *i.e.*, the buckling load and post-buckling deflection are obtained by direct substitution of time-varying properties in the elastic formulations of the problem.

Pan [21] has analyzed the dynamic response problem of isotropic viscoelastic plates by extending, for this case, Mindlin's shear-deformation plate theory. Librescu and Chandiramani [22] have presented a paper that deals with the dynamic stability analysis of transversely isotropic viscoelastic plates subjected to in-plane biaxial edge-load systems. Zenkour [23] has investigated quasi-static stability analysis of fiber-reinforced viscoelastic rectangular plates subjected to in-plane edge-load systems. Zenkour [24] investigated the static thermo-viscoelastic responses of fiber-reinforced composite plates using a refined shear deformation theory. Allam, Zenkour and El-Mekawy [25] used the FPT to present the bending response of inhomogeneous fiber-reinforced viscoelastic sandwich plates Zenkour [26] investigated the bending response of an exponentially graded fiber-reinforced viscoelastic (EGFV) sandwich plate using various plate theories. Zenkour and El-Mekawy [27] investigated the bending response of an inhomogeneous viscoelastic sandwich plate. Zenkour and El-Mekawy [28] used hyperbolic shear deformation theory to investigate the bending analysis of inhomogeneous elastic/viscoelastic/elastic (EVE) sandwich plates. Recently many researchers have discussed different theories for viscoelastic sandwich plates and have presented some graphical and tabulated results [29]-[37].

In this paper, two cases of sandwich plates are considered. In the first one, the core of the sandwich plate is made from an isotropic viscoelastic material, and the faces are made from an isotropic elastic material with the same elastic properties. In the other case we take the core of the sandwich plate as an isotropic elastic material the faces are isotropic viscoelastic material with the same viscoelastic modulus properties. With the help of the effective-moduli method [38] as well as Illyushin's approximation method [39], a wide variety of results are presented for the symmetric analysis of homogenous and non-homogeneous viscoelastic rectangular sandwich plates.

2. Problem Formulation

Let us consider the case of a flat sandwich plate composed of three inhomogeneous microscopically heterogeneous layers as shown in **Figure 1**.

Rectangular Cartesian coordinates (x, y, z) are used to describe infinitesimal

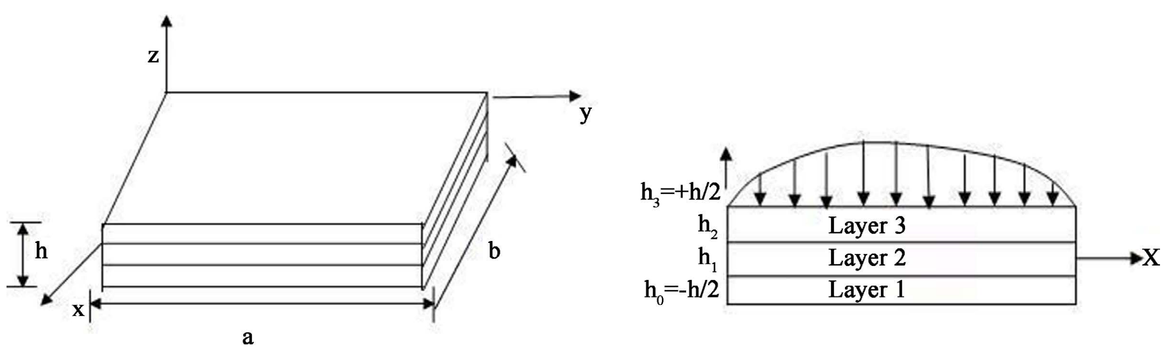

Figure 1. Geometry of rectangular plate composed of sandwich plate.

deformations of a three-layer sandwich elastic plate occupying the region $x=[0,a]$, $y=[0,b]$, $z=[-h/2,h/2]$ in the unstressed reference configuration. The mid-plane of the composite sandwich plate is defined by $z=0$ and its external bounding planes are defined by $z=\pm h/2$. The layers of the sandwich plate are made of an isotropic non-homogeneous material with material properties varying smoothly in the z (thickness) direction only. The vertical positions of the bottom surface, the two interfaces between the layers, and the top surface are denoted $h_0=-h/2$, $h_1=-h/6$, $h_2=h/6$ and $h_3=h/2$, respectively. Young's modulus for each layer can be expressed as:

$$E_{(k)}(z)=E_k \mathrm{e}^{\left(-\frac{nz}{h}\right)},\ k=1,2,3. \tag{1}$$

A normal traction $\sigma_z=q(x,y)$ is applied on the upper surface, while the lower surface is traction-free. The displacements of a material point located at (x,y,z) in the plate may be written according to the classical plate theory as:

$$u_x(x,y,z)=u-z\frac{\partial w}{\partial x},\quad u_y(x,y,z)=v-z\frac{\partial w}{\partial y},\quad u_z(x,y,z)=w, \tag{2}$$

where (u_x,u_y,u_z) are the displacements corresponding to the co-ordinates system and are functions of the spatial co-ordinates, (u,v,w) are the displacements along the axes x, y, and z, respectively, all of the generalized displacements (u,v,w) are independent of z axes.

The infinitesimal strains associated with the displacements Equation (2) are given by

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}+z\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix},\quad \varepsilon_z=0,\quad \gamma_{yz}=\gamma_{xz}=0, \tag{3}$$

where

$$\varepsilon_x^0=\frac{\partial u}{\partial x},\quad \varepsilon_y^0=\frac{\partial v}{\partial y},\quad \gamma_{xy}^{0\,(k)}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y},$$

$$\kappa_x=-\frac{\partial^2 w}{\partial x^2},\quad \kappa_y=-\frac{\partial^2 w}{\partial y^2},\quad \kappa_{xy}=-2\frac{\partial^2 w}{\partial x\partial y}.$$

By treating each layer as an individual non-homogeneous plate, the stress-strain relationships in the plate coordinates for the kth layer are written in the form:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\end{Bmatrix}^{(k)}=\frac{E_k \mathrm{e}^{\left(-\frac{nz}{h}\right)}}{1-\nu_k^2}\begin{bmatrix}1 & \nu_k\\ \nu_k & 1\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\end{Bmatrix}^{(k)},$$

$$\left\{\tau_{yz}\quad \tau_{xz}\quad \tau_{xy}\right\}^{(k)}=\frac{E_k \mathrm{e}^{\left(-\frac{nz}{h}\right)}}{2(1+\nu_k)}\left\{\gamma_{yz}\quad \gamma_{xz}\quad \gamma_{xy}\right\}^{(k)}. \tag{4}$$

The stress and moment resultants of the sandwich plate can be obtained by integrating Equation (4) over the thickness and are written as

$$\begin{Bmatrix} N_\alpha \\ M_\alpha \end{Bmatrix} = \frac{1}{1-\nu_k^2}\begin{bmatrix} C & D \\ D & F \end{bmatrix}\begin{Bmatrix} \bar{\varepsilon}_\alpha^0 \\ \bar{\kappa}_\alpha \end{Bmatrix}, \tag{5a}$$

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \end{Bmatrix} = \frac{1}{2(1-\nu_k)}\begin{bmatrix} C & D \\ D & F \end{bmatrix}\begin{Bmatrix} \gamma_{xy}^0 \\ \kappa_{xy} \end{Bmatrix}, \tag{5b}$$

where $\alpha = x, y$ and

$$\bar{\varepsilon}_x^0 = \varepsilon_x^0 + \nu_k \varepsilon_y^0, \quad \bar{\varepsilon}_y^0 = \varepsilon_y^0 + \nu_k \varepsilon_x^0,$$

$$\bar{\kappa}_x = \kappa_x + \nu_k \kappa_y, \quad \bar{\kappa}_y = \kappa_y + \nu_k \kappa_x. \tag{6}$$

Note that, N_x, N_y, and N_{xy} and M_x, M_y, and M_{xy} are the basic components of stress resultants. The coefficients C, D, and F are defined by

$$\{C \quad D \quad F\} = \sum_{k=1}^{3}\int_{h_{k-1}}^{h_k} E_k \mathrm{e}^{(-nz/h)}\{1 \quad z \quad z^2\}\mathrm{d}z. \tag{7}$$

Here h_k and h_{k-1} are the top and bottom z-coordinates of the kth layer.

3. Governing Equations

The governing equations of equilibrium can be derived by using the principle of virtual displacements as follows

$$\partial_x N_x + \partial_y N_{xy} = 0,$$

$$\partial_x N_{xy} + \partial_y N_y = 0,$$

$$\partial_x^2 M_x + 2\partial_x \partial_y M_{xy} + \partial_y^2 M_y + q = 0. \tag{8}$$

Substituting Equation (5) into Equation (8), we obtain the following operator equation

$$[p]\{\delta\} = \{f\}, \tag{9}$$

where $\{\delta\} = \{u, v, w\}^{\mathrm{T}}$, in which the superscript "T" denotes the transpose of the given vector, $\{f\} = \{0, 0, q(x, y)\}^{\mathrm{T}}$ is a generalized force vector, and $[p]$ is the symmetric matrix of differential operators,

$$p_{11} = \frac{C}{1-\nu_k^2}\left(\partial_x^2 + \frac{1-\nu_k}{2}\partial_y^2\right), \quad p_{12} = \frac{C}{2(1-\nu_k)}\partial_x \partial_y,$$

$$p_{13} = \frac{D}{1-\nu_k^2}\left(\partial_x^2 + \partial_y^2\right)\partial_x, \quad p_{22} = \frac{C}{1-\nu_k^2}\left(\frac{1-\nu_k}{2}\partial_x^2 + \partial_y^2\right),$$

$$p_{23} = \frac{D}{1-\nu_k^2}\left(\partial_x^2 + \partial_y^2\right)\partial_y, \quad p_{33} = \frac{D}{1-\nu_k^2}\nabla^4, \tag{10}$$

in which $\nabla^2 = \left(\partial_x^2 + \partial_y^2\right)$ is Laplace's operator.

4. Exact Solutions for Sandwich Plates

Here we are concerned with the exact solution of Equation (8) for simply supported non-homogenous sandwich plate. The following boundary conditions are imposed at the side edges.

$$v = w = N_x = M_x = 0, \quad \text{at } x = 0, a,$$
$$u = w = N_y = M_y = 0, \quad \text{at } y = 0, b. \tag{11}$$

To obtain the displacements and stresses of the laminate, we must satisfy the continuity and equilibrium conditions at the interfaces and the transverse stress boundary conditions at the face surfaces. There are two sets of internal boundary conditions at each interface $z = z_{k+1}$ between the kth and $(k + 1)$st layers ($k = 0$, 1, 2): the continuity of displacements

$$u^{(k)}(x, y, z_{k+1}) = u^{(k+1)}(x, y, z_{k+1}),$$
$$v^{(k)}(x, y, z_{k+1}) = v^{(k+1)}(x, y, z_{k+1}),$$
$$w^{(k)}(x, y, z_{k+1}) = w^{(k+1)}(x, y, z_{k+1}), \tag{12a}$$

and the continuity of transverse stresses

$$\sigma_z^{(k)}(x, y, z_{k+1}) = \sigma_z^{(k+1)}(x, y, z_{k+1}),$$
$$\tau_{yz}^{(k)}(x, y, z_{k+1}) = \tau_{yz}^{(k+1)}(x, y, z_{k+1}),$$
$$\tau_{xz}^{(k)}(x, y, z_{k+1}) = \tau_{xz}^{(k+1)}(x, y, z_{k+1}). \tag{12b}$$

The set of boundary conditions on the face surfaces has the form

$$\sigma_z^{(1)}(x, y, z_1) = \tau_{yz}^{(1)}(x, y, z_1) = \tau_{xz}^{(1)}(x, y, z_1) = 0, \tag{12c}$$

and

$$\sigma_z^{(3)}(x, y, z_3) = q(x, y), \tau_{yz}^{(3)}(x, y, z_3) = \tau_{xz}^{(3)}(x, y, z_3) = 0. \tag{12d}$$

To solve this problem, Navier presented the external force for the case of sinusoidally distributed load,

$$q(x, y) = q_0 \sin(\lambda x) \sin(\mu y), \tag{13}$$

where $\lambda = \pi/a, \mu = \pi/a$ and q_0 represents the intensity of the load at the plate center. Following the Navier solution procedure, we assume the following solution form for (u, v, w) that satisfies the boundary conditions,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} U \cos(\lambda x) \sin(\mu y) \\ V \sin(\lambda x) \cos(\mu y) \\ W \sin(\lambda x) \sin(\mu y) \end{Bmatrix}, \tag{14}$$

where U, V, and W are arbitrary parameters to be determined subject to the condition that the solution in Equation (14) satisfies the operator Equation (9). Substituting Equation (14) into Equation (9), we obtain

$$[P]\{\Delta\} = \{F\}, \tag{15}$$

where $\{\Delta\}$ and $\{F\}$ denote the columns

$$\{\Delta\}^{\mathrm{T}} = \{U, V, W\},$$
$$\{F\}^{\mathrm{T}} = \{0, 0, -q_0\}. \tag{16}$$

The elements $P_{ij} = P_{ji}$ of the coefficient matrix $[P]$ are given by

$$P_{11} = -\frac{C\left[2\lambda^2 + (1-\nu_k)\mu^2\right]}{2\left(1-\nu_k^2\right)}, \quad P_{12} = -\frac{C\lambda\mu}{2(1-\nu_k)},$$

$$P_{13} = \frac{D\lambda\left(\lambda^2+\mu^2\right)}{1-\nu_k^2}, \quad P_{22} = -\frac{C\left[(1-\nu_k)\lambda^2 + 2\mu^2\right]}{2\left(1-\nu_k^2\right)},$$

$$P_{23} = \frac{D\mu\left(\lambda^2+\mu^2\right)}{1-\nu_k^2}, \quad P_{33} = \frac{D\left(\lambda^2+\mu^2\right)^2}{1-\nu_k^2}. \tag{17}$$

Moreover, by substituting Equation (3) into Equation (4) with the help of Equation (14), one can obtain the stress components in terms of Young's modulus and the arbitrary parameters U, V, and W as follows:

$$\sigma_x^{(k)}(x,y,z) = \frac{E_k \mathrm{e}^{(-nz/h)}}{1-\nu_k^2}\left[\lambda U + \nu_k \mu V - z\left(\lambda^2 + \nu_k\mu^2\right)W\right]\sin(\lambda x)\sin(\mu y),$$

$$\sigma_y^{(k)}(x,y,z) = \frac{E_k \mathrm{e}^{(-nz/h)}}{1-\nu_k^2}\left[\lambda \nu_k U + \mu V - z\left(\nu_k\lambda^2 + \mu^2\right)W\right]\sin(\lambda x)\sin(\mu y),$$

$$\tau_{xy}^{(k)}(x,y,z) = \frac{E_k \mathrm{e}^{(-nz/h)}}{2(1-\nu_k)}\left[\mu U + \lambda V - 2z\lambda\mu W\right]\cos(\lambda x)\cos(\mu y). \tag{18}$$

5. Viscoelastic Solution

Here we discussed the following two problems.

5.1. Elastic-Viscoelastic-Elastic (E-V-E) Sandwich Plate

In the first one, the core of the sandwich plate is taken as an isotropic viscoelastic material and the faces are isotropic elastic material with the same elastic properties *i.e.*, $E_1 = E_3 = E$ and $\nu_1 = \nu_3 = \nu$. Note that the viscoelastic core modulus is given by

$$E_2 = \frac{9K\omega}{2+\omega}, \tag{19}$$

where K is the coefficient of volume compression (the bulk modulus) and it is assumed to be not relaxed, *i.e.* $K = \text{const.}$, and ω is the dimensionless kernel of relaxation function which is related to the corresponding Poisson's ratio by the formula

$$\nu_2 = \frac{1-\omega}{2+\omega}. \tag{20}$$

Substitution from Equations. (19) and (20) into Equations. (15)-(18), we get

$$\bar{\sigma}_i^{(j)}(\omega) = \sum\nolimits_{\ell=1}^{4} A_\ell^{(j)}[i]\, g_{\beta_\ell}(\omega),\ j = 1,2,3,\ i = 1,2,6 \tag{21}$$

where

$$A_\ell^{(j)}[i] = R\sum\nolimits_{m=1}^{4} \frac{(-1)^{2m+1} a_m[i,j]\beta_\ell^{m-1}}{\prod_{m=1,m\neq\ell}^{4}\left(\beta_\ell - \beta_m\right)},\ \ell = 1,2,3,4,\ i = 1,2,6,\ j = 1,2,3, \tag{22a}$$

$$R = \frac{e^{(-2Z+1)} s^2 a^2}{\left(s^2+1\right)^2 \pi^2 h^2}, \tag{22b}$$

and

$$g_{\beta_\ell} = \frac{1}{1+\beta_\ell \omega}, \quad \ell = 1,2,3,4, \tag{22c}$$

$\beta_\ell, \ell = 1,2,3,4$ are the roots of the equation $a_1\omega^4 + a_2\omega^3 + a_3\omega^2 + a_4\omega + a_5 = 0$, in which $a_1, \cdots, a_5$ and $a_m[i, j]$ are defined in Appendix A.

In the case of a viscoelastic homogenous material, $(n=0)$ the stresses take the form

$$\bar{\sigma}_i^{(j)}(x,y,z,\omega) = \sum_{\ell=1}^{3} A_\ell^{(j)}[i](x,y,z)\Phi_\ell^{(j)}(\omega), i = 1,2,6, \ j = 1,2,3, \tag{23}$$

where $\Phi_\ell(\omega)$ are some known kernels, constructed on the base of the kernel $\bar{\omega}$ and have the form

$$\Phi_1^{(j)}(\omega) = \begin{cases} 0 & j = 1,3 \\ 1 & j = 2 \end{cases}, \quad \Phi_i^{(j)}(\omega) = \bar{g}_{\beta_{i-1}}, \quad j = 1,2,3, i = 2,6, \tag{24}$$

and

$$\beta_1 = \frac{26\zeta + 3\left(1-\nu^2\right) - \sqrt{676\zeta^2 + 78\zeta\left(1-\nu^2\right) + 9\left(1-\nu^2\right)^2}}{26\zeta},$$

$$\beta_2 = \frac{26\zeta + 3\left(1-\nu^2\right) + \sqrt{676\zeta^2 + 78\zeta\left(1-\nu^2\right) + 9\left(1-\nu^2\right)^2}}{26\zeta}, \quad \zeta = \frac{E}{K}. \tag{25}$$

5.2. Viscoelastic-Elastic-Viscoelastic (V-E-V) Sandwich Plate

In this case, we take the core of the sandwich plate as an isotropic $E_2 = E$ and $\nu_2 = \nu$ and the face sides are isotropic viscoelastic material with the same visco-elastic modulus properties. Note that, the viscoelastic modulus is given by

$$E_1 = E_3 = \frac{9K\omega}{2+\omega}, \ \nu_1 = \nu_3 = \frac{1-\omega}{2+\omega}. \tag{26}$$

Substitution from the above equations into Equations (15)-(18), we get

$$\bar{\sigma}_i^{(j)}(\omega) = \sum_{\ell=1}^{4} B_\ell^{(j)}[i] g_{\chi_\ell}(\omega), \ell = 1,2,3, \tag{27}$$

where

$$B_\ell^{(j)}[i] = R\sum_{m=1}^{4} \frac{(-1)^{2m+1} b_m[i,j]\chi_\ell^{m-1}}{\prod_{m=1,m\neq\ell}^{4}(\chi_\ell - \chi_m)}, \ \ell = 1,2,3,4, \ i = 1,2,6, \ j = 1,2,3, \tag{28a}$$

and

$$g_{\chi_\ell} = \frac{1}{1+\chi_\ell \omega}, \ell = 1,2,3,4, \tag{28b}$$

$\chi_\ell, \ell = 1,2,3,4$ are the roots of the equation $b_1\omega^4 + b_2\omega^3 + b_3\omega^2 + b_4\omega + b_5 = 0$, in which $b_1,\cdots,b_5$ and $b_m[i,j]$ are defined in Appendix B.

In the case of a viscoelastic homogenous material $(n=0)$ the stresses take the form

$$\bar{\sigma}_i^{(j)}(x,y,z,\omega)=\sum_{\ell=1}^{3}B_\ell^{(j)}[i](x,y,z)\Phi_\ell^j(\omega),\ i=1,2,6,\ j=1,2,3, \tag{29}$$

where $\Phi_l(\omega)$ are some known kernels, constructed on the base of the kernel $\bar{\omega}$ and have the form

$$\Phi_1[j](\omega)=0,\ \Phi_2[j](\omega)=\bar{g}_{\chi_1},\ \Phi_3[j](\omega)=\bar{g}_{\chi_2},\ j=1,2,3,$$

$$\bar{g}_{\chi_1}=\frac{1}{1+\chi_1\omega},\ \bar{g}_{\chi_2}=\frac{1}{1+\chi_2\omega}, \tag{30}$$

and

$$\chi_1=\frac{\zeta+78(1-\nu^2)-\sqrt{\zeta^2+78\zeta(1-\nu^2)+6084(1-\nu^2)^2}}{\zeta},$$

$$\chi_2=\frac{\zeta+78(1-\nu^2)+\sqrt{\zeta^2+78\zeta(1-\nu^2)+6084(1-\nu^2)^2}}{\zeta},\quad \zeta=\frac{E}{K}. \tag{31}$$

The viscoelastic solution may now record to obtain explicit formulae for hoop stresses $\sigma_i^{(j)}$ as functions of the coordinates (x,y,z) and t. Then, for the first problem, we have

$$\bar{\sigma}_i^{(j)}(x,y,z,t)=\sum_{\ell=1}^{4}A_\ell^{(j)}[i]\int_0^1 g_{\beta_l}(t-\tau)\mathrm{d}q(\tau), \tag{32}$$

and in a homogenous form Equation (23) takes the form

$$\begin{aligned}\bar{\sigma}_i^{(j)}(x,y,z,t)&=A_1^{(j)}[i]\int_0^1 g_{\beta_1}(t-\tau)\mathrm{d}q(\tau)\\&\quad+A_2^{(j)}[i]\int_0^1 g_{\beta_2}(t-\tau)\mathrm{d}q(\tau),\quad j=1,3,\\ \bar{\sigma}_i^{(2)}(x,y,z,t)&=A_1^{(2)}[i]q(\tau)+A_2^{(2)}[i]\int_0^1 g_{\beta_1}(t-\tau)\mathrm{d}q(\tau)\\&\quad+A_3^{(2)}[i]\int_0^1 g_{\beta_2}(t-\tau)\mathrm{d}q(\tau).\end{aligned} \tag{33}$$

For the second problem, we have

$$\bar{\sigma}_i^{(j)}(x,y,z,t)=\sum_{\ell=1}^{4}B_\ell^{(j)}[i]\int_0^1 g_{\chi_\ell}(t-\tau)\mathrm{d}q(\tau) \tag{34}$$

and in a homogenous form Equation (29) takes the form

$$\begin{aligned}\bar{\sigma}_i^{(j)}(x,y,z,t)&=B_1^{(j)}[i]q(\tau)+B_2^{(j)}[i]\int_0^1 g_{\chi_1}(t-\tau)\mathrm{d}q(\tau)\\&\quad+B_3^{(j)}[i]\int_0^1 g_{\chi_2}(t-\tau)\mathrm{d}q(\tau).\end{aligned} \tag{35}$$

Taking

$$q(t)=Q_0H(t), \tag{36}$$

where $H(t)$ is the Heaviside's unit step function

$$H(t)=\begin{cases}1 & \text{if } t\ge 0,\\ 0 & \text{if } t<0.\end{cases} \tag{37}$$

Then Equations (32), (34) take the form

$$\bar{\sigma}_i^{(j)}(x,y,z,t)=Q_0\sum_{\ell=1}^{4}A_\ell^{(j)}[i]g_{\beta_\ell}(t),$$
$$\bar{\sigma}_i^{(j)}(x,y,z,t)=Q_0\sum_{\ell=1}^{4}B_\ell^{(j)}[i]g_{\chi_\ell}(t), \tag{38}$$

and in a homogenous form Equations (33) and (35) take the form

$$\bar{\sigma}_i^{(j)}(x,y,z,t)=Q_0\left[A_1^{(j)}[i]g_{\beta_1}(t)+A_1^{(j)}[i]g_{\beta_2}(t)\right],\quad j=1,3,$$
$$\bar{\sigma}_i^{(2)}(x,y,z,t)=Q_0\left[A_1^{(2)}[i]+A_2^{(2)}[i]g_{\beta_1}(t)+A_3^{(2)}[i]g_{\beta_2}(t)\right],$$
$$\bar{\sigma}_i^{(j)}(x,y,z,t)=Q_0\left[B_1^{(j)}[i]+B_2^{(j)}g_{\chi_1}(t)+B_3^{(j)}g_{\chi_2}(t)\right]. \tag{39}$$

Assuming an exponential relaxation function

$$\omega(t)=c_1+c_2\mathrm{e}^{-\alpha t}, \tag{40}$$

where c_1, c_2 and α are constants. The functions $g_{\beta_l}(t), g_{\chi_l}(t), l=1,2,3,4$ can be determined by deducing the Laplace-Carson transform of these functions from the known Laplace-Carson transform of the function $\omega(t)$, which can be written in the form

$$\omega^*(s)=c_1+c_2 s(\alpha+s)^{-1}. \tag{41}$$

Then we have

$$g_{\beta_l}(t)=\frac{1}{1+\beta_l c_1}\left[1-\frac{c_1\beta_l}{1+\beta_l(c_1+c_2)}\mathrm{e}^{\frac{-(1+\beta_l c_1)\tau}{1+\beta_l(c_1+c_2)}}\right],$$
$$g_{\chi_l}(t)=\frac{1}{1+\chi_l c_1}\left[1-\frac{c_1\chi_l}{1+\chi_l(c_1+c_2)}\mathrm{e}^{\frac{-(1+\chi_l c_1)\tau}{1+\chi_l(c_1+c_2)}}\right],\quad \tau=-\alpha t. \tag{42}$$

6. Numerical Results and Discussion

Numerical results for the stresses of simply supported sandwich plates are obtained. The relaxation time α is still unknown and the time parameter $\tau=-\alpha t$ given in terms of it. In addition, unless otherwise stated, it is assumed that
$b/a=0.5$, $\omega=0.1$, $\zeta=0.2$, $c_1=0.1$, $c_2=0.9$, $\nu_1=\nu_2=0.25$.

The following non-dimensional response characteristics are used throughout the figures:

$$\sigma_1=\frac{h^2}{b^2q_0}\bar{\sigma}_1\left(\frac{a}{2},\frac{b}{2},Z\right),\quad \sigma_2=\frac{h^2}{b^2q_0}\bar{\sigma}_2\left(\frac{a}{2},\frac{b}{2},Z\right),\quad \sigma_6=\frac{h^2}{b^2q_0}\bar{\sigma}_6(0,0,Z),$$

in which $Z=z/h$. Variations of stresses through the thickness with various aspect ratio b/a, constitutive parameter ζ, and time parameter τ, respectively, of uniformly loaded sandwich plates for the first case (E-V-E) are shown graphically in Figures 2-7. Figure 2 illustrates the variation of dimensionless stresses through the thickness of sandwich plates for homogenous and non-homogenous viscoelastic sandwich plates, respectively.

It can be seen that the dimensionless stresses vanished at $Z = 0$ for the homogenous case and at $Z \cong -0.0831$ for the non-homogenous case. The stresses are very sensitive to the variation of the thickness especially through the viscoelastic core layer. Figure 3 and Figure 4 illustrate the variation of dimensionless

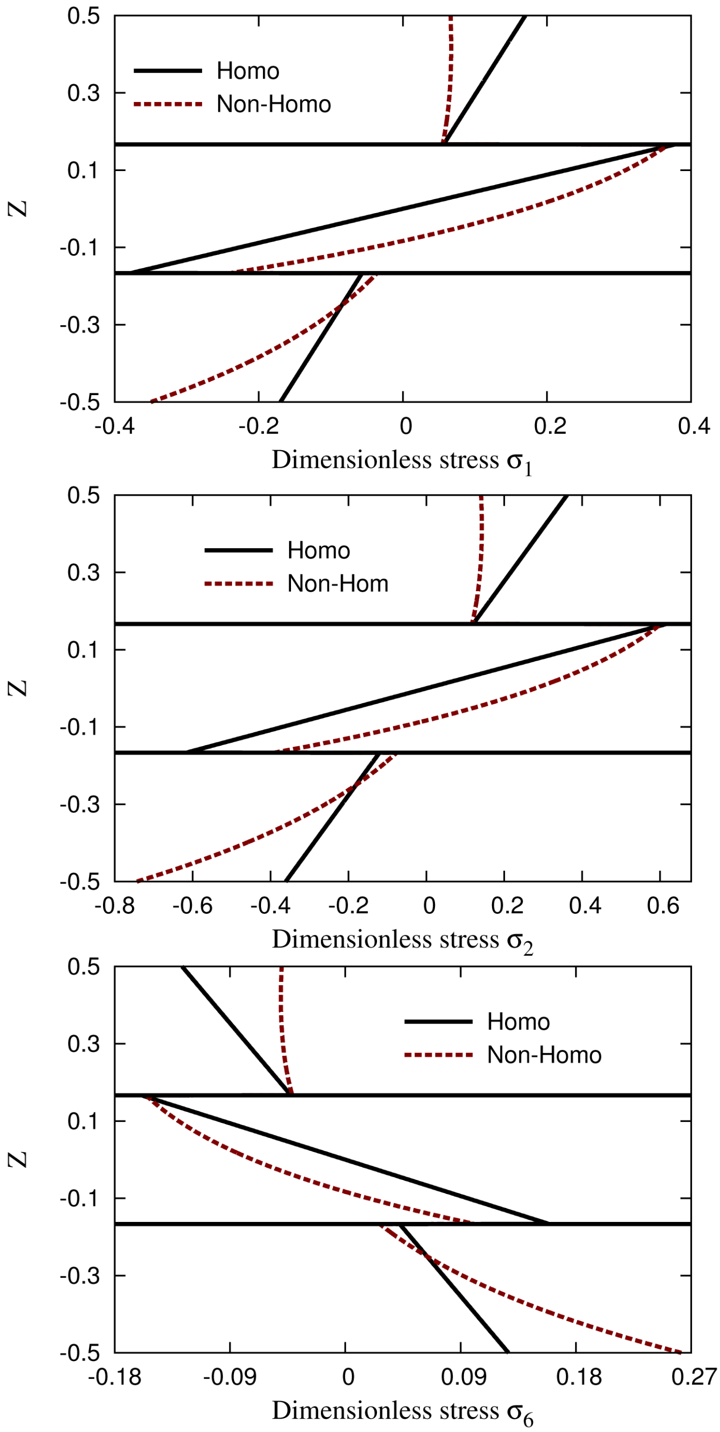

Figure 2. Variation of dimensionless stresses through the thickness of homogenous and non-homogenous rectangular (E-V-E) sandwich plates with $b/a = 0.5$, $\zeta = 0.1$ and $\omega = 0.1$.

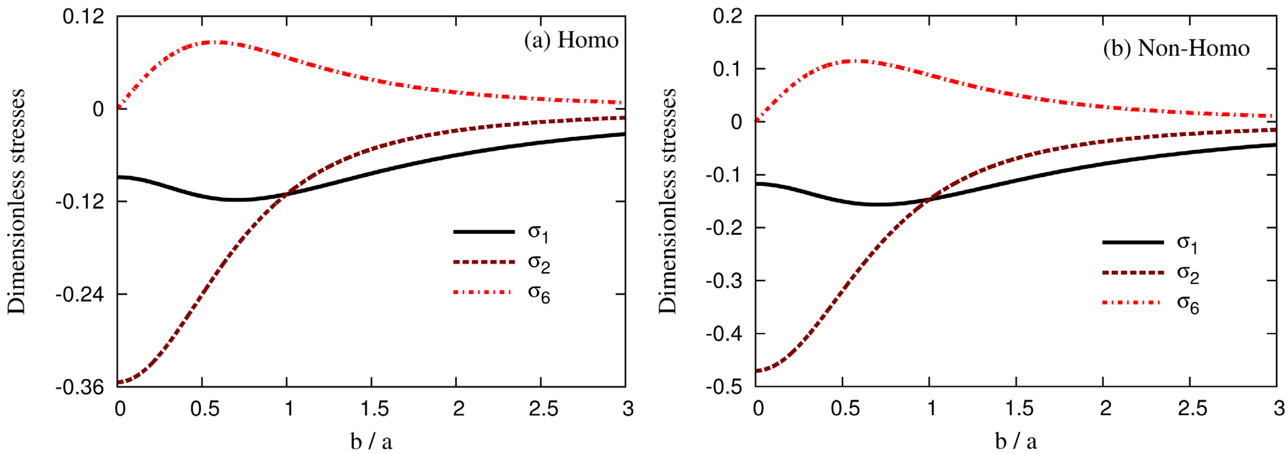

Figure 3. Stresses at $Z = -1/3$ as a function of b/a for homogenous and non-homogenous (E-V-E) sandwich plates.

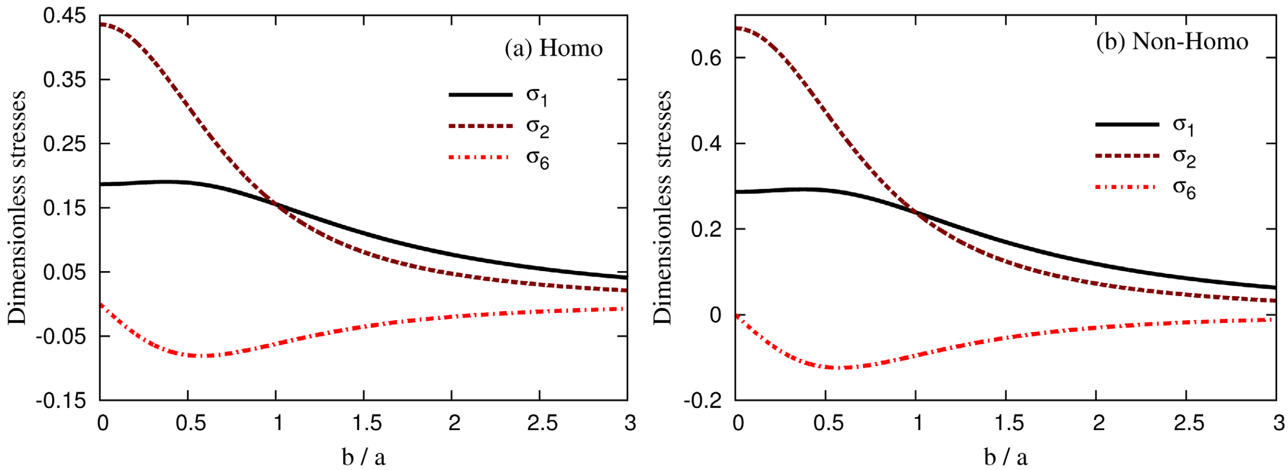

Figure 4. Stresses at $Z = 1/12$ as a function of b/a for homogenous and non-homogenous (E-V-E) sandwich plates.

stresses vs the aspect ratio b/a for homogenous and non-homogenous (E-V-E) sandwich plates. Stress behaviors are studied in the bottom face layer **Figure 3** and the viscoelastic core layer **Figure 4**.

The behaviors of the stresses may be changed with the change of the position in the bottom face or the core layers. However, in the same position, the non-homogenous case gives the largest stresses. **Figure 5** and **Figure 6** illustrate the variation of dimensionless stresses vs the constitutive parameter ζ for homogenous and non-homogenous (E-V-E) sandwich plates at the first (elastic) and the second (viscoelastic) layer, respectively.

It is to be noted that the dimensionless stresses σ_1 and σ_2 increase as constitutive parameters increase and the dimensionless stresses σ_6 decrease as the constitutive parameter increases and becomes approximation constant at $\zeta > 0.2$, and for the first layer the dimensionless stresses tended to zero at h tended to zero, this means that the layer becomes viscoelastic, and for the second layer (viscoelastic core) the dimensionless stresses σ_1 and σ_2 increase as constitutive parameters increase and the dimensionless stresses σ_6 decrease as the

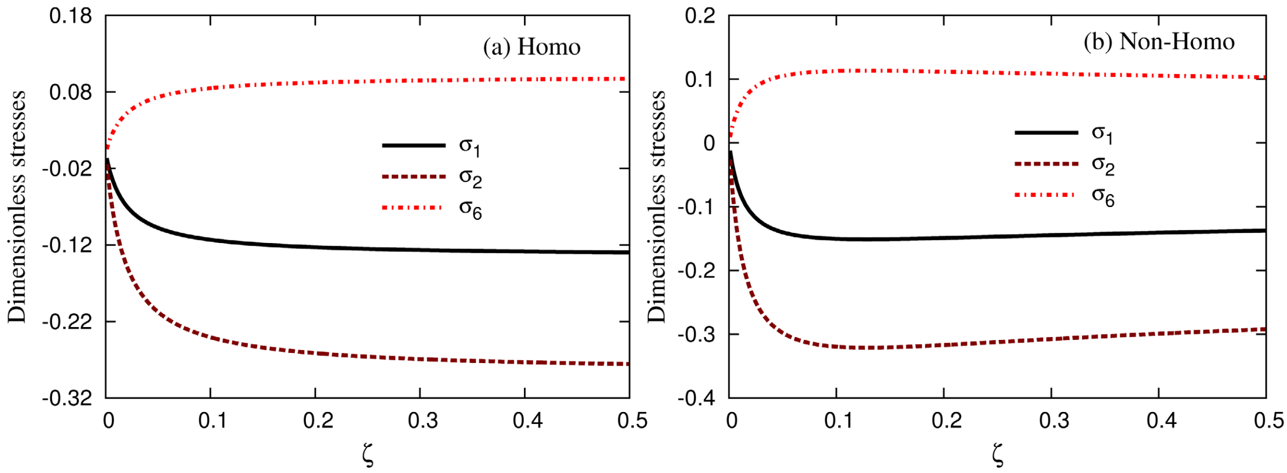

Figure 5. Stresses at $Z = -1/3$ as a function of ζ for homogenous and non-homogenous (E-V-E) sandwich plates.

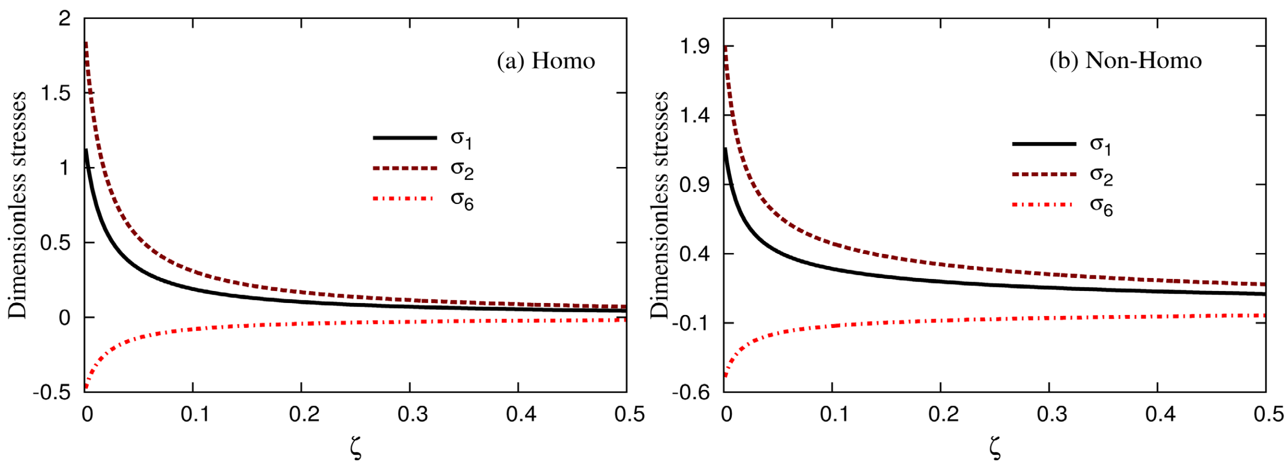

Figure 6. Stresses at $Z = 1/12$ as a function of ζ for homogenous and non-homogenous (E-V-E) sandwich plates.

constitutive parameter increases.

Figure 7 illustrates the variation of dimensionless stresses through time parameter τ for homogenous and non-homogenous sandwich plates respectively. The variation of dimensionless stresses for layer two (viscoelastic core) appears clearly with the variation time parameter τ and becomes constant where $\tau > 8$, and for the other face side (elastic) the variation of dimensionless stresses is nearly nil *i.e.* the variation is constant.

Variation of stresses through the thickness with various of the aspect ratio b/a, constitutive parameter ζ, and time parameter τ, respectively, of uniformly loaded sandwich plates for the first case (V-E-V) are shown graphically in Figures 8-13. Figure 8 illustrates the variation of dimensionless stresses through the thickness of viscoelastic sandwich plates for homogenous and non-homogenous sandwich plates, respectively. It can be seen that the dimensionless stresses vanished at $Z = 0$, for the homogenous case and at $Z \cong -0.1978$ for the non-homogenous case. The stresses are very sensitive to the variation of the thickness especially through the viscoelastic faces layer.

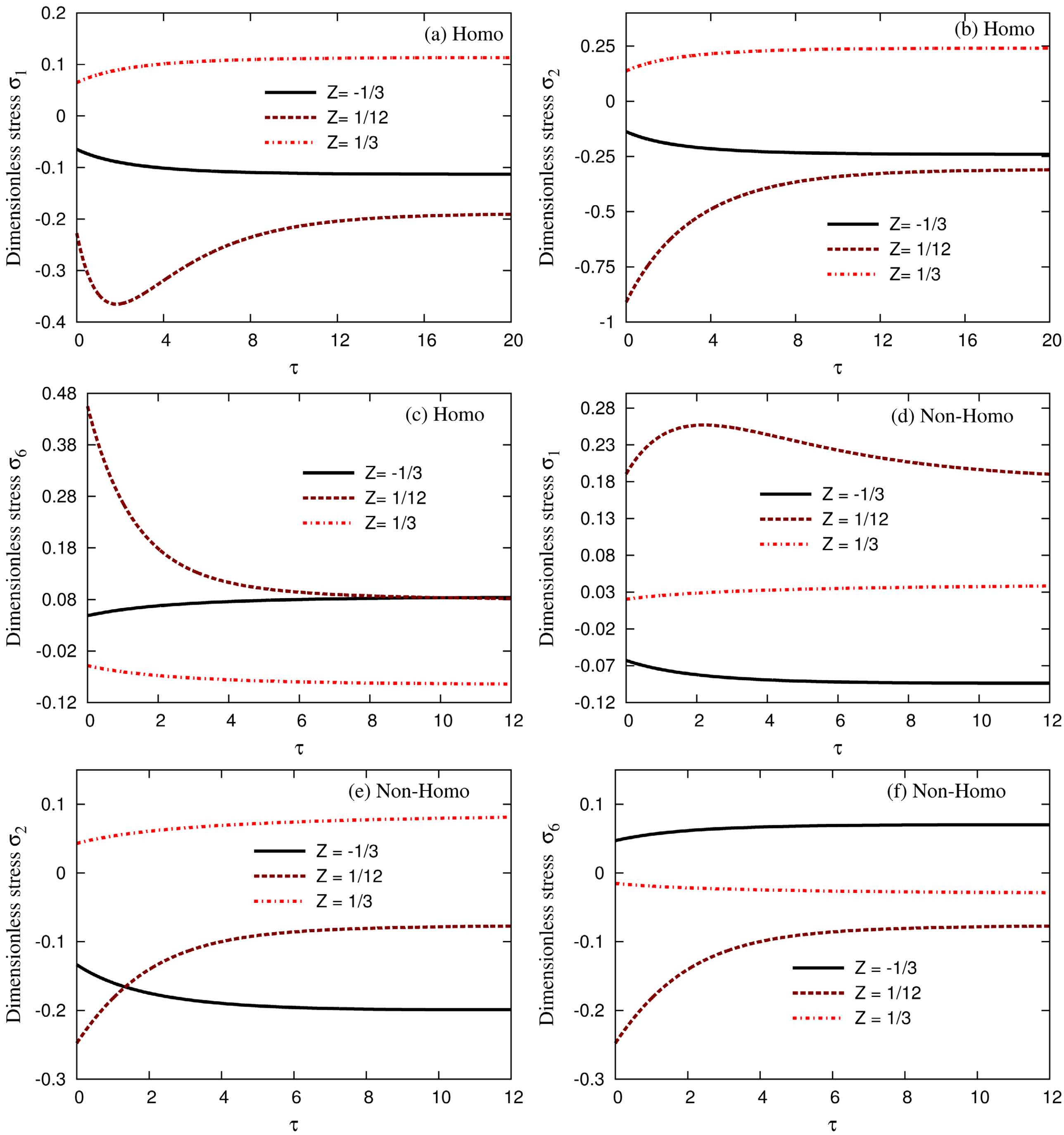

Figure 7. Variation of dimensionless stresses versus the time parameter τ for homogenous (a)-(c) and non-homogenous (d)-(f) sandwich plates (E-V-E).

Figure 9 and Figure 10 illustrate the variation of dimensionless stresses vs the aspect ratio b/a for homogenous and non-homogenous (V-E-V) sandwich. Stress behaviors are studied in the bottom viscoelastic face layer Figure 9 and the elastic core layer Figure 10. The behaviors of the stresses may be changed with the change of the position in the bottom face or the core layers. However, in the same position, the non-homogenous case gives the largest stresses. Figure 11 and

Figure 12 illustrate the variation of dimensionless stresses through the constitutive parameter ζ for homogenous and non-homogenous (V-E-V) sandwich plates at the first (viscoelastic) and the second (elastic core) layers, respectively. It

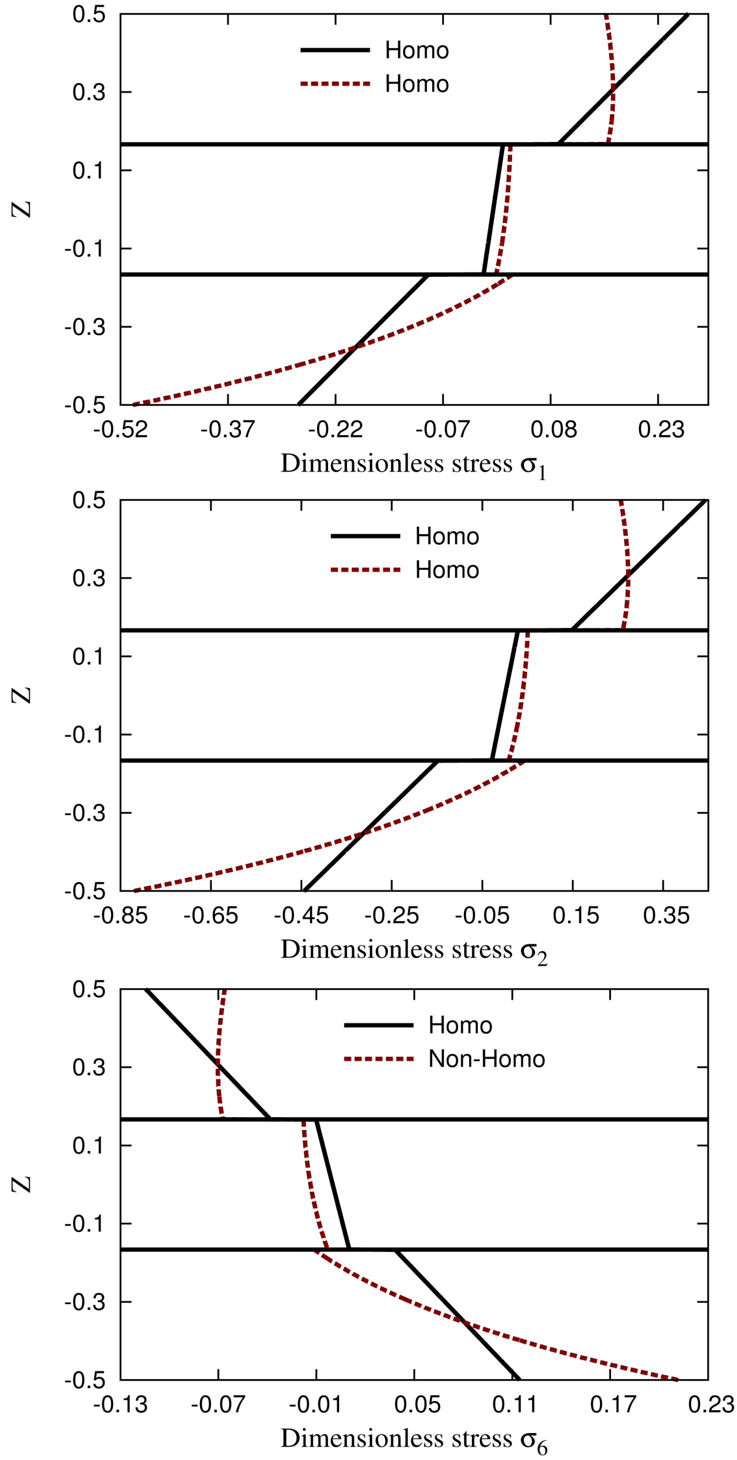

Figure 8. Variation of dimensionless stresses through the thickness of homogenous and non-homogenous rectangular (V-E-V) sandwich plates with $b/a = 0.5$, $\zeta = 0.1$ and $\omega = 0.1$.

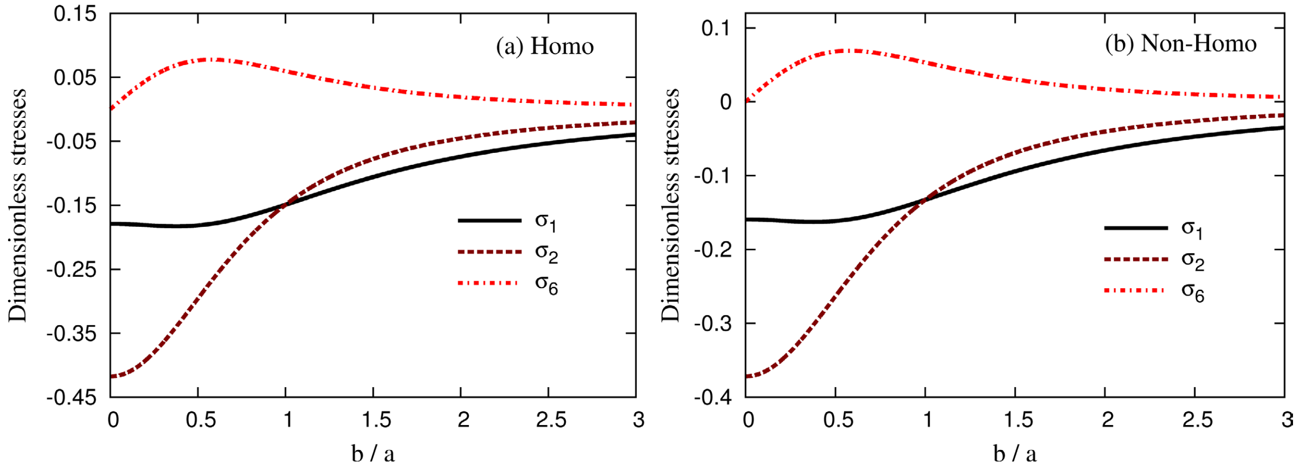

Figure 9. Stresses at $Z = -1/3$ as a function of b/a for homogenous and non-homogenous (V-E-V) sandwich plates.

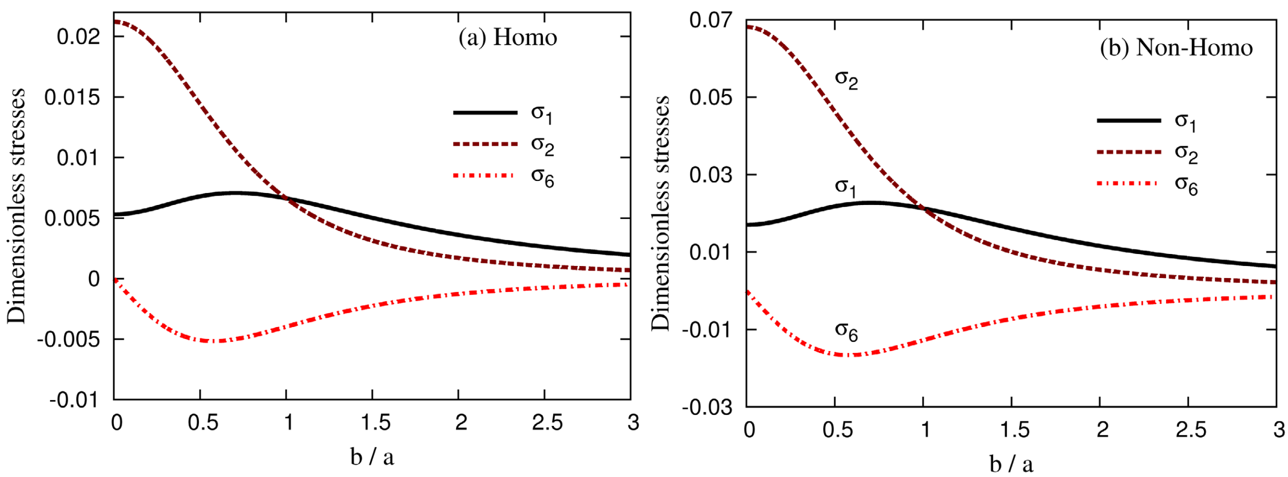

Figure 10. Stresses at $Z = 1/12$ as a function of b/a for homogenous and non-homogenous (V-E-V) sandwich plates.

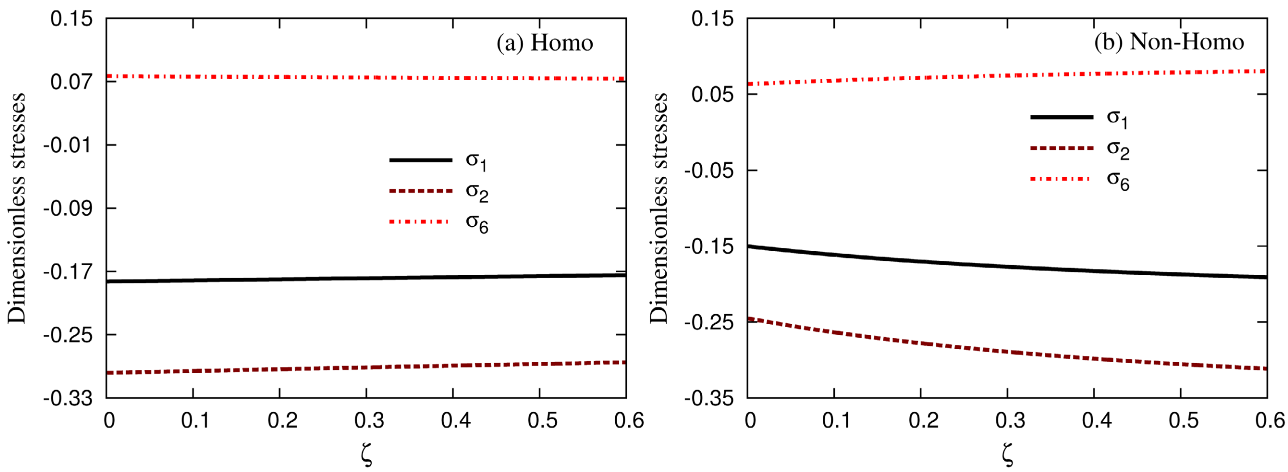

Figure 11. Stresses at $Z = -1/3$ as a function of ζ for homogenous and non-homogenous (V-E-V) sandwich plates.

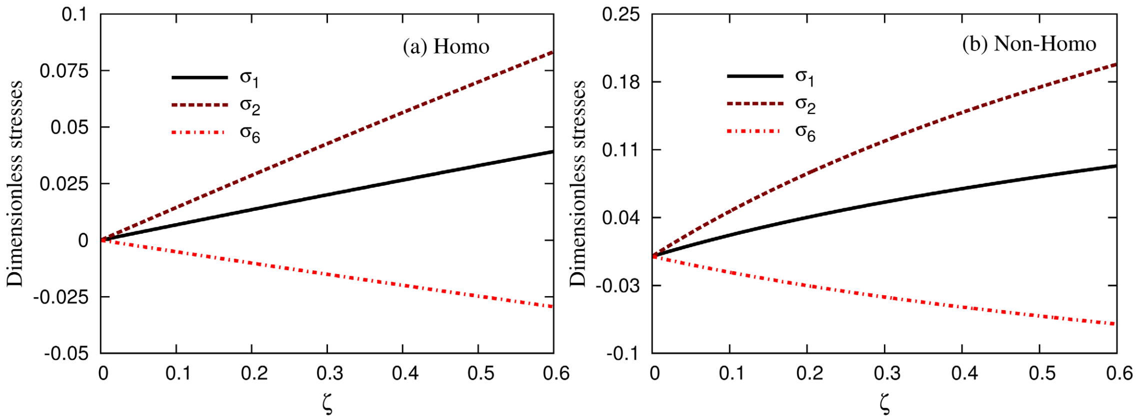

Figure 12. Stresses at $Z = 1/12$ as a function of ζ for homogenous and non-homogenous (V-E-V) sandwich plates.

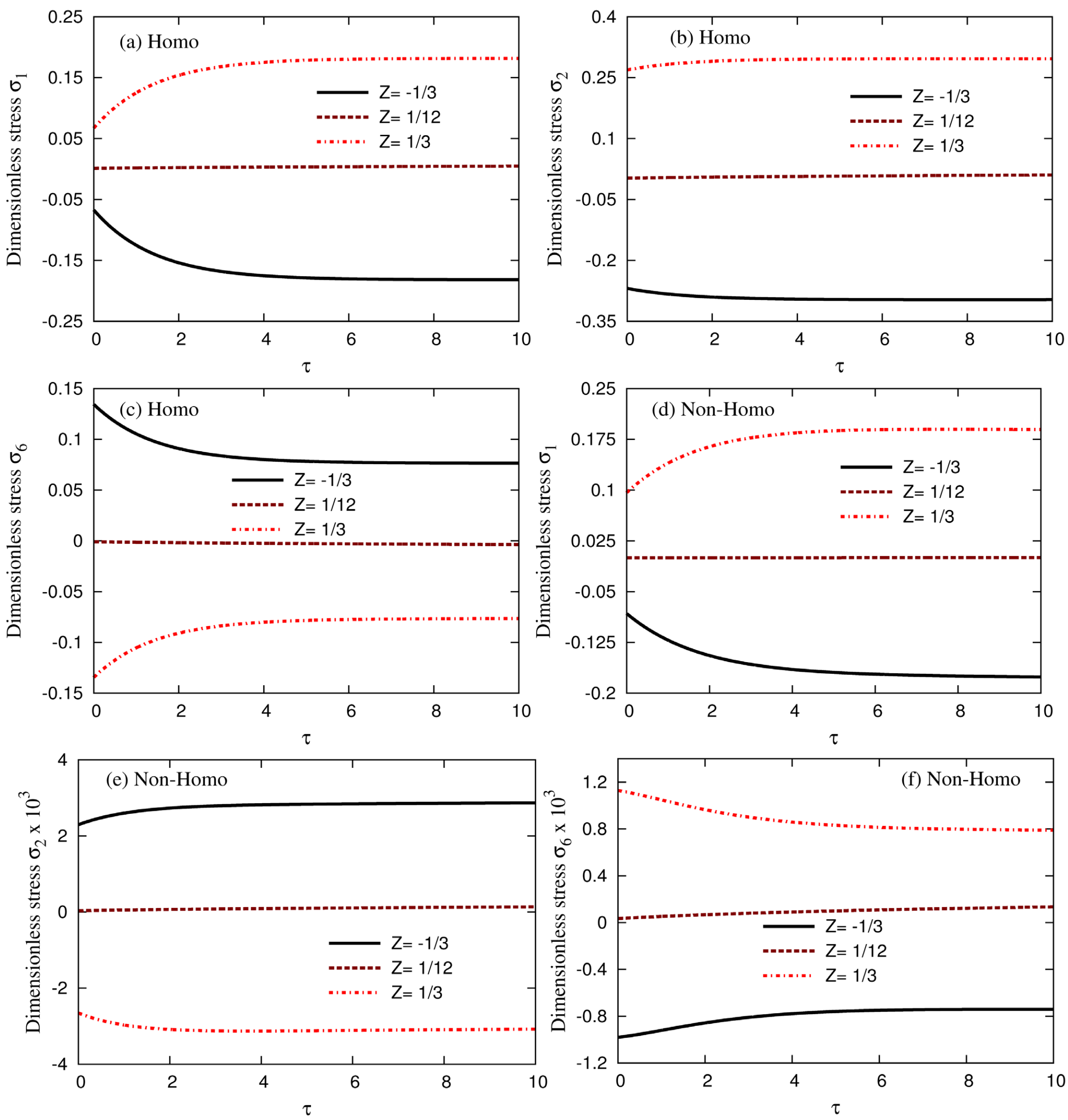

Figure 13. Variation of dimensionless stresses versus the time parameter τ for homogenous and non-homogenous (V-E-V) sandwich plate.

is to be noted that the variation of dimensionless stresses with the constitutive parameter is weak for the faces side of the sandwich plate and shown clearly at the core and increases as the constitutive parameter increases. **Figure 13** illustrates the variation of dimensionless stresses through time parameter τ for homogenous and non-homogenous sandwich plates respectively. The variation of dimensionless stresses for the faces side (viscoelastic) appears clearly with the variation time parameter τ and becomes constant where $\tau > 8$, and for the core layer (elastic) the variation of dimensionless stresses is nearly nil *i.e.* the variation is constant.

7. Concluding Remarks

A consistent classical thin plate theory for bending of homogenous and non-homogenous viscoelastic sandwich plates is presented with the help of using a method of effective moduli and assessed by comparing the obtained results with results of a homogenous viscoelastic type. The results obtained show that the effects of the weak non-homogenous response on the stresses are pronounced. Non-dimensional stresses and deflection are computed for a plate with two different cases of sandwich plate Variation of dimensionless stresses and deflection through the thickness, aspect ratio b/a, constitutive parameter ζ, and time parameter τ are presented, and show how it depends on the type of sandwich plate.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Plantema, F.J. (1966) Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells, John Wiley & Sons.

[2] Reissner, E. (1947) On Bending of Elastic Plates. *Quarterly of Applied Mathematics*, **5**, 55-68. https://doi.org/10.1090/qam/20440

[3] Hoff, N.J. (1950) Bending and Buckling of Rectangular Sandwich Plates, NACA TN 2225. https://ntrs.nasa.gov/api/citations/19930082873/downloads/19930082873.pdf

[4] Yu, Y. (1959) A New Theory of Elastic Sandwich Plates—One-Dimensional Case. *Journal of Applied Mechanics*, **26**, 415-421. https://doi.org/10.1115/1.4012054

[5] Ganowicz, R. (1967) Singular Solutions in the General Theory of Three-Layer Plates. *Mechanika Teoretyczna i Stosowana*, **3**, 293-307.

[6] Pagano, N.J. (1970) Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates. *Journal of Composite Materials*, **4**, 20-34. https://doi.org/10.1177/002199837000400102

[7] Noor, A.K., Burton, W.S. and Bert, C.W. (1996) Computational Models for Sandwich Panels and Shells. *Applied Mechanics Reviews*, **49**, 155-199. https://doi.org/10.1115/1.3101923

[8] Reddy, J.N. (1997) Mechanics of Laminated Composite Plates: Theory and Analysis. 2nd Edition, CRC Press.

[9] Vinson, J.R. (1999) The Behavior of Sandwich Structures of Isotropic and Composite

Materials. Routledge. https://doi.org/10.1201/9780203737101

[10] Zenkour, A.M. (1999) Transverse Shear and Normal Deformation Theory for Bending Analysis of Laminated and Sandwich Elastic Beams. *Mechanics of Advanced Materials and Structures*, **6**, 267-283. https://doi.org/10.1080/107594199305566

[11] Zenkour, A.M. (2005) A Comprehensive Analysis of Functionally Graded Sandwich Plates: Part 1—Deflection and Stresses. *International Journal of Solids and Structures*, **42**, 5224-5242. https://doi.org/10.1016/j.ijsolstr.2005.02.015

[12] Zenkour, A.M. (2005) A Comprehensive Analysis of Functionally Graded Sandwich Plates: Part 2—Buckling and Free Vibration. *International Journal of Solids and Structures*, **42**, 5243-5258. https://doi.org/10.1016/j.ijsolstr.2005.02.016

[13] DiTaranto, R.A. (1965) Theory of Vibratory Bending for Elastic and Viscoelastic Layered Finite-Length Beams. *Journal of Applied Mechanics*, **32**, 881-886. https://doi.org/10.1115/1.3627330

[14] Mead, D.J. and Markus, S. (1969) The Forced Vibration of a Three-Layer, Damped Sandwich Beam with Arbitrary Boundary Conditions. *Journal of Sound and Vibration*, **10**, 163-175. https://doi.org/10.1016/0022-460x(69)90193-x

[15] Douglas, B.E. and Yang, J.C.S. (1978) Transverse Compressional Damping in the Vibratory Response of Elastic-Viscoelastic-Elastic Beams. *AIAA Journal*, **16**, 925-930. https://doi.org/10.2514/3.7595

[16] Hashin, Z. (1965) Viscoelastic Behavior of Heterogeneous Media. *Journal of Applied Mechanics*, **32**, 630-636. https://doi.org/10.1115/1.3627270

[17] Ahmed, S. and Jones, F.R. (1990) A Review of Particulate Reinforcement Theories for Polymer Composites. *Journal of Materials Science*, **25**, 4933-4942. https://doi.org/10.1007/bf00580110

[18] Wilson, D.W. and Vinson, J.R. (1984) Viscoelastic Analysis of Laminated Plate Buckling. *AIAA Journal*, **22**, 982-988. https://doi.org/10.2514/3.8718

[19] Kim, C. and Hong, C. (1988) Viscoelastic Sandwich Plates with Crossply Faces. *Journal of Structural Engineering*, **114**, 150-164. https://doi.org/10.1061/(asce)0733-9445(1988)114:1(150)

[20] Huang, N.N. (1994) Viscoelastic Buckling and Postbuckling of Circular Cylindrical Laminated Shells in Hygrothermal Environment. *Journal of Marine Science and Technology*, **2**, Article 2. https://doi.org/10.51400/2709-6998.2483

[21] Pan, H. (1966) Vibrations of Viscoelastic Plates. *Journal de Mécanique*, **5**, 355-374.

[22] Librescu, L. and Chandiramani, N.K. (1989) Dynamic Stability of Transversely Isotropic Viscoelastic Plates. *Journal of Sound and Vibration*, **130**, 467-486. https://doi.org/10.1016/0022-460x(89)90070-9

[23] Zenkour, A.M. (2004) Buckling of Fiber-Reinforced Viscoelastic Composite Plates Using Various Plate Theories. *Journal of Engineering Mathematics*, **50**, 75-93. https://doi.org/10.1023/b:engi.0000042123.94111.35

[24] Zenkour, A.M. (2004) Thermal Effects on the Bending Response of Fiber-Reinforced Viscoelastic Composite Plates Using a Sinusoidal Shear Deformation Theory. *Acta Mechanica*, **171**, 171-187. https://doi.org/10.1007/s00707-004-0145-7

[25] Allam, M.N.M., Zenkour, A.M. and El-Mekawy, H.F. (2009) Bending Response of Inhomogeneous Fiber-Reinforced Viscoelastic Sandwich Plates. *Acta Mechanica*, **209**, 231-248. https://doi.org/10.1007/s00707-009-0157-4

[26] Zenkour, A.M. (2012) Viscoelastic Analysis of an Exponentially Graded Sandwich Plate. *Journal of Mechanical Science and Technology*, **26**, 889-898. https://doi.org/10.1007/s12206-011-1244-8

[27] Zenkour, A.M., and El-Mekawy H.F. (2014) Bending of Inhomogeneous Sandwich Plates with Viscoelastic Cores. *Journal of Vibroengineering*, **16**, 3260-3272. https://www.extrica.com/article/15273

[28] Zenkour, A.M. and El-Mekawy, H.F. (2018) Stresses in Inhomogeneous Elastic-Viscoelastic-Elastic Sandwich Plates via Hyperbolic Shear Deformation Theory. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, **40**, Article No. 363. https://doi.org/10.1007/s40430-018-1284-4

[29] Tekili, S., Khadri, Y. and Karmi, Y. (2020) Dynamic Analysis of Sandwich Beam with Viscoelastic Core under Moving Loads. *Mechanics*, **26**, 325-330. https://doi.org/10.5755/j01.mech.26.4.23956

[30] Huang, Z., Pan, J., Yang, Z., Wang, X. and Chu, F. (2021) Transverse Vibration of Viscoelastic Sandwich Structures: Finite Element Modeling and Experimental Study. *Materials*, **14**, Article 7751. https://doi.org/10.3390/ma14247751

[31] Permoon, M.R. and Farsadi, T. (2021) Free Vibration of Three-Layer Sandwich Plate with Viscoelastic Core Modelled with Fractional Theory. *Mechanics Research Communications*, **116**, Article 103766. https://doi.org/10.1016/j.mechrescom.2021.103766

[32] Baro, D.K., and Mahto, S. (2022) Dynamics of Viscoelastic Material Sandwich Beam Using FEM. *Advances in Materials and Processing Technologies*, **8**, 2350-2366. https://doi.org/10.1080/2374068X.2022.2038842

[33] Zenkour, A.M. and El-Shahrany, H.D. (2021) Hygrothermal Vibration of a Cross-Ply Composite Plate with Magnetostrictive Layers, Viscoelastic Faces, and a Homogeneous Core. *Engineering with Computers*, **38**, 4437-4456. https://doi.org/10.1007/s00366-021-01482-9

[34] Zenkour, A.M. and El-Shahrany, H.D. (2021) Active Control of a Sandwich Plate with Reinforced Magnetostrictive Faces and Viscoelastic Core, Resting on Elastic Foundation. *Journal of Intelligent Material Systems and Structures*, **33**, 1321-1337. https://doi.org/10.1177/1045389x211053047

[35] Tantawy, R.M., Zenkour, A.M. (2023) Bending Response of a Rotating Viscoelastic Functionally Graded Porous Disk with Variable Thickness. *Journal of Computational Applied Mechanics*, **54**, 482-500. https://doi.org/10.22059/jcamech.2023.366194.885

[36] Bennedjadi, M., Aldosari, S.M., Chikh, A., Kaci, A., Bousahla, A.A. and Bourada, F. (2023) Visco-Elastic Foundation Effect on Buckling Response of Exponentially Graded Sandwich Plates under Various Boundary Conditions. *Geomechanics and Engineering*, **32**, 159-177. https://doi.org/10.12989/gae.2023.32.2.159

[37] Messaoud, B. and Amrane, M.N. (2024) Vibration Analysis of Damaged Viscoelastic Composite Sandwich Plate. *Periodica Polytechnica Mechanical Engineering*, **68**, 85-96. https://doi.org/10.3311/ppme.19466

[38] Illyushin, A. and Pobedria, B.E. (1970) Foundations of Mathematical Theory of Thermo-Viscoelasticity. Nauka.

[39] Pobedrya, B.E. (1977) Structural Anisotropy in Viscoelasticity. *Polymer Mechanics*, **12**, 557-561. https://doi.org/10.1007/bf00857005

Appendix A

$$a_1 = 9.999195\left(1-\nu^2\right)^2,$$

$$a_2 = 39.996783\left(1-\nu^2\right)^2 + 234.083982\zeta\left(1-\nu^2\right),$$

$$a_3 = 39.996783\left(1-\nu^2\right)^2 + 1053.377926\zeta\left(1-\nu^2\right) + 2204.97487\zeta^2,$$

$$a_4 = 35.552696\left(1-\nu^2\right)^2 + 585.2099555\zeta\left(1-\nu^2\right) + 244.9972080\zeta^2,$$

$$a_5 = 61.24930170\zeta^2,$$

$$a_1[1,1] = -24\times 6\zeta\left(\nu+s^2\right)\left(1-\nu^2\right)Z_1,$$

$$a_2[1,1] = -15\times 24\zeta\left(\nu+s^2\right)\left(1-\nu^2\right)Z_1 - 4\times 24\zeta^2\left(\nu+s^2\right)Z_2,$$

$$a_3[1,1] = -6\times 24\zeta\left(\nu+s^2\right)\left(1-\nu^2\right)Z_1 - 4\times 24\zeta^2\left(\nu+s^2\right)Z_2,$$

$$a_4[1,1] = -12\zeta^2\left(\nu+s^2\right)Z_2,$$

$$a_1[2,1] = -24\times 6\zeta\left(1+\nu s^2\right)\left(1-\nu^2\right)Z_1,$$

$$a_2[2,1] = -15\times 24\zeta\left(1+\nu s^2\right)\left(1-\nu_1^2\right)Z_1 - 4\times 24\zeta^2\left(1+\nu s^2\right)Z_2,$$

$$a_3[2,1] = -6\times 24\zeta\left(1+\nu s^2\right)\left(1-\nu^2\right)Z_1 - 4\times 24\zeta^2\left(1+\nu s^2\right)Z_2,$$

$$a_4[2,1] = -24\zeta^2\left(1+\nu s^2\right)Z_2,$$

$$a_1[6,1] = 24\times 6\zeta s(1-\nu)\left(1-\nu^2\right)Z_1,$$

$$a_2[6,1] = 15\times 24\zeta s(1-\nu)\left(1-\nu^2\right)Z_1 + 24\times 4\zeta^2 s(1-\nu)Z_2,$$

$$a_3[6,1] = 6\times 24\zeta s(1-\nu)\left(1-\nu^2\right)Z_1 + 24\times 4\zeta^2 s(1-\nu)Z_2,$$

$$a_4[6,1] = 24\zeta^2 s(1-\nu)Z_2,$$

$$a_1[1,2] = -24\times 9\left(4s^2-1\right)\left(1-\nu^2\right)^2 Z_1 - 24\times 6\zeta\left(s^2-1\right)\left(1-\nu^2\right)Z_2,$$

$$a_2[1,2] = -18\times 24\left(2s^2-1\right)\left(1-\nu^2\right)^2 Z_1 - 3\times 24\zeta\left(5s^2+1\right)\left(1-\nu^2\right)Z_2,$$

$$a_3[1,2] = -3\times 24\zeta\left(2s^2+1\right)\left(1-\nu^2\right)Z_2,$$

$$a_4[1,2] = 0,$$

$$a_1[2,2] = a_1[6,2] = 15\times 24s\left(1-\nu^2\right)^2 Z_1 + 12\times 24\zeta s\left(1-\nu^2\right)Z_2,$$

$$a_2[2,2] = a_2[6,2] = 6\times 24s\left(1-\nu^2\right)^2 Z_1 + 12\times 24\zeta s\left(1-\nu^2\right)Z_2,$$

$$a_3[2,2] = a_3[6,2] = 3\times 24\zeta s\left(1-\nu^2\right)Z_2,$$

$$a_4[2,2] = a_4[6,2] = 0,$$

$$Z_1 = 5.537801557Z + 0.1018001900,$$

$$Z_2 = 13.62936674Z + 2.898199811,$$

$$A_1^{(1)}[1]=\frac{162(\beta_1-2)(s^2+\nu)s^2a^2Z}{13(\beta_1-\beta_2)\pi^2(1+s^2)^2h^2},$$

$$A_2^{(1)}[1]=\frac{162(\beta_2-2)(s^2+\nu)s^2a^2Z}{13(\beta_2-\beta_1)\pi^2(1+s^2)^2h^2},$$

$$A_1^{(1)}[2]=\frac{\nu s^2+1}{s^2+\nu}A_1[1,1],$$

$$A_2^{(1)}[2]=\frac{\nu s^2+1}{s^2+\nu}A_2[1,1],$$

$$A_1^{(1)}[6]=\frac{(\nu-1)s}{s^2+\nu}A_1[1,1],$$

$$A_2^{(1)}[6]=\frac{(\nu-1)s}{s^2+\nu}A_2[1,1],$$

$$A_1^{(2)}[1]=\frac{972(\nu^2-1)(s^2-1)s^2a^2Z}{13\zeta\beta_1\beta_2\pi^2(1+s^2)^2h^2},$$

$$A_2^{(2)}[1]=\frac{972(\beta_1s^2+2\beta_1+s^2-1)(\nu-1)s^2a^2Z}{13\zeta\beta_1(\beta_1-\beta_2)\pi^2(1+s^2)^2h^2},$$

$$A_3^{(2)}[1]=\frac{972(\beta_2s^2+2\beta_2+s^2-1)(\nu-1)s^2a^2Z}{13\zeta\beta_2(\beta_2-\beta_1)\pi^2(1+s^2)^2h^2},$$

$$A_1^{(2)}[2]=-A_1^{(2)}[1],\quad A_2^{(2)}[6]=-A_2^{(2)}[1],\quad A_3^{(2)}[6]=-A_3^{(2)}[1],$$

$$A_1^{(2)}[6]=\frac{-972(\nu^2-1)s^3a^2Z}{13\zeta\beta_1\beta_2\pi^2(1+s^2)^2h^2},$$

$$A_2^{(2)}[6]=\frac{-972(-\beta_1+2)(\nu^2-1)s^3a^2Z}{13\zeta(\beta_1-\beta_2)\pi^2(1+s^2)^2h^2},$$

$$A_3^{(2)}[6]=\frac{-972(-\beta_2+2)(\nu^2-1)s^3a^2Z}{13\zeta(\beta_2-\beta_1)\pi^2(1+s^2)^2h^2}.$$

Appendix B.

$$b_1=-551.243718(1-\nu^2)^2,$$

$$b_2=-2204.97487(1-\nu^2)^2-234.083982\zeta(1-\nu^2),$$

$$b_3=-2204.97487(1-\nu^2)^2-585.20996\zeta(1-\nu^2)-4.4440870\zeta^2,$$

$$b_4=-234.0839824\zeta(1-\nu^2)-4.4440870\zeta^2,$$

$$b_5=-1.1110217\zeta^2,$$

$$B_1^{(1)}[1]=\frac{972(\nu^2-1)(s^2-1)s^2a^2Z}{\zeta\beta_1\beta_2\pi^2(1+s^2)^2h^2},$$

$$B_2^{(1)}[1]=\frac{972(\beta_1s^2+2\beta_1+s^2-1)(\nu^2-1)s^2a^2Z}{\zeta\beta_1(\beta_1-\beta_2)\pi^2(1+s^2)^2h^2},$$

$$B_3^{(1)}[1]=\frac{972(\beta_2s^2+2\beta_2+s^2-1)(\nu^2-1)s^2a^2Z}{\zeta\beta_2(\beta_2-\beta_1)\pi^2(1+s^2)^2h^2},$$

$$B_1^{(1)}[2]=-B_1^{(1)}[1],\ B_2^{(1)}[2]=-B_2^{(1)}[1],\ B_3^{(1)}[2]=-B_3^{(1)}[1],$$

$$B_1^{(1)}[6]=-sB_1^{(1)}[1],\ B_2^{(1)}[6]=-sB_2^{(1)}[1],\ B_3^{(1)}[6]=-sB_3^{(1)}[1],$$

$$B_1^{(2)}[1]=\frac{-324(\beta_1-2)(s^2+\nu)s^2a^2Z}{(\beta_1-\beta_2)\pi^2(1+s^2)^2h^2},$$

$$B_2^{(2)}[1]=\frac{-324(\beta_2-2)(s^2+\nu)s^2a^2Z}{(\beta_2-\beta_1)\pi^2(1+s^2)^2h^2},$$

$$B_1^{(2)}[2]=\frac{\nu s^2+1}{s^2+\nu}B_1^{(2)}[1],$$

$$B_2^{(2)}[2]=\frac{\nu s^2+1}{s^2+\nu}B_2^{(2)}[1],$$

$$B_1^{(2)}[6]=\frac{(\nu-1)s}{s^2+\nu}B_1^{(2)}[1],$$

$$B_2^{(2)}[6]=\frac{(\nu-1)s}{s^2+\nu}B_2^{(2)}[1],$$

$$b_1[1,1]=-6\times24\zeta(s^2-1)(1-\nu^2)Z_1-9\times24(4s^2-1)(1-\nu^2)^2Z_2,$$

$$b_2[1,1]=-72\zeta(5s^2+1)(1-\nu^2)Z_1-18\times24(2s^2+1)(1-\nu^2)^2Z_2,$$

$$b_3[1,1]=-72\zeta(2s^2+1)(1-\nu^2)Z_1,$$

$$b_4[1,1]=0,$$

$$b_1[2,1]=6\times24\zeta(s^2-1)(1-\nu^2)Z_1+9\times24(s^2-4)(1-\nu^2)^2Z_2,$$

$$b_2[2,1]=-72\zeta(s^2+5)(1-\nu^2)Z_1-18\times24(s^2+2)(1-\nu^2)^2Z_2,$$

$$b_3[2,1]=-72\zeta(s^2+2)(1-\nu^2)Z_1,$$

$$b_4[2,1]=0,$$

$$b_1[6,1]=12\times24\zeta s(1-\nu^2)Z_1+45\times24s(1-\nu^2)^2Z_2,$$

$$b_2[6,1]=12\times24\zeta s(1-\nu^2)Z_1+12\times24s(1-\nu^2)^2Z_2,$$

$$b_3[6,1]=72\zeta s(1-\nu^2)Z_1,$$

$$b_4[6,1]=0,$$

$$b_1[1,2]=-6\times 24\zeta\left(s^2+\nu\right)\left(1-\nu^2\right)Z_2,$$

$$b_2[1,2]=-4\times 24\zeta^2\left(s^2+\nu\right)Z_1-15\times 24\zeta\left(s^2+\nu\right)\left(1-\nu^2\right)Z_2,$$

$$b_3[1,2]=-2\times 24\zeta^2\left(s^2+\nu\right)Z_1-6\times 24\zeta\left(s^2+\nu\right)\left(1-\nu^2\right)Z_2,$$

$$b_4[1,2]=-24\zeta\left(s^2+\nu\right)\zeta^2 Z_1,$$

$$b_1[2,2]=b_1[6,2]=6\times 24\zeta s\left(1-\nu^2\right)(1-\nu)Z_2,$$

$$b_2[2,2]=b_2[6,2]=4\times 24s\zeta^2(1-\nu)Z_1+15\times 24\zeta s\left(1-\nu^2\right)(1-\nu)Z_2,$$

$$b_3[2,2]=b_3[6,2]=2\times 24\zeta^2 s(1-\nu)Z_1+6\times 24\zeta s\left(1-\nu^2\right)(1-\nu)Z_2,$$

$$b_4[2,2]=b_4[6,2]=24s\zeta^2(1-\nu)Z_1.$$