Journal of Applied Mathematics and Physics, 2025, 13(3), 689-708
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Evolutionary Nonconservative Field Theories

Bogdana A. Georgieva

Department of Mechatronics, Robotics and Mechanics, Faculty of Mathematics and Informatics, Sofia University "Kl. Ohridski", Sofia, Bulgaria
Email: georgieva@fmi.uni-sofia.bg

How to cite this paper: Georgieva, B.A. (2025) Evolutionary Nonconservative Field Theories. *Journal of Applied Mathematics and Physics*, **13**, 689-708.
https://doi.org/10.4236/jamp.2025.133038

Received: October 24, 2024
Accepted: March 16, 2025
Published: March 19, 2025

Abstract

This paper introduces a new evolutionary system which is uniquely suitable for the description of nonconservative systems in field theories, including quantum mechanics, but is not limited to it only. This paper also introduces a new exact method of solution for such nonconservative systems. These are significant contributions because the vast majority of nonconservative systems with several independent variables do *not* have self-adjoint Frechet derivatives and because of that *can not* benefit from the exact methods of the classical calculus of variations. The new evolutionary system is rigorously mathematically derived and the new method for solution is mathematically proved to be applicable to systems of PDEs of second order for nonconservative process. As examples of applications, the method is applied to several nonconservative systems: the propagation of electromagnetic fields in a conductive medium, the nonlinear Schrodinger equation with electromagnetic interactions, and others.

Keywords

Mathematical Methods in Quantum Theory, Nonconservative Quantum Systems, Nonconservative Systems, Exact Methods for Solution of Pdes, Nonconservative Systems, Integrable Nonconservative Systems, Nonconservative Systems of Variational Origin, Nonconservative Processes

1. Introduction

Nonconservative processes with several independent variables *can not* be given a variational description via the classical variational principle (in which the functional is defined by an integral), because the Frechet derivative of the system of equations is *not* self-adjoint. Thus, nonconservative processes can not make use of the classical methods and machinery available from the Calculus of Variations. In 2003, Bogdana Georgieva, Ronald Guenther and Theodore Bodurov introduce

the generalized variational principle with several independent variables [1] (Journal of Mathematical Physics, Vol. 44 (2003), No. 9, pp. 3911-3927) which *can* give a variational description of non-conservative processes *even when the Frechet derivative of its equations is not self-adjoint.* Noether-type theorems were found [1]-[3], symmetries and corresponding identities were found [4]-[6], and a number of applications were published of this *new* variational principle, which contains the classical one as a strictly special case, but is a lot more general than it. More specifically, these applications are in: integrable systems, the calculus of variations, dynamical systems, symmetry methods, control theory, optimal control theory, dissipative dynamics, field theories, Hamiltonian dynamics, nonlinear dynamics, thermodynamics, contemporary quantum mechanics, electricity and magnetism, nonholonomic systems, stochastic processes, celestial mechanics, the study of the Universe, Vakonomic dynamics, Birkhofian systems, waves propagation, Bregman dynamics, contemporary general relativity, dark energy, virosology, oncology and other branches of science.

As is well known, Hamiltonian systems emerged historically as a method of solution for the second order Euler-Lagrange system of equations for a classical variational functional defined by an integral in the case of one independent variable. The method consists of converting the second order Euler-Lagrange system into a first order system of ODEs, the Hamiltonian system. The transformation of Legendre, which performs this "conversion" opened a gate from the Lagrangian mechanics to the Hamiltonian mechanics.

This paper introduces a new exact method of solution for evolutionary systems which are nonconservative and have variational origin. This is significant, because nonconservative systems with several independent variables do *not* have self-adjoint Frechet derivatives and because of that *can not* benefit from the methods of the classical calculus of variations. Additional asset of this new method is that the described process can be controlled through control functions. These are the functions $u(t,x)$, $u \equiv (u^1,\cdots,u^m)$. Here t and $x \equiv (x^1,\cdots,x^n)$ are the independent variables. Thus, this method covers nonconservative systems in control and optimal control theories. This paper also rigorously derives the evolutionary system which corresponds to the generalized Euler-Lagrange equations (a system of second order PDEs) for the nonconservative process.

More specifically, the new exact method is a method of solution for the second order system of PDEs which are the generalized Euler-Lagrange equations for the functional introduced in [1] by B. Georgieva, R. Guenther and Th. Bodurov. This method consists of "converting" this second order system of PDEs (which describes the nonconservative process) into a first order system of evolution equations. The present paper proves rigorously the equivalence of the system of generalized Euler-Lagrange equations to this system of evolution equations and rigorously derives the specific form of this system of evolution equations.

1.1. Some Historical Remarks

In 1932 Gustav Herglotz gave a series of lectures on the relationship between contact transformations and the generalized Hamiltonian system

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}x_j = \frac{\partial H}{\partial p_j},\\
&\frac{\mathrm{d}}{\mathrm{d}t}z = p_j\frac{\partial H}{\partial p_j} - H,\\
&\frac{\mathrm{d}}{\mathrm{d}t}p_j = -\frac{\partial H}{\partial x_j} - p_j\frac{\partial H}{\partial z}, \quad j = 1,\cdots,n,
\end{aligned} \tag{1}$$

where H is a function of $x_1,\cdots,x_n,z,p_1,\cdots,p_n$. See [7]. This system is closely related to the variational principle of Herglotz [8] [9]. In that variational principle the functional z, whose extrema are sought, is defined by an ordinary differential equation rather than by an integral:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = L\left(t,x,\dot{x},z\right), \quad 0 \le t \le s$$

where t is the only independent variable, $x \equiv \left(x^1,\cdots,x^n\right)$ are the argument functions of t, $\dot{x} = \mathrm{d}x/\mathrm{d}t$. For any arbitrary but fixed set of functions $x(t)$ and a fixed initial value for z at $t = 0$ the solution of this ordinary differential equation depends both on t and on $x(t)$. If we take $t = s = \text{const.}$ then the solution $z = z[x;s]$ is a functional of the set of functions $x^k(t)$. For more detailed description of the definition of a functional via this class of ordinary differential equations see [2].

Herglotz variational principle is uniquely suitable for giving a variational description of nonconservative processes involving one independent variable. It is more general than the classical variational principle with one independent variable and contains it as a special case, which occurs when L does not depend on z.

Herglotz showed that the value of this functional is an extremum when its argument-functions $x^k(t)$ are solutions of the generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x^k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k} = 0, \quad k = 1,\cdots,n.$$

In his lectures Herglotz revealed the remarkable geometry which underlines the generalized Hamiltonian system. Herglotz's work was motivated by ideas from S. Lie [10] [11] and others. Furta *et al.* show in [12] a close link between the Herglotz variational principle and control and optimal control theories. For historical remarks through 1935 see Caratheodory [13]. The contact transformations, which are related to the generalized variational principle, have found applications in thermodynamics. Mrugala shows in [14] that the processes in equilibrium thermodynamics can be described by successions of contact transformations acting in a suitably defined thermodynamic phase space. The latter is endowed with a contact structure, closely related to the symplectic structure.

1.2. Contemporary Preliminary Remarks

In [2] and [3] B. Georgieva and R. Guenther formulated and proved first and second Noether-type theorems which yields a first integral corresponding to a known symmetry of the functional defined by the Herglotz variational principle; and an identity corresponding to an infinite-dimensional symmetry of the Herglotz functional. For a summary of the resent results related to the variational principle of Herglotz see [15].

In [1] B. Georgieva, R. Guenther and Th. Bodurov introduce a new variational principle, which extends the Herglotz principle to one with several independent variables (In honor of Gustav Herglotz we named it in his name). This new variational principle contains as special cases both the classical variational principle with several independent variables and the Herglotz variational principle. It can describe not only all physical processes which the classical variational principle can, but also many others for which the classical variational principle is not applicable. It can give a variational description of nonconservative processes involving physical fields, *even when the Frechet derivative of the equation(s) is NOT self-adjoint*. In other words, if a PDE (or a system of PDEs) does NOT have a variational description via the classical variational principle because its Frechet derivative is not self-adjoint, then the generalized variational principle which Georgieva, Guenther and Bodurov introduce in [1] can (in most cases) be used to give this equation/equations a variational description. Another asset of this variational principle is that it is *not* "sensitive" to nonlinearity. It is so general in form that it handles nonlinear processes with the same ease as it does linear ones. For the convenience of the reader I state the precize definition of the *Generalized Variational Principle With Several Independent Variables*:

Let the functional $z = z[u;s]$ *of* $u = u(t,x)$ *be defined by an integro-differential equation of the form*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \int_{\Omega} \mathcal{L}(t,x,u,u_t,u_x,z)\mathrm{d}^n x, \quad 0 \le t \le s \tag{2}$$

where t and $x \equiv (x^1,\cdots,x^n)$ are the independent variables, $u \equiv (u^1,\cdots,u^m)$ are the argument functions, $u_x \equiv (u_x^1,\cdots,u_x^m)$, $u_t \equiv (u_t^1,\cdots,u_t^m)$ and $u_x^i \equiv (u_{x^1}^i,\cdots,u_{x^n}^i)$, $i = 1,\cdots,m$, $\mathrm{d}^n x \equiv \mathrm{d}x^1\cdots\mathrm{d}x^n$, and where the function $\mathcal{L}$ is at least twice differentiable with respect to u_x, u_t and once differentiable with respect to t,x,z. Let $\eta \equiv (\eta^1(t,x),\cdots,\eta^m(t,x))$ have continuous first derivatives and otherwise be arbitrary except for the boundary conditions:

$$\eta(0,x) = \eta(s,x) = 0$$
$$\eta(t,x) = 0 \quad \text{for } x \in \partial\Omega,\ 0 \le t \le s$$

where $\partial\Omega$ is the boundary of Ω. Then, the value of the functional $z[u;s]$ is an extremum for functions u which satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} z[u+\varepsilon\eta;s]\bigg|_{\varepsilon=0} = 0.$$

The function $\mathcal{L}$, just as in the classical case, is called the *Lagrangian density*. It should be observed that when a variation $\varepsilon\eta$ is applied to u, the integro-differential equation defining the functional z must be solved with the same fixed initial condition $z(0)$ at $t = 0$ and the solution evaluated at the same fixed final time $t = s$ for all varied argument functions $u + \varepsilon\eta$.

Every function $u \equiv \left(u^1, \cdots, u^m\right)$, for which the functional z defined by the integro-differential Equation (2) has an extremum, is a solution of

$$\frac{\partial \mathcal{L}}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial u_t^i} - \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}^i} + \frac{\partial \mathcal{L}}{\partial u_t^i}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\mathrm{d}x = 0, \quad i = 1, \cdots, m. \tag{3}$$

These equations are called (in correspondence with the classical case) the generalized Euler-Lagrange equations. The summation convention on repeated indices is assumed in the entire the paper.

It is important to observe that the definition of the functional z by the integro-differential Equation (2) contains as a strictly special case the classical definition of a functional by an integral. This special case occurs if $\mathcal{L}$ does not depend on z. Similarly, the generalized Euler-Lagrange equations reduce to the classical Euler-Lagrange equations if $\mathcal{L}$ does not depend on z.

Georgieva, Guenther and Bodurov, [8], formulated and proved a Noether-type theorem which yields which gives an identity corresponding to each symmetry of the functional defined by this new variational principle. From this identity a first integral is readily obtained.

1.3. Example of a Nonconservative Field Theory of a Variational Origin

A nonconservative system in the context of field theories is a field theory for which the energy density function, represented by the Hamiltonian $\mathcal{H} \equiv \sum_{j=1}^m p^j u_t^j - \mathcal{L}$, is not constant on solutions of the defining equations of the field theory.

Consider a specific example. Let's look at the nonlinear Klein-Gordon equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + G\left(uu^*\right)u = 0 \tag{4}$$

describing the real or complex field $u = u\left(x, t\right)$, where u^* denotes the complex conjugate of u, G is a differentiable function and v is a constant. Its linear version, with $G = \text{constant}$ plays an important role in relativistic field theories. The one-dimensional version of (4) with real u and $G\left(u^2\right)u = \sin u$ is the sine-Gordon equation. The field equations of the form (4) can be derived from the Lagrangian density

$$\mathcal{L}\left(u, u_t, \nabla u\right) = \nabla u \cdot \nabla u^* - \frac{1}{v^2}\frac{\partial u}{\partial t}\frac{\partial u^*}{\partial t} - F\left(uu^*\right) \tag{5}$$

where

$$\frac{\mathrm{d}F\left(\rho\right)}{\mathrm{d}\rho} = G\left(\rho\right) \text{ and } F\left(0\right) = 0.$$

We consider as physically meaningful only those solutions of (4) which are free of singularities and for which

$$\left|\int_\Omega \mathcal{L}\left(t, x, u, u_t, u_x\right)\mathrm{d}x\right| < \infty$$

holds over the entire time domain. The processes described with an equation of the form (4) are conservative since the Lagrangian (5) does not explicitly depend on time.

One is also interested in nonconservative processes involving fields. The simplest modification of (4) which makes it suitable to describe nonconservative processes is to include in it a term proportional to the time-derivative of the field. Thus, a physically meaningful nonconservative version of (4) is

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} + G\left(uu^*\right)u = 0 \tag{6}$$

where k is a constant. With $k > 0$ the process described by (6) is generative, and with $k < 0$ it is dissipative. When u is a real field, equations of the form (6) can be derived via the present generalized variational principle from the Lagrangian density

$$\mathcal{L} = \nabla u \cdot \nabla u - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - F\left(u^2\right) + \alpha\left(x\right)z \tag{7}$$

where $\partial F\left(\rho\right)/\partial\rho = G\left(\rho\right)$, and $\alpha = \alpha\left(x\right)$ is a given function of the coordinates $x = \left(x^1, \cdots, x^n\right)$ which satisfies the condition

$$\left|\int_\Omega \alpha\left(x\right)\mathrm{d}^n x\right| < \infty$$

Indeed, inserting the Lagrangian (7) into the generalized Euler-Lagrange equations (3)

$$\begin{aligned}&\frac{\partial\mathcal{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial u_t} - \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial\mathcal{L}}{\partial u_{x^k}} + \frac{\partial\mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial\mathcal{L}}{\partial z}\mathrm{d}^n x \\ &= -2u\frac{\partial F}{\partial\left(u^2\right)} + \frac{2}{v^2}\frac{\partial^2 u}{\partial t^2} - 2\nabla^2 u - \frac{2}{v^2}\frac{\partial u}{\partial t}\int_\Omega \alpha\left(x\right)\mathrm{d}^n x = 0\end{aligned}$$

we see that the last expression is the same as (6) with

$$k = \frac{1}{v^2}\int_\Omega \alpha\left(x\right)\mathrm{d}^n x = \text{const.} \tag{8}$$

Consequently, we may apply the first Noether-type theorem 5.1 to obtain conserved quantities. In particular, observing that the Lagrangian (7) is invariant under translations in time we may apply Corollary 7.1 in [1] to obtain the conserved quantity

$$\exp\left(-kv^2 t\right)\int_\Omega\left(\frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 + \nabla u\cdot\nabla u - F\left(u^2\right) + \alpha\left(x\right)z\right)\mathrm{d}x = \text{const.} \tag{9}$$

where z is the solution of the defining Equation (2). In accordance with the conservative case, we can interpreted the quantity

$$\frac{\partial\mathcal{L}}{\partial u_t}u_t - \mathcal{L} = -\frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - \nabla u\cdot\nabla u + F\left(u^2\right) - \alpha\left(x\right)z$$

as the energy density of the field $u\left(t,x\right)$. Then Equation (9) states that the total field energy in not constant but instead increases exponentially when $k > 0$ and decreases when $k < 0$.

For the convenience of the reader we state from [1] Corollary 7.1:

Let

$$\left|\int_{\mathbb{R}} \mathcal{L}\left(t, x, u\left(t, x\right), u_t, u_x, z\right) \mathrm{d}^n x\right| < \infty$$

over the entire time domain and let the functional z, defined by the Equation (2), be invariant with respect to translations in time. Then the quantity

$$E\left(t\right)\int_{\mathbb{R}}\left(u_t^i \frac{\partial \mathcal{L}}{\partial u_t^i} - L\right)\mathrm{d}^n x = \text{const.} \quad i = 1, \cdots, m$$

with $E\left(t\right)$ given by

$$E\left(t\right) \equiv \exp\left(-\int_0^t \int_{\mathbb{R}} \frac{\partial \mathcal{L}}{\partial z} \mathrm{d}^n x \mathrm{d}\theta\right),$$

is conserved on solutions of the generalized Euler-Lagrange equations (3).

2. One-Parameter Families of Contact Transformations

In his lectures on contact transformations, see [9], Gustav Herglotz derives the generalized Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t} X_j = \frac{\partial \mathcal{H}}{\partial P_j}, \quad \frac{\mathrm{d}}{\mathrm{d}t} Z = P_j \frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}, \quad \frac{\mathrm{d}}{\mathrm{d}t} P_j = -\frac{\partial \mathcal{H}}{\partial X_j} - P_j \frac{\partial \mathcal{H}}{\partial Z}, \quad j = 1, \cdots, n \quad (10)$$

as a result of a demand on the solution of the general system

$$\dot{X} = \xi\left(X, Z, P, t\right), \quad \dot{Z} = \zeta\left(X, Z, P, t\right), \quad \dot{P} = \pi\left(X, Z, P, t\right)$$

for $2n+1$ unknowns $X = \left(X_1, \cdots, X_n\right), Z, X = \left(P_1, \cdots, P_n\right)$ to define a family of contact transformations of the initial conditions. This remarkable calculation is a result of an extraordinary intellectual insight, mathematical might and geometrical vision beyond the ordinary. By its nature this accomplishment is of no less value and significance to science than is the general theory of relativity, which in much the same way is a consequence of geometry. Let us recall Herglotz's derivation.

Consider the system of $2n+1$ differential equations

$$\dot{X} = \xi\left(X, Z, P, t\right), \quad \dot{Z} = \zeta\left(X, Z, P, t\right), \quad \dot{P} = \pi\left(X, Z, P, t\right) \quad (11)$$

for $2n+1$ unknowns $X = \left(X_1, \cdots, X_n\right), Z, X = \left(P_1, \cdots, P_n\right)$ which satisfy the initial conditions

$$X = x, \quad Z = z, \quad P = p, \quad \text{when } t = 0. \quad (12)$$

The functions $\xi = \left(\xi_1, \cdots, \xi_n\right), \zeta, \pi = \left(\pi_1, \cdots, \pi_n\right)$ are all assumed to be continously differentiable. The solutions

$$X = X\left(x, z, p, t\right), \quad Z = Z\left(x, z, p, t\right), \quad P = P\left(x, z, p, t\right) \quad (13)$$

to (11), (12) determine a family of transformations,

$$S_t : \left(x, z, p\right) \to \left(X, Z, P\right). \quad (14)$$

In the present section we give the necessary and sufficient conditions for the transformations (14) to be contact transformations uniformly in t.

Theorem 2.1. *In order for the solution (13) of the system (11) to represent a one-parameter family of contact transformations containing the identity, it is necessary that (11) be a canonical system, that is, that there exists a function,* $\mathcal{H}=\mathcal{H}(X,Z,P,t)$ *called the characteristic function, such that the system (11) has the form (10).*

Proof: From the theory of contact transformations is known that a one–to–one, continuously differentiable transformation

$$X=X(x,y,p),\quad Z=Z(x,y,p),\quad P=P(x,y,p)$$

from a cylindrical domain in the xzp-space, $\{(x,z,p):(x,z)\in D,|p|\le\tilde{p},\tilde{p}\text{ a const.}\}$ onto a domain in XZP-space, with a non-vanishing Jacobian, $\partial(X,Z,P)/\partial(x,z,p)\ne 0$ is called an *element transformation*. We now recall the following criterion for an element transformation to be a contact transformation: *An element transformation is a contact transformation if and only if there exists a function* $\rho(x,z,p)$*, such that:*

i. $\rho(x,z,p)\ne 0$ on the domain of definition of the transformation;

ii. the functions ρ, X, Z and P satisfy

$$P\cdot \mathrm{d}X-\mathrm{d}Z=\rho(p\cdot \mathrm{d}x-\mathrm{d}z),\quad \rho\ne 0. \tag{15}$$

Equation (15) is supposed to hold when the differentials are calculated only with respect to the spatial variables. When X,Z,P also depend on t, then $\mathrm{d}Z$ is given by

$$\mathrm{d}Z=\frac{\partial Z}{\partial x_j}\mathrm{d}x_j+\frac{\partial Z}{\partial z}\mathrm{d}z+\frac{\partial Z}{\partial p_j}\mathrm{d}p_j+\frac{\partial Z}{\partial t}\mathrm{d}t.$$

A similar assertion holds for the $\mathrm{d}X_i$. Thus, the condition (15) must be replaced by

$$P_i\mathrm{d}X_i-\mathrm{d}Z-\left(P_i\frac{\partial X_i}{\partial t}-\frac{\partial Z}{\partial t}\right)\mathrm{d}t=\rho(p_i\mathrm{d}x_i-\mathrm{d}z). \tag{16}$$

By (11), $\partial X_i/\partial t=\xi_i(X,Z,P,t)$, $\partial Z_i/\partial t=\zeta(X,Z,P,t)$. Let us introduce the function

$$\mathcal{H}\equiv\mathcal{H}(X,Z,P,t)\equiv P_i\xi_i(X,Z,P,t)-\zeta(X,Z,P,t). \tag{17}$$

Then the relation (16) takes the form

$$P\cdot \mathrm{d}X-\mathrm{d}Z=\rho(p\cdot \mathrm{d}x-\mathrm{d}z)+\mathcal{H}\mathrm{d}t. \tag{18}$$

If $\mathrm{d}t=0$, Equation (18) reduces to (15). Equation (18) represents a system of $2n+2$ equations relating the variables (X,Z,P,t) with those of (x,z,p,t), which is obtained by expanding the differentials and comparing coefficients. To obtain the conditions we seek, we shall rewrite these conditions in the (X,Z,P,t) variables. This is most simply done by working directly with (18). First differentiate (18) with respect to t and note that the differential operator, d, commutes with the differentiation $\mathrm{d}/\mathrm{d}t$. This leads to

$$\pi_j \mathrm{d}X_j + P_j \mathrm{d}\xi_j - \mathrm{d}\zeta = \dot{\rho}\left(p_j \mathrm{d}x_j - \mathrm{d}z\right) + \dot{\mathcal{H}}\mathrm{d}t \tag{19}$$

where $\partial P_j / \partial t = \pi_j\left(X, Z, P, t\right)$, the dot, as usual, represents $\mathrm{d}/\mathrm{d}t$. From (18) and (19) we obtain

$$\pi_j \mathrm{d}X_j + P_j \mathrm{d}\xi_j - \mathrm{d}\zeta - \dot{\mathcal{H}}\mathrm{d}t = \frac{\dot{\rho}}{\rho}\left(P_j \mathrm{d}X_j - \mathrm{d}Z - \mathcal{H}\mathrm{d}t\right). \tag{20}$$

From (17) we find $\mathrm{d}\mathcal{H} = \xi_j \mathrm{d}P_j + P_j \mathrm{d}\xi_j - \mathrm{d}\zeta$ so that (20) takes the form

$$\mathrm{d}\mathcal{H} + \pi_j \mathrm{d}X_j - \xi_j \mathrm{d}P_j = \frac{\dot{\rho}}{\rho}\left(P_j \mathrm{d}X_j - \mathrm{d}Z\right) + \left(\dot{\mathcal{H}} - \frac{\dot{\rho}}{\rho}\mathcal{H}\right)\mathrm{d}t. \tag{21}$$

Expand $\mathrm{d}\mathcal{H}$ in the form

$$\mathrm{d}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial X_j}\mathrm{d}X_j + \frac{\partial \mathcal{H}}{\partial Z}\mathrm{d}Z + \frac{\partial \mathcal{H}}{\partial P_j}\mathrm{d}P_j + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t$$

insert the result into (21) and compare coefficients to obtain the following system:

$$\begin{aligned} &\frac{\partial \mathcal{H}}{\partial X_j} = -\pi_j + \frac{\dot{\rho}}{\rho}P_j, \quad &&\frac{\partial \mathcal{H}}{\partial P_j} = \xi_j, \\ &\frac{\partial \mathcal{H}}{\partial Z} = -\frac{\dot{\rho}}{\rho}, &&\frac{\partial \mathcal{H}}{\partial t} = \dot{\mathcal{H}} - \frac{\dot{\rho}}{\rho}\mathcal{H}. \end{aligned} \tag{22}$$

The ξ_j and π_j are obtained directly from (22) by eliminating the quotient $\dot{\rho}/\rho$ and solving. To obtain ζ combine (17) with (22). We find

$$\begin{aligned} &\xi_j = \frac{\partial \mathcal{H}}{\partial P_j}, \\ &\zeta = P_j \xi_j - \mathcal{H} = P_j \frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}, \\ &\pi_j = -\frac{\partial \mathcal{H}}{\partial X_j} - P_j \frac{\partial \mathcal{H}}{\partial Z}, \quad j = 1, \cdots, n. \end{aligned} \tag{23}$$

which is the system (10).□

We see that the geometric demand that the solution is a family of contact transformations of the initial conditions completely determines the form of the system.

The converse of this theorem is also valid. We state and prove.

Theorem 2.2. *The solution to the canonical equations (10), which satisfy the initial conditions (12), generates a one-parameter family of contact transformations, which for* $t = 0$ *contains the identity.*

Proof. We must show that every solution of (10) and (12) satisfies the strip condition (18). For notational purposes let us set $\Omega = \Omega(t) \equiv P_j \mathrm{d}X_j - \mathrm{d}Z - \mathcal{H}\mathrm{d}t$ and $\Omega(0) \equiv \omega = p_j \mathrm{d}x_j - \mathrm{d}z$. Then the strip condition, (18) takes the form $\Omega(t) = \rho\omega$. Set up a differential equation for Ω making use of (10). The proof is simply a calculation. We find $\dot{\Omega} = \dot{P}_j \mathrm{d}X_j + P_j \mathrm{d}\dot{X}_j - \mathrm{d}\dot{Z} - \dot{\mathcal{H}}\mathrm{d}t$. Since $\mathcal{H} = P_j \mathrm{d}\dot{X}_j - \mathrm{d}\dot{Z}$,

$$\begin{aligned} \dot{\Omega} &= \dot{P}_j \mathrm{d}X_j - \dot{X}_j P_j \\ &= -\left(\frac{\partial \mathcal{H}}{\partial X_j} + P_j \frac{\partial \mathcal{H}}{\partial Z}\right)\mathrm{d}X_j - \frac{\partial \mathcal{H}}{\partial P_j}\mathrm{d}P_j \end{aligned}$$

$$= -\left(\frac{\partial \mathcal{H}}{\partial X_j}\mathrm{d}X_j + \frac{\partial \mathcal{H}}{\partial P_j}\mathrm{d}P_j\right) - \frac{\partial \mathcal{H}}{\partial Z}\left(P_j\,\mathrm{d}X_j\right)$$

$$= -\left(\frac{\partial \mathcal{H}}{\partial X_j}\mathrm{d}X_j + \frac{\partial \mathcal{H}}{\partial P_j}\mathrm{d}P_j\right) + \frac{\partial \mathcal{H}}{\partial Z}\mathrm{d}Z + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t$$

$$-\frac{\partial \mathcal{H}}{\partial Z}\left(P_j\,\mathrm{d}X_j\right) + \frac{\partial \mathcal{H}}{\partial Z}\mathrm{d}Z + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t$$

$$= -\mathrm{d}\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\left(P_j\,\mathrm{d}X_j - \mathrm{d}Z - \mathcal{H}\mathrm{d}t\right) - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\mathrm{d}t + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t$$

$$= -\mathrm{d}\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\Omega - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\mathrm{d}t + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t.$$

Thus we obtain the ODE for Ω

$$\dot{\Omega} = -\mathrm{d}\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\Omega - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\mathrm{d}t + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t.$$

Next we calculate, using (10)

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial X_j}\frac{\mathrm{d}X_j}{\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial P_j}\frac{\mathrm{d}P_j}{\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial Z}\frac{\mathrm{d}Z}{\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial t}$$

$$= \frac{\partial \mathcal{H}}{\partial X_j}\frac{\partial \mathcal{H}}{\partial P_j} - \frac{\partial \mathcal{H}}{\partial P_j}\left(\frac{\partial \mathcal{H}}{\partial X_j} + P_j\frac{\partial \mathcal{H}}{\partial Z}\right) + \frac{\partial \mathcal{H}}{\partial Z}\left(P_j\frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}\right) + \frac{\partial \mathcal{H}}{\partial t}$$

$$= -\mathcal{H}\frac{\partial \mathcal{H}}{\partial Z} + \frac{\partial \mathcal{H}}{\partial t}.$$

Thus,

$$\mathrm{d}\mathcal{H} = -\mathcal{H}\frac{\partial \mathcal{H}}{\partial Z}\mathrm{d}t + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t.$$

and so from the previous calculation

$$\dot{\Omega} = -\frac{\partial \mathcal{H}}{\partial Z}\Omega.$$

We integrate to obtain $\Omega = \rho\omega$ where

$$\rho = \exp\left(-\int_0^t \frac{\partial \mathcal{H}}{\partial Z}\mathrm{d}t\right) \tag{24}$$

which proves the assertion. □

We close this section with a few remarks on the characteristic function

$$\mathcal{H} = \mathcal{H}\left(X, Z, P, t\right).$$

From the forth equation in (22), we have

$$\rho\frac{\partial \mathcal{H}}{\partial t} = \rho\dot{\mathcal{H}} - \dot{\rho}\mathcal{H}.$$

Divide by ρ^2 to find

$$\frac{1}{\rho}\frac{\partial \mathcal{H}}{\partial t} = \frac{\rho\dot{\mathcal{H}} - \dot{\rho}\mathcal{H}}{\rho^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathcal{H}}{\rho}\right).$$

Integrate with respect to t to find

$$\frac{\mathcal{H}}{\rho}-\frac{\mathcal{H}^0}{\rho^0}=\int_0^t \frac{1}{\rho}\frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t, \tag{25}$$

where the subscript indicates that the arguments of $\mathcal{H}$ and ρ are to be taken at $t=0$: $\rho^0=\rho(x,z,p,0)$, $\mathcal{H}^0=\mathcal{H}(x,z,p,0)$. The fact that $\rho^0=1$ is a consequence of (24).

We consider two special cases.

Case 1. $\partial\mathcal{H}/\partial t=0$ so that $\mathcal{H}$ does not depend explicitely on t.

Then the family, $\{S_t\}$, represents a one-parameter group of contact transformations (The proof can be found in [7]). The relation (25) implies that

$$\mathcal{H}(X,Z,P)=\mathcal{H}^0(x,z,p)\rho(x,z,p). \tag{26}$$

Equation (26) has a geometric interpretation. Let us think of the parameter t as the time and the curve along which $(X,Z,P)=S_t(x,z,p)$ moves in $\mathbf{R}^{2n+1}$ as its orbit under the group of contact transformations. Along this orbit the function $\mathcal{H}(X,Z,P)$, up to the factor H^0, coincides with $\rho(X,Z,P)$.

If in particular $H^0=0$ at a point (x,z,p), then $\mathcal{H}(X,Z,P)=0$ along the whole orbit through it. The strip condition is along the orbit. If we think of (X,Z,P) as an element in $\mathbf{R}^{n+1}$, then we refer to the orbit as an orbital strip of the group of contact transformations in $\mathbf{R}^{n+1}$. For points on the orbital strip, the second equation in (10) simplifies to

$$\frac{\mathrm{d}Z}{\mathrm{d}t}=P_j\frac{\partial\mathcal{H}}{\partial P_j},\quad j=1,\cdots,n.$$

Case 2. $\partial\mathcal{H}/\partial Z=0$ so that $\mathcal{H}$ does not depend explicitely on Z and by (24) $\rho=\rho(X,Z,P,t)\equiv 1$.

The canonical Equations (10) reduce to

$$\frac{\mathrm{d}X_j}{\mathrm{d}t}=\frac{\partial\mathcal{H}}{\partial P_j},\quad \frac{\mathrm{d}P_j}{\mathrm{d}t}=-\frac{\partial\mathcal{H}}{\partial X_j} \tag{27}$$

together with the additional equation

$$\frac{\mathrm{d}Z}{\mathrm{d}t}=P_j\frac{\partial\mathcal{H}}{\partial P_j}-\mathcal{H},\quad j=1,\cdots,n \tag{28}$$

for the construction of Z.

The transformations determined by (27) are the special, or xp-transformations which commute with translations along the z-axis. The Equation (18) in this case reads

$$P_j\mathrm{d}X_j-p_j\mathrm{d}x_j=\mathrm{d}(Z-z)+\mathcal{H}\mathrm{d}t.$$

If in addition, $\partial\mathcal{H}/\partial t=0$, then $\mathcal{H}=\mathcal{H}^0$. The family determined by solutions to (27) is a group of contact transformations which on the orbit passing through (x,z,p) satisfies $\mathcal{H}(X,Z,P)=\mathcal{H}^0(x,z,p)$.

3. Evolutionary System for Nonconservative Processes

This section introduces the new exact method of solution for nonconservative systems. It also introduces the evolutionary system of PDEs which is equivalent to the nonconservative system of second order PDEs with which the nonconservative process is described. Theorem 3.2 proves with deepest rigor the equivalence of theses two systems. The same theorem demonstrates how starting with this second order PDEs system we can rewrite it with its equivalent system of evolutionary PDEs and proceed with solving this equivalent system. The solution obtained is also a solution of the original nonconservative system. The key to obtaining this valuable new exact method of solution for these systems, which in general can be solved only numerically, is the generalization of the transformation of Legendre with the definition (40), which gives the transition from the Lagrangian to the Hamiltonian formalism. Theorem 3.2 rigorously proves that this new *generalized Legendre transformation* transforms the original nonconservative system of second order PDEs to the equivalent to it evolutionary system (38). Thus, the present section makes 2 new contributions to mathematics and all of science which uses mathematics: the exact method of solution for nonconservative systems and the exact form of the system of evolution equations for nonconservative processes.

It is important to observe that this new evolutionary system contains as a strictly special case the generalized Hamiltonian system (1) for nonconservative processes with one independent variable, and has the same relation to the generalized variational principle with several independent variables which the generalized Hamiltonian system (1) has with the Herglotz variational principle (which has one independent variable).

Consider a generally nonconservative process in which t and $\mathrm{d}x^1\cdots\mathrm{d}x^n$ are the independent variables and $u\equiv\left(u^1,\cdots,u^m\right)$ are the dependent variables. Let the functional z be the functional giving the variational description of this process. Then the functional $z=z[u;s]$ of $u=u(t,x)$ is defined by an integro-differential equation of the form

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\int_\Omega \mathcal{L}\left(t,x,u,u_t,u_x,z\right)\mathrm{d}^n x,\quad 0\le t\le s \tag{29}$$

where $u\equiv\left(u^1,\cdots,u^m\right)$ are the argument functions, $u_x\equiv\left(u_x^1,\cdots,u_x^m\right)$, $u_t\equiv\left(u_t^1,\cdots,u_t^m\right)$ and $u_x^i\equiv\left(u_{x^1}^i,\cdots,u_{x^n}^i\right)$, $i=1,\cdots,m$, $\mathrm{d}^n x\equiv\mathrm{d}x^1\cdots\mathrm{d}x^n$, and where the Lagrangian function $\mathcal{L}$ is at least twice differentiable with respect to u_x, u_t and once differentiable with respect to t,x,z.

Theorem 3.1. *Every function* $u\equiv\left(u^1,\cdots,u^m\right)$*, for which the functional* z *defined by the integro-differential Equation (29) has an extremum, is a solution of*

$$\frac{\partial\mathcal{L}}{\partial u^i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial u_t^i}-\frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial\mathcal{L}}{\partial u_{x^k}^i}+\frac{\partial\mathcal{L}}{\partial u_t^i}\int_\Omega\frac{\partial\mathcal{L}}{\partial z}\mathrm{d}x=0,\quad i=1,\cdots,m \tag{30}$$

This theorem was formulated and proved by Georgieva, Guenther and Bodurov in [1]. We recall its proof here for convenience of the reader.

Proof: We will show that Equation (30) are consequence of the condition:

The value of the functional $z[u;s]$ is an extremum for functions u which satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} z[u+\varepsilon\eta; s]\Big|_{\varepsilon=0} = 0. \tag{31}$$

For this purpose let $\varepsilon\eta$ be the variation of the argument of the functional z and denote by $\zeta = \zeta(t)$ the quantity

$$\zeta(t) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} z[u+\varepsilon\eta; t]\Big|_{\varepsilon=0}. \tag{32}$$

To find the differential equation for ζ we apply the variation $\varepsilon\eta$ to the argument function in the defining Equation (29), *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} z[u+\varepsilon\eta; t] = \int_\Omega \mathcal{L}(t, x, u+\varepsilon\eta, u_t+\varepsilon\eta_t, u_x+\varepsilon\eta_x, z)\,\mathrm{d}^n x \tag{33}$$

and differentiate the result with respect to ε. Then, we set $\varepsilon = 0$ to obtain

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\eta^i_t + \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\eta^i_{x^k} \right)\mathrm{d}^n x + \zeta \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\mathrm{d}^n x, \quad i = 1, \cdots, m \tag{34}$$

where in the last term we have used the fact that ζ does not depend on x. For convenience we denote by $A(t)$ and $B(t)$ the quantities

$$A(t) = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\eta^i_t + \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\eta^i_{x^k} \right)\mathrm{d}^n x, \quad B(t) = \int_\Omega \frac{\partial \mathcal{L}}{\partial z}\mathrm{d}^n x.$$

With this notation Equation (34) becomes

$$\frac{\mathrm{d}\zeta(t)}{\mathrm{d}t} = A(t) + B(t)\zeta(t)$$

which is the sought equation for $\zeta(t)$. Its solution $\zeta(s)$, evaluated at the end of the time interval $t = s$, is the variation of $z[u; s]$ and is given by

$$\exp\left(-\int_0^s B(\theta)\mathrm{d}\theta\right)\zeta(s) = \int_0^s \exp\left(-\int_0^t B(\theta)\mathrm{d}\theta\right)A(t)\mathrm{d}t \tag{35}$$

since $\zeta(0) = 0$. We are interested in those functions u which leave the functional $z[u; s]$ stationary, *i.e.* those for which the variation $\zeta(s)$ is identically zero. Hence, (35) becomes

$$\int_0^s \exp\left(-\int_0^t B(\theta)\mathrm{d}\theta\right)A(t)\mathrm{d}t = 0. \tag{36}$$

Inserting expressions $A(t)$ and $B(t)$ into (36), denoting the exponential function by

$$E(t) \equiv \exp\left(-\int_0^t B(\theta)\mathrm{d}\theta\right)$$

and integrating by parts the terms containing $\eta^i_{x^k}$ produces the equation

$$\int_0^s E(t)\int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\eta^i_t + \frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\eta^i\right) - \eta^i \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{L}}{\partial u^i_{x^k}} \right)\mathrm{d}^n x\mathrm{d}t = 0. \tag{37}$$

By Gauss' theorem the space integral of the third term in (37)

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\eta^i\right)\mathrm{d}^n x = \int_{\partial\Omega} \eta^i \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\mathrm{d}a_k = 0$$

vanishes because $\eta = 0$ on $\partial\Omega$, by definition. Here, $\mathrm{d}^n a$ stands for the k-component of the surface element $\mathrm{d}^n a = (\mathrm{d}a_1, \cdots, \mathrm{d}a_n)$ of $\partial\Omega$. Next, we integrate by parts (with respect to t) the terms involving η_t^i in (37) to obtain

$$\int_0^s E(t)\int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\right)\eta^i \mathrm{d}^n x \mathrm{d}t - \int_\Omega \int_0^s \eta^i \frac{\mathrm{d}}{\mathrm{d}t}\left(E(t)\frac{\partial \mathcal{L}}{\partial u^i_t}\right)\mathrm{d}t\mathrm{d}^n x = 0$$

because $\eta(0,x) = 0$ and $\eta(s,x) = 0$, by definition. Finally, expanding the second integrand in the last equation and collecting terms, we get

$$\int_0^s \int_\Omega E(t)\left(\frac{\partial L}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial L}{\partial u^i_{x^k}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u^i_t} + \frac{\partial L}{\partial u^i_t}\int_\Omega \frac{\partial L}{\partial z}\mathrm{d}^n x\right)\eta^i \mathrm{d}^n x \mathrm{d}t = 0.$$

Taking in consideration that η is arbitrary and that $E(t) > 0$ for all t, we obtain equations (30), which concludes the proof. □

Theorem 3.2. *The functions* (u,z) *for which the functional* z *has stationary values satisfy the following evolutionary system:*

$$\begin{aligned} u_t^i &= \frac{\partial \mathcal{H}}{\partial p^i}, \quad i = 1, \cdots, m \\ z_t &= \int_\Omega \left(p^i \frac{\partial \mathcal{H}}{\partial p^i} - \mathcal{H}\right)\mathrm{d}x \\ p_t^i &= -\frac{\partial \mathcal{H}}{\partial u^i} + \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{H}}{\partial u^i_{x^k}} - p^i \int_\Omega \frac{\partial \mathcal{H}}{\partial z}\mathrm{d}x, \quad i = 1, \cdots, m. \end{aligned} \tag{38}$$

Proof. From the previous theorem we know that the functions $u \equiv (u^1, \cdots, u^m)$, for which the functional z defined by the integro-differential Equation (29) has an extremum, are solutions of the generalized Euler-Lagrange Equation (30):

$$\frac{\partial \mathcal{L}}{\partial u^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial u^i_t} - \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{L}}{\partial u^i_{x^k}} + \frac{\partial \mathcal{L}}{\partial u^i_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\mathrm{d}x = 0, \quad i = 1, \cdots, m.$$

Define

$$p^i \equiv \frac{\partial \mathcal{L}}{\partial u^i_t}, \quad i = 1, \cdots, m. \tag{39}$$

The implicit function theorem can now be invoked to show that the system of m algebraic Equation (39) relating the variables $p^i, t, x^j, u^i, u^i_t, u^i_{x^k}, z$ can (in principle) be solved for the u^i_t variables in terms of the variables $p^i, t, x^j, u^i, u^i_{x^k}, z$, provided the $m \times m$ Hessian matrix

$$\left(\frac{\partial^2 \mathcal{L}}{\partial u^i_t \partial u^l_t}\right), \quad i, l = 1, \cdots, m$$

is nonsingular, *i.e.* satisfies the Jacobian condition

$$\frac{\partial(p^1, \cdots, p^m)}{\partial(u^1_t, \cdots, u^m_t)} \neq 0.$$

Now we define the function $\mathcal{H}$ by

$$\mathcal{H}\left(t,x,u,p,u_x,z\right) \equiv \sum_{j=1}^{m} p^j u_t^j - \mathcal{L}\left(t,x,u,u_t,u_x,z\right) \tag{40}$$

which we name *generalized Hamiltonian*, where the variables u_t^i are regarded as functions of $p^1,\cdots,p^m,u^1,\cdots,u^m,u_{x^k}^1,\cdots,u_{x^k}^m,z,x^1,\cdots,x^n,t$.

Let us observe that

$$\frac{\partial \mathcal{H}}{\partial p^i} = \sum_{j=1}^{m}\left(-\frac{\partial \mathcal{L}}{\partial u_t^j} + p^j\right)\frac{\partial u_t^j}{\partial p^i} + u_t^i = u_t^i, \quad i = 1,\cdots,m. \tag{41}$$

Performing the above change of variables from the $\left(t,x,u,u_t,u_x,z\right)$ variables of the Lagrangian to the $\left(t,x,u,p,u_x,z\right)$ variables of the Hamiltonian in the generalized Euler-Lagrange Equation (30) we obtain the evolutionary equation for p^i:

$$p_t^i = -\frac{\partial \mathcal{H}}{\partial u^i} + \frac{\mathrm{d}}{\mathrm{d}x^k}\frac{\partial \mathcal{H}}{\partial u_{x^k}^i} - p^i \int_\Omega \frac{\partial \mathcal{H}}{\partial z}\mathrm{d}x, \quad i = 1,\cdots,m. \tag{42}$$

Finally, expressing $\mathcal{L}$ via $\mathcal{H}$, p^i and $\partial \mathcal{H}/\partial p^i$, we obtain the evolutionary equation for z, namely

$$z_t = \int_\Omega \left(p^i \frac{\partial \mathcal{H}}{\partial p^i} - \mathcal{H}\right)\mathrm{d}x. \tag{43}$$

To conclude the derivation, we observe that Equations (41)-(43) are the system (38) which was sought. □

We observe that the evolutionary system (38) contains as a special case the generalized Hamiltonian system (1). This special case occurs when no "space variables" $x^1 \cdots x^n$ are present in the described process.

Theorem 3.2 rigorously proves the equivalence of the newly introduces system of evolution Equation (38) to the system (30) of second order PDEs with which the nonconservative process is originally described!

In conclusion, we observe that

The evolutionary system (38) serves two purposes: first—it is uniquely suitable for the description of nonconservative processes via an evolutionary system and second—it provides a method for solution of the original generalized Euler-Lagrange equations of the process. A great deal of machinery is present for studying evolutionary systems and that machinery now becomes available to tackle the complicated and often formidable system of the generalized Euler-Lagrange equations (30).

4. Applications

In this section we demonstrate the usefulness of the evolutionary system (38) to the description of nonconservative processes and how it is used to obtain exact solutions of such nonconservative systems.

1) Propagation of plane waves in a medium with losses or gains

The first process which we like to describe with this system is the propagation of waves with losses or gains. Consider the equation

$$\frac{\partial^2 q}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} + b\frac{\partial q}{\partial t} = 0, \quad b = \text{const}, \quad c = \text{const},$$

which describes the propagation of plane waves in a medium with losses or gains. In the case of sound waves with gasses $q(t,x)$ is the acoustic pressure of the gas (the local deviation from ambient pressure) at the point x at time t, c is the speed of the sound wave and b is the rate of losses ($b<0$) or gains ($b>0$). This equation does not have a variational description via the classical variational principle, but can be described variationally with the variational principle proposed by Georgieva, Guenther and Bodurov in [1]. This equation is the generalized Euler-Lagrange equation for the functional $z[q(t,x)]$ defined with the integro-differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \int_\Omega \left(\left(\frac{\partial q}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial q}{\partial t}\right)^2 + \alpha(x) z \right) \mathrm{d}x,$$

where $\Omega = [0,\infty)$ and $\alpha(x)$ is such that $\int_\Omega \alpha(x)\mathrm{d}x = bc^2$. Let us describe this process with the evolutionary system (38). Here the Lagrangian is $\mathcal{L} = \left(\frac{\partial q}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial q}{\partial t}\right)^2 + \alpha(x) z$. Define p with

$$p = \frac{\partial \mathcal{L}}{\partial q_t}.$$

The generalized Hamiltonian is

$$\mathcal{H} = pq_t - q_x^2 + \frac{1}{c^2} q_t^2 + \alpha(x) z = -\frac{c^2 p^2}{4} - q_x^2 + \alpha(x) z.$$

The evolutionary system (38) is

$$q_t = \frac{\partial \mathcal{H}}{\partial p}$$

$$z_t = \int_\Omega \left(p\frac{\partial \mathcal{H}}{\partial p} - H \right)\mathrm{d}x = \int_\Omega \left(\left(\frac{\partial q}{\partial x}\right)^2 - \frac{c^2}{4} p^2 + \alpha(x) z \right)\mathrm{d}x$$

$$p_t = -\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial q_x} - p\int_\Omega \frac{\partial \mathcal{H}}{\partial z}\mathrm{d}x = -2q_{xx} - bc^2 p.$$

2) Propagation of electromagnetic waves in a conductive medium

The second nonconservative process which we like to describe with the evolutionary system (38) is that of propagation of electromagnetic waves in a conductive medium.

Consider the equations

$$c^2 \nabla^2 \boldsymbol{E} - \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{\sigma}{\varepsilon}\frac{\partial \boldsymbol{E}}{\partial t} = 0 \tag{44}$$

where $\boldsymbol{E} = (E^1, E^2, E^3)$ is the electric field vector, c is the velocity of the electromagnetic waves, σ is the electrical conductivity and ε is the dielectric constant of the medium. Exactly the same equation holds for the magnetic field vector $\boldsymbol{B} = (B^1, B^2, B^3)$. These equations describe the propagation of electromagnetic waves in a conductive medium and are direct consequence of the Maxwell's equations in conjunction with the medium's property equations $\boldsymbol{J}=\sigma\boldsymbol{E}$ and $\rho=0$, where $\boldsymbol{J}=\left(J^1,J^2,J^3\right)$ is the current density and ρ is the charge density.

Equations (44) are the system of Euler-Lagrange equations (30) for the functional z defined by Equation (29) with Lagrangian

$$\mathcal{L}=c^2\frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j}-\frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t}+\alpha\left(x\right)z,\quad i,j=1,2,3$$

$$\frac{\sigma}{\varepsilon}=\int_\Omega\alpha\left(x\right)\mathrm{d}^3x=\text{const.}$$

We define p with

$$p^i=\frac{\partial\mathcal{L}}{\partial E_t^i},\quad i=1,2,3.$$

The generalized Hamiltonian is

$$\mathcal{H}=p^iE_t^i-\mathcal{L}=p^iE_t^i-c^2\frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j}+\frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t}-\alpha\left(x\right)z,\quad i,j=1,2,3.$$

The evolutionary system (38) is

$$E_t^i=\frac{\partial\mathcal{H}}{\partial p},\quad i=1,2,3$$

$$z_t=\int_\Omega\left(p^i\frac{\partial\mathcal{H}}{\partial p^i}-\mathcal{H}\right)\mathrm{d}x=\int_\Omega\left(p^iE_t^i-p^iE_t^i+c^2\frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j}-\frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t}+\alpha\left(x\right)z\right)\mathrm{d}x$$

$$p_t^i=-\frac{\partial\mathcal{H}}{\partial E^i}+\frac{\partial}{\partial x^k}\frac{\partial\mathcal{H}}{\partial E_{x^k}^i}-p^i\int_\Omega\frac{\partial\mathcal{H}}{\partial z}\mathrm{d}x=-2c^2\frac{\partial^2E^i}{\partial\left(x^k\right)^2}-\frac{\sigma}{\varepsilon}p^i.$$

3) The sine-Gordon equation

As a third example consider the one-dimensional nonconservative sine-Gordon equation

$$\frac{\partial^2u}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2u}{\partial t^2}+k\frac{\partial u}{\partial t}+\sin u=0.$$

There is no classical Lagrangian for this equation, because its Frechet derivative is not self-adjoint. But it has a variational description with the generalized variational principle (29), (30), introduced in [1]. It is the Euler-Lagrange Equation (30) for the variational functional z defined by (29) with

$$\mathcal{L}=\left(\frac{\partial u}{\partial x}\right)^2-\frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2-2\cos u+\alpha\left(x\right)z.$$

Here k and v are constants, v represents the velocity of the wave and α is a differentiable function such that $\int_\Omega\alpha\left(x\right)\mathrm{d}x=kv^2$. Define p with

$$p=\frac{\partial\mathcal{L}}{\partial u_t}.$$

Then

$$u_t = -\frac{v^2 p}{2}$$

and the generalized Hamiltonian is

$$\mathcal{H} = pu_t - \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 + 2\cos u - \alpha(x) z$$
$$= -\frac{v^2 p^2}{4} - u_x^2 + 2\cos u - \alpha(x) z.$$

The evolutionary system (38) is

$$u_t = \frac{\partial \mathcal{H}}{\partial p}$$
$$z_t = \int_\Omega \left(p\frac{\partial \mathcal{H}}{\partial p} - \mathcal{H}\right) \mathrm{d}x$$
$$= \int_\Omega \left(pu_t - pu_t + \left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - 2\cos u + \alpha(x) z\right) \mathrm{d}x = \int_\Omega \mathcal{L} \mathrm{d}x$$
$$p_t = -\frac{\partial \mathcal{H}}{\partial u} + \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial u_x} - p\int_\Omega \frac{\partial \mathcal{H}}{\partial z} \mathrm{d}x = 2\sin u - 2\frac{\partial}{\partial x} u_x + pkv^2$$
$$= -2u_{xx} + pkv^2 + 2\sin u.$$

Problem. Carry out the above procedure, for deriving the evolutionary system, to obtain the evolutionary system for the non-linear non-conservative Klein–Gordon equation.

4) The nonlinear Schrodinger equation with electromagnetic interactions

As a last example let us consider the nonlinear Schrodinger equation

$$i\frac{\partial \Psi}{\partial t} - \Phi\Psi + \mu\left(\frac{\partial}{\partial x^k} - iA_k\right)^2 \Psi - G\left(\Psi\Psi^*, x\right)\Psi - \beta\frac{i}{2}\Psi = 0, \quad \beta = \text{const.} \quad (45)$$

for the wave function $\Psi\left(t, x^1, x^2, x^3\right)$ with electromagnetic interaction and losses or gains, where there is summation over $k = 1,2,3$. Here $\left(\Phi\left(t, x^1, x^2, x^3\right), \boldsymbol{A}\left(t, x^1, x^2, x^3\right)\right)$ is the electromagnetic potential, G is a real-valued function, and $\boldsymbol{A} = \left(A_1, A_2, A_3\right)$ is the vector potential. The losses ($\beta > 0$) or the gains ($\beta < 0$) are represented with the term $-\beta\frac{i}{2}\Psi$. This equation does not have a variational description via the classical variational principle, because its Frechet derivative is not self-adjoint. Equation (45) has a variational description via the generalized variational principle (29), (30), introduced by Georgieva, Guenther and Bodurov in [1]. The functional z of the argument functions Ψ and Ψ^* is defined by the integro-differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \int_\Omega \left(\frac{i}{2}\left(\Psi^*\left(\frac{\partial \Psi}{\partial t} + i\Phi\Psi\right) - \left(\frac{\partial \Psi^*}{\partial t} - i\Phi\Psi^*\right)\Psi\right) - \mu\left(\frac{\partial \Psi^*}{\partial x^k} + iA_k\Psi^*\right)\left(\frac{\partial \Psi}{\partial x^k} - iA_k\Psi\right) - \Gamma\left(\Psi^*\Psi, x\right) + \alpha(x) z\right) \mathrm{d}^3x \quad (46)$$

where the function $\alpha(x)$ is such that $\int_\Omega \alpha(x)\mathrm{d}^3x = \beta$ and the function

$\Gamma(\rho, x)$ is such that $G(\rho, x) = \partial\Gamma(\rho, x)/\partial\rho$. Now that we have the generalized Lagrangian $\mathcal{L}$ (the integrand in Equation (46)) for the Schrodinger Equation (45), we define p with

$$p = \frac{\partial \mathcal{L}}{\partial \Psi_t^*}$$

where the t-subscript denotes partial differentiation with respect to time. Then p^* is

$$p^* = \frac{\partial \mathcal{L}}{\partial \Psi_t}.$$

Now we define the generalized Hamiltonian function as

$$\mathcal{H} = \frac{i}{2}\left(p^*\Psi_t - p\Psi_t^*\right) - \frac{i}{2}\left(\Psi^*\left(\Psi_t + i\Phi\Psi\right) - \left(\Psi_t^* - i\Phi\Psi^*\right)\Psi\right) + \mu\left(\frac{\partial\Psi^*}{\partial x^k} + iA_k\Psi^*\right)\left(\frac{\partial\Psi}{\partial x^k} - iA_k\Psi\right) + \Gamma\left(\Psi^*\Psi, x\right) - \alpha(x)z$$

The evolutionary system is

$$\begin{aligned} \Psi_t &= -i\left(\frac{\partial\mathcal{H}}{\partial p^*} + \left(\frac{\partial\mathcal{H}}{\partial p}\right)^*\right) \\ z_t &= \int_\Omega \left(p\frac{\partial\mathcal{H}}{\partial p} + p^*\frac{\partial\mathcal{H}}{\partial p^*} - \mathcal{H}\right)\mathrm{d}x = \int_\Omega \mathcal{L}\mathrm{d}^3x \\ p_t &= -\frac{\partial\mathcal{H}}{\partial\Psi^*} + \frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial\Psi_x^*} - p\int_\Omega \frac{\partial\mathcal{H}}{\partial z}\mathrm{d}x = \left(\beta + \frac{i}{2}\right) - \frac{i}{2}p_x. \end{aligned} \tag{47}$$

The equation for p in the system (47) is a linear PDE and is immediately solvable with the method of characteristics!

In these applications we demonstrated how the generalized Hamiltonian system (38) which this paper introduces can be used to successfully simplify and in many cases solve the otherwise formidable generalized Euler-Lagrange equations (30) for a nonconservative process with several independent variables. Useful references which treat the matters discussed in the present paper are [16]-[21].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Georgieva, B., Guenther, R. and Bodurov, T. (2003) Generalized Variational Principle of Herglotz for Several Independent Variables. First Noether-Type Theorem. *Journal of Mathematical Physics*, **44**, 3911-3927. https://doi.org/10.1063/1.1597419

[2] Georgieva, B. and Guenther, R.B. (2002) First Noether-Type Theorem for the Generalized Variational Principle of Herglotz. *Topological Methods in Nonlinear Analysis*, **20**, 261-273. https://doi.org/10.12775/tmna.2002.036

[3] Georgieva, B. and Guenther, R.B. (2005) Second Noether-Type Theorem for the Generalized Variational Principle of Herglotz. *Topological Methods in Nonlinear Analy-*

sis, **26**, 307-314. https://doi.org/10.12775/tmna.2005.034

[4] Georgieva, B. (2011) Symmetries of the Generalized Variational Functional of Herglotz for Several Independent Variables. *Zeitschrift für Analysis und ihre Anwendungen*, **30**, 253-268. https://doi.org/10.4171/zaa/1434

[5] Georgieva, B. (2010) Symmetries of the Herglotz Variational Principle in the Case of One Independent Variable. *Annual of Sofia University "St. Kliment Ohridski"*, **100**, 113-122.

[6] Georgieva, B. and Bodurov, T. (2013) Identities from Infinite-Dimensional Symmetries of Herglotz Variational Functional. *Journal of Mathematical Physics*, **54**, Article ID: 062901. https://doi.org/10.1063/1.4807728

[7] Guenther, R., Gottsch, A. and Guenther, C. (1996) The Herglotz Lectures on Contact Trans-Formations and Hamiltonian Systems. Juliusz Center for Nonlinear Studies.

[8] Herglotz, G. (1979) Gesammelte Schriften, Göttingen. Vandenhoeck & Ruprecht.

[9] Herglotz, G. (1930) Berührungstransformationen. Lectures at the University of Göttingen.

[10] Lie, S. (1927) Gesammelte Abhandlungen, Vol. 6. Benedictus Gotthelf Teubner, 649-663.

[11] Lie, S. (1897) Die Theorie der Integralinvarianten ist ein Korollar der Theorie der Differentialinvarianten. *Berichte*, **49**, 342-357.

[12] Furta, K., Sano, A. and Atherton, D. (1988) State Variable Methods in Automatic Control. John Wiley, New York.

[13] Caratheodory, C. (1989) Calculus of Variations and Partial Differential Equations of the First Order. 2nd Edition, Chelsea Publishing.

[14] Mrugala, R. (1980) Contact Transformations and Brackets in Classical Thermodynamics. *Acta Physica Polonica*, **A58**, 19-29.

[15] Georgieva, B. (2010) The Variational Principle of Herglotz and Related Results. Proceedings of the Twelfth International Conference on Geometry, Integrability and Quantization, Varna, 4-9 June 2010, 214-225.

[16] Bodurov, T. and Georgieva, B. (2000) Complex Hamiltonian Evolution Equations. *International Journal of Differential Equations and Applications*, **1A**, 15-22.

[17] Eisenhart, L. (1933) Continuous Groups of Transformations. Princeton University Press.

[18] Killing, W. (1892) Ueber die Grundlagen der Geometrie. *CRLL*, **1892**, 121-186. https://doi.org/10.1515/crll.1892.109.121

[19] Logan, J. (1977) Invariant Variational Principles. Academic Press.

[20] Noether, E. (1918) Invariante Variationsprobleme. Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 235-257.

[21] Noether, E. (1918) Invarianten beliebiger Differentialausdrücke. Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 37-44.