Journal of Applied Mathematics and Physics, 2025, 13(3), 871-888
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Symmetry and Monotonicity of Positive Solutions for a Class of Choquard Equation with Hardy Potential

Ying Cai

School of Mathematics, Yunnan Normal University, Kunming, China
Email: yingcai0412@163.com

How to cite this paper: Cai, Y. (2025) Symmetry and Monotonicity of Positive Solutions for a Class of Choquard Equation with Hardy Potential. *Journal of Applied Mathematics and Physics*, **13**, 871-888.
https://doi.org/10.4236/jamp.2025.133046

Received: February 10, 2025
Accepted: March 22, 2025
Published: March 25, 2025

Abstract

In this paper, a class of Choquard equation with Hardy potential in the whole space $\mathbb{R}^N$ will be considered, where the parameters p, q and α in the convolution-nonlinearity term are under a suitable range. Without the hypothesis of pointwise decay estimates for solutions, the symmetry of positive solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \{0\})$ to such an equation is obtained via some subtle integral estimates and moving plane techniques.

Keywords

Choquad Equation, Hardy Potential, Positive Solutions, Moving Plane Method

1. Introduction

The following Schrödinger equation with convolution nonlinearity

$$-\Delta u(x) + V(x)u(x) = \left(\int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^{N-\alpha}} \mathrm{d}y \right) f(u(x)), \tag{1}$$

where $0 < \alpha < N$ with $V \geq 0$ as usual, is called as Choquard equation. The Choquard equation, as a significant class of nonlinear partial differential equations, plays a crucial role in self-gravitational quantum field theory. It can characterize some special physical phenomena, such as interactions among particles. When the potential function V is singular, the Equation (1) involving singular potential can simulate singular interactions of the motions of particles in an environment with a singular potential field. That is one of reasons why such Schrödinger equations containing such potentials in $\mathbb{R}^N$ have been paid great attention by many scholars in recent years.

When $F(t) = t^p$ and $f(t) = t^q$, Moroz and Van Schaftingen [1] respectively discussed existence, nonexistence and sharp decay estimates of super-solutions to (1) in exterior domain with four different types of potentials: unperturbed Laplacian $V(x) = 0$, fast decay potentials $V(x) = \frac{\lambda}{|x|^\gamma}$ with $\lambda \in \mathbb{R}$ and $\gamma > 2$, Hardy potentials $V(x) = \frac{\nu^2 - \left(\frac{N-2}{2}\right)^2}{|x|^2}$ with $\nu > 0$ and slow decay potentials $V(x) = \frac{\lambda^2}{|x|^\gamma}$ with $\lambda \in \mathbb{R}$ and $\gamma < 2$. As for symmetry results involving such Choquard-type nonlinearity, it should be noted that Le [2] established the classification of solutions to the integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}} \int_{\mathbb{R}^N} \frac{u^q(z)}{|y-z|^{N-\beta}} \mathrm{d}z\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $0 < \alpha, \beta < N$ and $p + q = \frac{N + \alpha + \beta}{N - \alpha}$, by the method of moving planes in integral form.

For $V(x) = K(x) - \frac{\nu}{|x|^2}$ with $K \geq 0$ and $\nu > 0$, Zhu and Tang [3] considered existence and asymptotic behavior of solutions for (1) by virtue of variational methods and some analysis techniques. And analogous cases may be seen in [4] [5]. Moreover, Guo and Tang [6] addressed the indefinite case in which K is 1-periodic.

As $\alpha \to 0$, the Equation (1) degenerates to

$$-\Delta u(x) + V(x)u(x) = f(u(x)). \tag{2}$$

There are many noted works involving so-called Hardy-type potentials in the Whole space $\mathbb{R}^N$. For instance, the study on entire solutions for (2) with $V(x) = -\frac{h(x)}{|x|^2}$ has been concerned for a long time. Based on variational method for minimizing problems together with sophisticated versions of the moving plane technique, Terracini [7] derived different results concerning positive solutions of (2), where $h \in C^1(\mathbb{S}^{N-1}, \mathbb{R})$, $N \geq 3$, and f is a super-linear function. The author first proved there is no any positive solution in $D^{1,2}(\mathbb{R}^N)$, provided that $f > 0$ and the first eigenvalue of the associated linear problem is non-positive. And it was shown that there can be solutions in $D^{1,2}(\mathbb{R}^N) \cap L^p$ if and only if $p = 2^*$ as $f(x,u) = u^{p-1}$ and there is a positive solution $u \notin D^{1,2}(\mathbb{R}^N)$ which is homogeneous of degree $-\frac{N-2}{2}$. For the case of $h \equiv \text{constant} < 0$, it was proved that there is a constant $c < 0$ such that if $h < c$ there are two positive solutions in $D^{1,2}(\mathbb{R}^N)$, where one is radially symmetric and another is not. Felli and Pistoia [8] used the perturbation methods to obtain the singular or blowing-up solutions for the critical case about the Equation (2). For the power-nonlinearity case, Cîrstea and Fărcăşeanu [9] studied the existence and non-existence, uniqueness and non-uniqueness and the behaviors near zero or at infinity of solutions in $\mathbb{R}^N \setminus \{0\}$ when $h \equiv \mu$ with $\mu \in \left(0, \frac{(N-2)^2}{4}\right)$ and $f = -|x|^{-\theta} u$ with $\theta > 0$ whose parameters are under distinct ranges. When $h \equiv \mu \in \left[0, (N-2)^2/4\right)$, Chen and Zou [10] founded the existence and symmetry results of the doubly critical Schödinger system. Kang [11] studied the existence of radial solutions for elliptic systems involving critical power-nonlinearity. And Esposito, López-Soriano, and Sciunzi [12] completed the classification of positive solutions for such systems with the singular-critical power nonlinearity by a suitable version of moving plane method. The case of normalized solutions can be found in [13] and references therein. Such Schrödinger-Hardy equations were also discussed in [3] [14]-[16] about the case involving nonlinearities satisfying Berestycki-Lions type conditions.

In addition, when $V(x)$ serves as a generalization of Hardy-type potentials, there are many noted works arising from (2). Felli [17] dealt with the critical case with potential term formed $\sum_{i=1}^{k} \frac{h_i\left(\frac{x-a_i}{|x-a_i|}\right)}{|x-a_i|^2}$ with $h_i \in C^1\left(\mathbb{S}^{N-1}\right)$ and proved the existence of positive solutions by means of minimizing the associated Rayleigh quotient. Franca and Sfecci [18] studied the corresponding dynamical systems of (2) in the case where f is radial and h is radial function with different sign on the interior and exterior of a ball by applying to the Fowler transformation.

The aim of this paper is to study the symmetry of solutions to the following Schrödinger equation involving Hardy potential and Choquard-type nonlinearity, as one of special cases of (1),

$$\begin{cases} -\Delta u(x) - \dfrac{\mu}{|x|^2} u(x) = \left(\displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}} \mathrm{d}y \right) u^q(x), & x \in \mathbb{R}^N \setminus \{0\}, \\ u(x) > 0, & x \in \mathbb{R}^N \setminus \{0\}, \end{cases} \tag{3}$$

where $0 < \alpha < 2$, $N \ge 3$ and $0 < \mu < \left(\frac{N-2}{2}\right)^2$. And the pair of parameters (p, q) lies in the set

$$\left\{ (p,q) \in \mathbb{R}^2 : q > 1, p + q = \frac{N + 2\alpha + 2}{N-2} \right\}.$$

Clearly, the set is a part of a ray in 2-dimensional plane, which lies in a region where allow the subsolution of (3) exists, see [1].

It should be pointed out that this is the first time the moving plane method has been applied to address partial differential equations involving convolution-nonlinear term in weak or integral sense, in order to obtain the symmetry and monotonicity of solutions. Indeed, many noted works about Choquard-type equations involving such singular potentials like Hardy potentials in $\mathbb{R}^N$ focus only on existence or non-existence of solutions. Especially, there are no optimal existence results about equation (3) under different ranges of parameters p, q α and μ, albeit with the fact that there are partial results discussed in [1]. However, this paper only concentrates on the behavior (more explicitly, symmetry) of solutions under suitable range of those parameters. That is, althogh the range of (α, μ, p, q) in this paper is not optimal for the existence result of solutions, it is sharp for the symmetry result of solutions about (p, q) as (3) possesses a positive solution in $D^{1,2}(\mathbb{R})$.

And this work is motivated by the study of symmetry of solutions in the works above, particularly the spirit of [7] which was faced with the same issue that all of difficulties resulting from the lack of the C^1-regularity of some possible solutions force one to replace the pointwise decay estimates on the equation itself by integral estimates as well as make the moving plane procedure work in weak or distributional sense. However, when dealing with the symmetric property, one prefer constructing suitable test functions in the concrete calculations rather than making Kelvin transform as [7] to get rid of the decay hypothesis on solutions at the infinity, since such transform would make the calculations more complex and add more extra assumptions on the integrablility of solutions in the presence of convolution nonlinearity. Furthermore, the equation cannot be reduced to ordinary differential equations like [7]. So it is hard in this paper to directly derive the symmetry results unless combining necessary integral estimates about the convolution term and moving plane technique briefly mentioned in [7]. It is also necessary for the application of strong maximum principle about weak solutions. Meanwhile, in order to circumvent the possible impact on the singularity of origin and infinity point, some standard cutoff functions in analysis will be also utilized subtly.

And the main result is as below.

Theorem 1. *Let* $N \geq 3$, $\alpha \in (0,2)$ *and* $\mu \in \left(0, \frac{(N-2)^2}{4}\right)$. *And let* (p,q) *satisfiy* $q > 1$ *with* $p + q = \frac{N + 2\alpha + 2}{N - 2}$. *Suppose that* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \{0\})$ *be a solution to* (3), *no matter whether* u *is singular at the origin or not. Then it follows that* u *is symmetric with respect to the hyperplane* $\{x \in \mathbb{R}^N : x_1 = 0\}$ *and increasing in the* x_1*-direction in* $\{x_1 < 0\}$. *Further, if* u *is of* C^1, *we have*

$$u_{x_1} > 0 \text{ in } \{x_1 < 0\}.$$

In particular, the solution is radial and radially decreasing about the origin.

Remark. As is known that moving plane technique firstly proposed by Axelander [19] and Serrin [20] and subsequently developed by many others, see e.g. Sciunzi for [21] and Chen and Li for [22], etc. In addition, unfortunately, there is no any available approach to address the case of $2 \leq \alpha < N$ due to the deficiency of the analysis techniques used in this paper and therefore it is still open for this case.

The organizations in this paper are as follows. Some notations and preliminary results are stated in section 2 while the proof of main results are arranged in section 3. Moreover, the whole paper always denotes $C(\cdot)$ or $C_{(\cdot)}$ by any constant whose value may be distinct from line to line, and, as for $(\cdot)$, only the related dependence is specified in what follows.

2. Notations and Preliminary Results

In this section, some useful preliminary results will be given as below. To begin with, for any fixed $\lambda \in \mathbb{R}$, to set

$$\Sigma_\lambda = \left\{x \in \mathbb{R}^N : x_1 < \lambda\right\},$$

$$R_\lambda(0) := 0_\lambda = (2\lambda, 0, \cdots, 0).$$

And the reflection of a point $x \in \mathbb{R}^N$ though the hyperplane $T_\lambda = \{x_1 = \lambda\}$ is defined by x_λ and R_λ denotes by the reflection operator with respect to T_λ. In other words, the change of the parameter λ corresponds to the movement of the plane T_λ. In the meantime,

$$u_\lambda(x) = u(x_\lambda) = u(2\lambda - x_1, x_2, \cdots, x_N).$$

Owing to the lack of decay estimates of solutions to (3), it is necessary to find a function to "cut off" the infinity to ensure that integral estimates mentioned below make sense. And note that the solution u has no definition at the point zero, which makes it possible for 0_λ to be a non-removable singular point of u_λ. Hence, we must also truncate any small neighborhood of this possible isolated singularity so as to make the following estimates meaningful.

It is easy to see that there exist $\varphi_\varepsilon^\lambda \in C_c^\infty\left(\mathbb{R}^N \setminus \{0_\lambda\}\right)$ satisfying

(i) $\varphi_\varepsilon^\lambda \equiv 1$ on $\mathbb{R}^N \setminus \mathcal{B}_{2\varepsilon}^\lambda$ and $\varphi_\varepsilon^\lambda \equiv 0$ on $\mathbb{R}^N \setminus \mathcal{B}_\varepsilon^\lambda$,
where $\mathcal{B}_\varepsilon^\lambda := \left\{x \in \mathbb{R}^N \mid dist(0_\lambda, x) < \varepsilon\right\}$;

(ii) $0 \le \varphi_\varepsilon^\lambda \le 1$ on $\mathbb{R}^N$;

(iii) $\varphi_\varepsilon^\lambda \in W^{1,\infty}\left(\mathbb{R}^N\right)$;

(iv) $\int_{\mathbb{R}^N} \left|\nabla \varphi_\varepsilon^\lambda\right|^2 \mathrm{d}x < \varepsilon$

and $\varphi_R \in C_R^{0,1}\left(\mathbb{R}^N \setminus \{0_\lambda\}\right)$ satisfying.

(i) $\varphi_R \equiv 0$ on $\mathbb{R}^N \setminus B_{2R}$ and $\varphi_R \equiv 1$ on $\mathbb{R}^N \setminus B_R$,
where $\mathcal{B}_R := \left\{x \in \mathbb{R}^N \mid dist(0_\lambda, x) < R\right\}$;

(ii) $0 \le \varphi_R \le 1$ on $\mathbb{R}^N$;

(iii) $\left|\nabla \varphi_R\right| < \frac{2}{R}$.

Here ε is arbitrary sufficient small positive number and $R > 0$ is any large number as well as both $\varepsilon > 0$ and $R > 0$ are independent with each other.

For convenience, Hardy's and Hardy-Littlewood-Sobolev inequalities respectively are as follows.

Lemma 2. (Hardy's inequality) ([8] [23] [Lemma 1.1]) If $u \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right)$, then

$\frac{u}{|x|} \in L^2(\mathbb{R}^N)$ and

$$C_N \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x,$$

where $C_N = \left(\frac{N-2}{2}\right)^2$ is optimal and not attained.

Lemma 3. (The (weighted) Hardy-Littlewood-Sobolev inequality) ([24] [25]) Let $1 < t, r < +\infty$, $0 < \mu < N$ and $\alpha + \beta \ge 0$ satisfying $0 < \alpha + \beta + \mu \le N$. Assume that $f \in L^t(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, then there exists a positive constant $C(t, r, \alpha, \beta, \mu, N)$ not depending on f and g such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x) g(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} \mathrm{d}x\mathrm{d}y \le C(t, r, \alpha, \beta, \mu, N) \|f\|_t \|g\|_r,$$

where $\frac{1}{t} + \frac{1}{r} + \frac{\alpha+\beta+\mu}{N} = 2$ and $1 - \frac{1}{t} - \frac{\mu}{N} < \frac{\alpha}{N} < 1 - \frac{1}{t}$.

And the reader can refer [26] [27] and the references therein for more explicit details about cut-off functions.

We say $u \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus) \cap C(\mathbb{R}^N \setminus \{0\})$ solves (3) if u fulfills

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \mathrm{d}x - \int_{\mathbb{R}^N} \frac{\mu}{|x|^2} u \varphi \mathrm{d}x = \int_{\mathbb{R}^N} v(x) u^q \varphi \mathrm{d}x, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \tag{4}$$

where $v(x) = \int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}} \mathrm{d}y$ and the parameters α, p and q are supposed as mentioned in the introduction.

3. Main Results and Their Proofs

In this section, the proof of the main result will be shown later. Firstly, note that $u_\lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \{0_\lambda\})$ is the solution of

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u_\lambda(x) \varphi(x) \mathrm{d}x - \int_{R_\lambda(\mathbb{R}^N)} \frac{\mu}{|x_\lambda|^2} u_\lambda(x) \varphi(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^N} v(x_\lambda) u_\lambda^q(x) \varphi(x) \mathrm{d}x \end{aligned} \tag{5}$$

for any test function $\varphi(x) \in C_c^\infty(\mathbb{R}^N \setminus \{0_\lambda\})$.

Lemma 4. Under the assumption of Theorem 1, for $\lambda < 0$, we have that

$$\int_{\Sigma_\lambda} \left|\nabla (u - u_\lambda)^+\right|^2 \mathrm{d}x \le C\left(\mu, \alpha, N, \|u\|_{L^\infty(\mathcal{O}^\lambda)}, \|u\|_{L^{2^*}(\mathbb{R}^N)}\right), \tag{6}$$

where $w^+ := \max\{w, 0\}$, $\mathcal{O}^\lambda = \left\{x : dist(R_\lambda(0), x) < \frac{|\lambda|}{2}\right\}$. Apparently, $(u - u_\lambda)_\lambda^+ \chi_{\Sigma_\lambda} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ for any $\lambda < 0$. Here χ_{Σ_λ} is the characteristic function of set Σ_λ.

Remark. The auxiliary set $\mathcal{O}^\lambda$ proposed in the term $\|u\|_{L^\infty(\mathcal{O}^\lambda)}$ in the formula (6) is not necessary, unless 0 is a non-removable singular point of u. In addition, without loss of generality, we may assume $2\varepsilon < \frac{|\lambda|}{2}$ for simplicity, anyway.

Proof. For $\lambda < 0$, without loss of generality, we simplistically denote by $\varphi_\varepsilon := \varphi_\varepsilon^\lambda$ in the subsequent statements.

Define $\varphi := (u-u_\lambda)^+ \varphi_R^2 \varphi_\varepsilon^2 \chi_{\Sigma_\lambda}$. By the density argument, one can plug φ as a test function into (4) and (5), after subtracting (5) from (4), we have

$$\begin{aligned} I &:= \int_{\Sigma_\lambda} \left|\nabla (u-u_\lambda)^+\right|^2 \varphi_\varepsilon^2 \varphi_R^2 \mathrm{d}x \\ &= -2\int_{\Sigma_\lambda} \nabla (u-u_\lambda)^+ \nabla \varphi_\varepsilon (u-u_\lambda)^+ \varphi_\varepsilon \varphi_R^2 \mathrm{d}x \\ &\quad -2\int_{\Sigma_\lambda} \nabla (u-u_\lambda)^+ \nabla \varphi_R (u-u_\lambda)^+ \varphi_\varepsilon^2 \varphi_R \mathrm{d}x \\ &\quad +\int_{\Sigma_\lambda} \left(\frac{\mu}{|x|^2} u(x) - \frac{\mu}{|x_\lambda|^2} u_\lambda(x)\right)(u-u_\lambda)^+ \varphi_R^2 \varphi_\varepsilon^2 \mathrm{d}x \\ &\quad +\int_{\Sigma_\lambda} \left(v(x) u^q(x) - v(x_\lambda) u_\lambda^q(x)\right)(u-u_\lambda)^+ \varphi_R^2 \varphi_\varepsilon^2 \mathrm{d}x \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{7}$$

In order to get the summability of $\nabla (u-u_\lambda)^+ \chi_{\Sigma_\lambda}$ in $L^2(\mathbb{R}^N)$, it suffices to estimate I_1, I_2, I_3 and I_4 term by term. First, via Young's and Hölder's inequalities, we have

$$\begin{aligned} |I_1| &= 2\left|\int_{\Sigma_\lambda} \nabla (u-u_\lambda)^+ \nabla \varphi_\varepsilon (u-u_\lambda)^+ \varphi_\varepsilon \varphi_R^2 \mathrm{d}x\right| \\ &\le \delta \int_{\Sigma_\lambda \cap \left(\mathcal{B}_{2\varepsilon}^\lambda \setminus \mathcal{B}_\varepsilon^\lambda\right)} \left|\nabla (u-u_\lambda)^+\right| \varphi_\varepsilon^2 \varphi_R^2 \mathrm{d}x + \frac{1}{\delta} \int_{\Sigma_\lambda \cap \left(\mathcal{B}_{2\varepsilon}^\lambda \setminus \mathcal{B}_\varepsilon^\lambda\right)} u^2 |\nabla \varphi_\varepsilon|^2 \varphi_R^2 \mathrm{d}x \\ &\le \delta I + \frac{1}{\delta} \|u\|_{L^\infty\left(\mathcal{B}_{2\varepsilon}^\lambda\right)}^2 \int_{\mathcal{B}_{2\varepsilon}^\lambda} |\nabla \varphi_\varepsilon|^2 \mathrm{d}x \\ &\le \delta I + \frac{1}{\delta} \varepsilon \|u\|_{L^\infty\left(\mathcal{O}^\lambda\right)}^2 \\ &\le \delta I + \|u\|_{L^\infty\left(\mathcal{O}^\lambda\right)}^2 \end{aligned}$$

for ε small enough with $\varepsilon < \min\{\delta, |\lambda|/4\}$, where we used the fact that $0 \le (u-u_\lambda)^+ \le u$ and $\mathcal{B}_{2\varepsilon}^\lambda \subset \mathcal{O}^\lambda$. And δ is a fixed number between 0 and $\frac{(N-2)^2 - 4\mu}{2(N-2)^2}$. Likewise,

$$|I_2| \le \delta I + \frac{1}{\delta}\left(\int_{\Sigma_\lambda \cap (B_{2R} \setminus B_R)} u^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}} \left(\int_{B_{2R} \setminus B_R} |\nabla \varphi_R|^N \mathrm{d}x\right)^{\frac{2}{N}}.$$

Since

$$\int_{B_{2R} \setminus B_R} |\nabla \varphi_R|^N \mathrm{d}x \le \left(2^N / R^N\right)|B_{2R} \setminus B_R| = \frac{2^N\left(2^N - 1\right)\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)},$$

$$|I_2| \le \delta I + C(N)\|u\|_{L^{2^*}(\Sigma_\lambda)}^2.$$

It follows from Hardy's inequality that

$$
\begin{aligned}
I_3 &= \int_{\Sigma_\lambda}\left(\frac{\mu}{|x|^2}u(x)-\frac{\mu}{|x_\lambda|^2}u_\lambda(x)\right)(u-u_\lambda)^+\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x\\
&\le \mu\int_{\Sigma_\lambda}\frac{\left[(u-u_\lambda)^+\right]^2(x)\varphi_\varepsilon^2\varphi_R^2}{|x|^2}\mathrm{d}x \le \mu\int_{\Sigma_\lambda}\frac{\left[(u-u_\lambda)^+\right]^2(x)}{|x|^2}\mathrm{d}x\\
&\le \frac{4\mu}{(N-2)^2}\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)\right|^2\mathrm{d}x.
\end{aligned}
$$

At last, we estimate I_4, being similar to the calculation as above.

$$
\begin{aligned}
I_4 &= \int_{\Sigma_\lambda}\left(v(x)u^q(x)-v(x_\lambda)u_\lambda^q(x)\right)(u-u_\lambda)^+\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x\\
&= \int_{\Sigma_\lambda} v(x_\lambda)\left(u^q(x)-u_\lambda^q(x)\right)(u-u_\lambda)^+\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x\\
&\quad+\int_{\Sigma_\lambda} u^q(x)\left(v(x)-v(x_\lambda)\right)(u-u_\lambda)^+\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x\\
&=: I_{4,1}+I_{4,2}.
\end{aligned}
$$

Write $\Sigma_\lambda^u := \{x\in\Sigma_\lambda \mid u(x)>u_\lambda(x)\}$, and the Lagrange mean value theorem yields that

$$
u^q(x)-u_\lambda^q(x) = q\xi_\lambda^{q-1}(x)(u-u_\lambda)(x) \le qu^{q-1}(x)(u-u_\lambda)(x)
$$

for fixed $x\in\Sigma_\lambda^u$, where ξ_λ is a number depending only on x between u and u_λ. By virtue of Hardy-Littlewood-Sobolev (short for HLS) inequality, we have

$$
\begin{aligned}
I_{4,1} &\le q\int_{\Sigma_\lambda^u}\left(\int_{\Sigma_\lambda}\frac{u^p(y)+u_\lambda^p(y)}{|x-y|^{N-\alpha}}\mathrm{d}y\right)u^{q-1}(x)\left[(u-u_\lambda)^+\right]^2(x)\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x\\
&\le q\int_{\Sigma_\lambda}\left(\int_{\Sigma_\lambda}\frac{u^p(y)+u_\lambda^p(y)}{|x-y|^{N-\alpha}}\mathrm{d}y\right)u^{q+1}\mathrm{d}x\\
&\le qC(\alpha,N)\left(\|u\|^p_{L^{2^*}(\Sigma_\lambda)}+\|u_\lambda\|^p_{L^{2^*}(\Sigma_\lambda)}\right)\|u\|^{q+1}_{L^{2^*}(\Sigma_\lambda)}\\
&\le qC(\alpha,N)\|u\|^p_{L^{2^*}(\mathbb{R}^N)}\|u\|^{q+1}_{L^{2^*}(\Sigma_\lambda)}.
\end{aligned}
$$

Next, set

$$
A := \left\{(x,y)\in\Sigma_\lambda\times\Sigma_\lambda \mid (u-u_\lambda)^+(y) < (u-u_\lambda)^+(x)\right\}
$$

and

$$
A^c = \Sigma_\lambda\times\Sigma_\lambda \setminus A.
$$

Since

$$
\begin{aligned}
&v(x)-v(x_\lambda)\\
&= \int_{\Sigma_\lambda}\left(u^p(y)-u_\lambda^p(y)\right)\left(\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x_\lambda-y|^{N-\alpha}}\right)\mathrm{d}y,
\end{aligned}
$$

similar to the estimate of $I_{4,1}$ by mean value theorem, HLS inequality and Hölder's inequality, then we have

$$
\begin{aligned}
I_{4,2} &\le p\left\{\int_A + \int_{A^c}\right\}\left\{\frac{u^{p-1}(y)(u-u_\lambda)^+(y)}{|x-y|^{N-\alpha}}u^q(x)(u-u_\lambda)^+(x)\right\}\mathrm{d}x\mathrm{d}y \\
&\le p\int_A \frac{u^{p-1}(y)u^q(x)\left[(u-u_\lambda)^+\right]^2(x)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\quad + p\int_{A^c} \frac{u^{p-1}(y)u^q(x)\left[(u-u_\lambda)^+\right]^2(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\le p\int_{\Sigma_\lambda\times\Sigma_\lambda} \frac{u^{p-1}(y)u^q(x)\left[(u-u_\lambda)^+\right]^2(x)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\quad + p\int_{\Sigma_\lambda\times\Sigma_\lambda} \frac{u^{p-1}(y)u^q(x)\left[(u-u_\lambda)^+\right]^2(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\le pC(\alpha,N)\left(\int_{\Sigma_\lambda} u^{2^*}\mathrm{d}y\right)^{\frac{p-1}{2^*}}\left(\int_{\Sigma_\lambda}\left|u^q\left[(u-u_\lambda)^+\right]^2\right|^{\frac{2^*}{q+2}}\mathrm{d}x\right)^{\frac{q+2}{2^*}} \\
&\quad + pC(\alpha,N)\left(\int_{\Sigma_\lambda} u^{2^*}\mathrm{d}x\right)^{\frac{q}{2^*}}\left(\int_{\Sigma_\lambda}\left|u^{p-1}\left[(u-u_\lambda)^+\right]^2\right|^{\frac{2^*}{p+1}}\mathrm{d}x\right)^{\frac{p+1}{2^*}} \\
&\le 2pC(\alpha,N)\|u\|_{L^{2^*}(\Sigma_\lambda)}^{p+q+1}.
\end{aligned}
$$

Summing up all estimates of I_1, I_2, I_3 and I_4 above, we get

$$
\begin{aligned}
I &\le 2\delta I + \|u\|_{L^\infty(\mathcal{O}^\lambda)}^2 + C(N)\|u\|_{L^{2^*}(\Sigma_\lambda)}^2 \\
&\quad + \frac{4\mu}{(N-2)^2}\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x + C\left(\alpha, N, \|u\|_{L^{2^*}(\mathbb{R}^N)}\right).
\end{aligned}
$$

Using Lebesgue's dominated convergence theorem to take the limit on the right-hand side (and Fatou's lemma on the left-hand side), it yields that

$$
\begin{aligned}
(1-2\delta)\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x &\le (1-2\delta)\lim_{\substack{\varepsilon\to0^+\\ R\to+\infty}} I \\
&\le \frac{4\mu}{(N-2)^2}\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x + C\left(\alpha, N, \|u\|_{L^{2^*}(\mathbb{R}^N)}\right).
\end{aligned}
$$

Namely,

$$
\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x \le C\left(\mu,\alpha,N,\|u\|_{L^\infty(\mathcal{O}^\lambda)},\|u\|_{L^{2^*}(\mathbb{R}^N)}\right).
$$

This completes the proof.

With the above lemma, we give the next lemma, which makes sure the moving plane can start from negative infinity along x_1-axis.

Lemma 5. Under the assumption of lemma 4, for any $\lambda < 0$ with $|\lambda|$ sufficiently large, we have

$$\Lambda \neq \varnothing,$$

where

$$\Lambda := \left\{\lambda < 0 \mid u \le u_t \text{ in } \Sigma_t \setminus \{0_t\}, \forall t \in (-\infty, \lambda]\right\}.$$

Proof. Analogous to the proof in the Lemma 4, taking the same test function φ. We give more explicit estimates for I_1, I_2, I_3 and I_4. At first, for I_1 and I_2, choosing the parameter $\delta = \frac{(N-2)^2 + 4\mu}{4[(N-2)^2 + 4\mu]}$, then Young's and Hölder's inequalities yields that

$$\begin{aligned} |I_1| &\le \delta I + \frac{1}{\delta}\|u\|^2_{L^\infty(\mathcal{B}^\lambda_{2\varepsilon})} \int_{\mathcal{B}^\lambda_{2\varepsilon}} |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x \\ &\le \delta I + \frac{1}{\delta}\varepsilon \|u\|^2_{L^\infty(\mathcal{O}^\lambda)} \end{aligned} \tag{8}$$

and

$$\begin{aligned} |I_2| &\le \delta I + \frac{1}{\delta}\left(\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} u^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} \left(\int_{B_{2R}\setminus B_R} |\nabla \varphi_R|^N \,\mathrm{d}x\right)^{\frac{2}{N}} \\ &\le \delta I + C(N)\|u\|^2_{L^{2^*}(\Sigma_\lambda \cap (B_{2R}\setminus B_R))}. \end{aligned} \tag{9}$$

And $u \in L^{2^*}(\mathbb{R}^N)$ implies

$$\int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} |u|^{2^*} \,\mathrm{d}x \to 0$$

as $R \to +\infty$ due to the decay property of Lebesgue's integral. Next, by virtue of Hardy's inequality, we get

$$\begin{aligned} I_3 &\le \mu \int_{\Sigma_\lambda} \frac{u(x) - u_\lambda(x)}{|x|^2} (u - u_\lambda)^+(x) \varphi_\varepsilon^2 \varphi_R^2 \mathrm{d}x \\ &= \mu \int_{\Sigma_\lambda} \frac{\left[(u - u_\lambda)^+\right]^2 (x)}{|x|^2} \varphi_\varepsilon^2 \varphi_R^2 \mathrm{d}x \\ &\le \frac{\mu}{\left(\frac{N-2}{2}\right)^2} \int_{\Sigma_\lambda} \left|\nabla\left[(u - u_\lambda)^+ \varphi_\varepsilon \varphi_R\right]\right|^2 \mathrm{d}x \end{aligned} \tag{10}$$

while

$$\begin{aligned} \left|\nabla\left[(u - u_\lambda)^+ \varphi_\varepsilon \varphi_R\right]\right|^2 &= \left|\nabla (u - u_\lambda)^+\right|^2 \varphi_\varepsilon^2 \varphi_R^2 + \left[(u - u_\lambda)^+\right]^2 \left(\varphi_R^2 |\nabla \varphi_\varepsilon|^2 + \varphi_\varepsilon^2 \nabla \varphi_R^2\right) \\ &\quad + 2\nabla (u - u_\lambda)^+ \nabla \varphi_\varepsilon (u - u_\lambda)^+ \varphi_\varepsilon \varphi_R^2 \\ &\quad + 2\nabla (u - u_\lambda)^+ \nabla \varphi_R (u - u_\lambda)^+ \varphi_\varepsilon^2 \varphi_R \\ &\quad + 2\nabla \varphi_\varepsilon \nabla \varphi_R \left[(u - u_\lambda)^+\right]^2 \varphi_R \varphi_\varepsilon. \end{aligned}$$

Thus, putting it into (10), it follows that

$$\begin{aligned} I_3 \le \frac{4\mu}{(N-2)^2} \Bigg\{ & I + \int_{\Sigma_\lambda \cap \left(\mathcal{B}^\lambda_{2\varepsilon} \setminus \mathcal{B}^\lambda_\varepsilon\right)} u^2 |\nabla \varphi_\varepsilon|^2 \,\mathrm{d}x + \int_{\Sigma_\lambda \cap (B_{2R}\setminus B_R)} u^2 |\nabla \varphi_R|^2 \,\mathrm{d}x \\ & + |I_1| + |I_2| + 2\int_{\Sigma_\lambda} \left((u - u_\lambda)^+ |\nabla \varphi_\varepsilon| \varphi_\varepsilon\right)\left((u - u_\lambda)^+ |\nabla \varphi_R| \varphi_R\right) \mathrm{d}x \Bigg\}. \end{aligned} \tag{11}$$

Write the last term in the curly braces as $J_{\varepsilon,R}$ as below, *i.e.*

$$\begin{aligned} J_{\varepsilon,R} &= 2\int_{\Sigma_\lambda}\left((u-u_\lambda)^+|\nabla\varphi_\varepsilon|\varphi_\varepsilon\right)\left((u-u_\lambda)^+|\nabla\varphi_R|\varphi_R\right) \\ &\le \int_{\Sigma_\lambda\cap\left(\mathcal{B}_{2\varepsilon}^\lambda\setminus\mathcal{B}_\varepsilon^\lambda\right)} u^2|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} u^2|\nabla\varphi_R|^2\,\mathrm{d}x \\ &\le \|u\|_{L^\infty(\mathcal{O}^\lambda)}^2\left(\int_{\mathcal{B}_{2\varepsilon}^\lambda\setminus\mathcal{B}_{2\varepsilon}^\lambda}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\right) \\ &\quad + \left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} u^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}\left(\int_{B_{2R}\setminus B_R}|\nabla\varphi_R|^N\,\mathrm{d}x\right)^{\frac{2}{N}} \\ &\le \varepsilon\|u\|_{L^\infty(\mathcal{O}^\lambda)}^2 + C(N)\|u\|_{L^{2^*}(\Sigma_\lambda\cap(B_{2R}\setminus B_R))}^2, \end{aligned}$$

Let $(\varepsilon,R)\to(0^+,+\infty)$ as mentioned above, and taking the double limit, it follows that $J_{\varepsilon,R}\to 0$. Moreover, the second and third term in the curly braces are both infinitesimals as $\varepsilon\to 0^+$ and $R\to+\infty$, respectively. Indeed, we have the estimates as before:

$$\int_{\Sigma_\lambda\cap\left(\mathcal{B}_{2\varepsilon}^\lambda\setminus\mathcal{B}_\varepsilon^\lambda\right)} u^2|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x \le \varepsilon\|u\|_{L^\infty(\mathcal{O}^\lambda)}^2$$

and

$$\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} u^2|\nabla\varphi_R|^2\,\mathrm{d}x \le C(N)\|u\|_{L^{2^*}(\Sigma_\lambda\cap(B_{2R}\setminus B_R))}^2,$$

where $C(N)$ is a constant depending only on dimension N. Up to now, there is still term I_4 left. Based on the estimates for $I_{4,1}$ and $I_{4,2}$ in the preceding lemma, together with the Sobolev's embedding theorem, we have

$$\begin{aligned} I_{4,1} &\le q\int_{\Sigma_\lambda^u}\left(\int_{\Sigma_\lambda}\frac{u^p(y)+u_\lambda^p(y)}{|x-y|^{N-\alpha}}\mathrm{d}y\right)u^{q-1}(x)\left[(u-u_\lambda)^+\right]^2(x)\varphi_R^2\varphi_\varepsilon^2\mathrm{d}x \\ &\le qC(\alpha,N)\left(\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + \|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right) \\ &\quad\times\left(\int_{\Sigma_\lambda\cap B_{2R}}\left|u^{q-1}(x)\left[(u-u_\lambda)^+\right]^2\varphi_\varepsilon^2\varphi_R^2\right|^{\frac{2^*}{q+1}}\mathrm{d}x\right)^{\frac{q+1}{2^*}} \\ &\le qC(\alpha,N)\left(\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + \|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right) \\ &\quad\times\left(\int_{\Sigma_\lambda\cap B_{2R}} u^{2^*}\mathrm{d}x\right)^{\frac{q-1}{2^*}}\left(\int_{\Sigma_\lambda\cap B_{2R}}\left[(u-u_\lambda)^+\right]^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\le qS^2C(\alpha,N)\left(\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + \|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right) \\ &\quad\times\|u\|_{L^{2^*}(\Sigma_\lambda)}^{q-1}\left(\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x\right), \end{aligned} \tag{12}$$

where S is the Sobolev embedding constant from $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$. Likewise, we have

$$
\begin{aligned}
I_{4,2} &\le pC(\alpha,N)\left(\int_{\Sigma_\lambda} u^{2^*}\mathrm{d}y\right)^{\frac{p-1}{2^*}}\left(\int_{\Sigma_\lambda}\left|u^q\left[(u-u_\lambda)^+\right]^2\right|^{\frac{2^*}{q+2}}\mathrm{d}x\right)^{\frac{q+2}{2^*}} \\
&\quad + pC(\alpha,N)\left(\int_{\Sigma_\lambda} u^{2^*}\mathrm{d}x\right)^{\frac{q}{2^*}}\left(\int_{\Sigma_\lambda}\left|u^{p-1}\left[(u-u_\lambda)^+\right]^2\right|^{\frac{2^*}{p+1}}\mathrm{d}x\right)^{\frac{p+1}{2^*}} \\
&\le 2pC(\alpha,N)\|u\|_{L^{2^*}(\Sigma_\lambda)}^{p+q-1}\left(\int_{\Sigma_\lambda}\left[(u-u_\lambda)^+\right]^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \\
&\le 2pS^2C(\alpha,N)\|u\|_{L^{2^*}(\Sigma_\lambda)}^{p+q-1}\left(\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x\right).
\end{aligned}
\tag{13}
$$

Inserting (8)-(13) into (7) and making it up, then we deduce that

$$
\begin{aligned}
\left(1-\frac{4\mu}{(N-2)^2}\right)I &\le 2\left(1+\frac{4\mu}{(N-2)^2}\right)\delta I + C(\mu,N)\varepsilon\|u\|_{L^\infty(\mathcal{O}^\lambda)}^2 \\
&\quad + C(\mu,N)\left(\int_{\Sigma_\lambda\cap(B_{2R}\setminus B_R)} u^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}} \\
&\quad + qC(\alpha,N)S^2\left(\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + \|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right) \\
&\quad \times\|u\|_{L^{2^*}(\Sigma_\lambda)}^{q-1}\left(\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x\right) \\
&\quad + 2pC(\alpha,N)S^2\|u\|_{L^{2^*}(\Sigma_\lambda)}^{p+q-1}\left(\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x\right) \\
&\le \left(1+\frac{4\mu}{(N-2)^2}\right)\delta I + C(\mu,N)\|u\|_{L^\infty(\mathcal{O}^\lambda)}^2\varepsilon + C(\mu,N)\|u\|_{L^{2^*}(\Sigma_\lambda\cap(B_{2R}\setminus B_R))}^2 \\
&\quad + \Big\{C(\alpha,N)S^2\left[(2p+q)\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + q\|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right] \\
&\quad \times\|u\|_{L^{2^*}(\Sigma_\lambda)}^{q-1}\left(\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x\right)\Big\}.
\end{aligned}
\tag{14}
$$

Employing Fatou's lemma again, it follows (14) that

$$
\begin{aligned}
\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x &\le 2C(\mu,\alpha,N)S^2\left[(2p+q)\|u\|_{L^{2^*}(\Sigma_\lambda)}^p + q\|u\|_{L^{2^*}(\mathbb{R}^N)}^p\right] \\
&\quad \times\|u\|_{L^{2^*}(\Sigma_\lambda)}^{q-1}\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x.
\end{aligned}
$$

Note that $q>1$ and $\|u\|_{L^{2^*}(\Sigma_\lambda)}\to 0$ as $|\lambda|\to+\infty$ with $\lambda<0$. As a result, there exists $M>1$ such that for $\lambda<-M$, it holds that

$$
2C(\mu,\alpha,N)S^2(p+q)\|u\|_{L^{2^*}(\mathbb{R}^N)}^p\|u\|_{L^{2^*}(\Sigma_\lambda)}^{q-1}<1/2.
$$

This means that

$$
\int_{\Sigma_\lambda}\left|\nabla(u-u_\lambda)^+\right|^2\mathrm{d}x = 0
$$

for any $\lambda < -M$. Namely, $(u-u_\lambda)^+ \equiv \text{constant}$. Observe that $u \equiv u_\lambda$ on T_λ, so $u \le u_\lambda$ for any $\lambda < -M$ due to the continuity. This is as desired.

Before the proof of the main result, we give a weaker version of strong comparison principle whose proof is similar to the classical strong maximum principle for elliptic equations or inequalities with divergence structure.

Lemma 6. Let ω be a weakly superharmonic function in $\Sigma_\lambda \setminus \{0_\lambda\}$. If $\omega \in \mathcal{D}^{1,2}(\Sigma_{\Sigma_\lambda})$ is continuous over $\Sigma_\lambda \setminus \{0_\lambda\}$, then either $\omega > 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$ or $\omega \equiv 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$.

Proof. Since $\omega \ge 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$, if $\omega > 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$, we have done. Otherwise, setting

$$\Sigma'_\lambda := \left\{x \in \Sigma_\lambda \setminus \{0_\lambda\} \mid \omega(x) = 0\right\}$$

and it means $\Sigma'_\lambda \ne \varnothing$. Thus, it suffices to prove that $\Sigma'_\lambda = \Sigma_\lambda \setminus \{0_\lambda\}$. And ω weakly solves the following differential inequality

$$-\Delta\omega \ge 0 \text{ in } \Sigma_\lambda \setminus \{0_\lambda\}.$$

That is, for any nonnegative $\varphi(x) \in C_c^\infty(\Sigma_\lambda \setminus \{0_\lambda\})$, it holds

$$\int_{\Sigma_\lambda} \nabla\omega\nabla\varphi \mathrm{d}x \ge 0.$$

On the other hand, note that Σ'_λ is relatively closed to $\Sigma_\lambda \setminus \{0\}$ due to the continuity of ω. Let $x_0 \in \Sigma'_\lambda$ with $\rho > 0$ small enough satisfying $B_{4\rho}(x_0) \subset \Sigma_\lambda \setminus \{0_\lambda\}$, then

$$\int_{B_{4\rho}(x_0)} \nabla\omega\nabla\varphi \mathrm{d}x \ge 0$$

for any nonnegative $\varphi \in C_c^\infty(B_{4\rho}(x_0))$. So ω restricted to $B_{4\rho}(x_0)$ is also a weak superharmonic function in $B_{4\rho}(x_0)$. And $\omega \in \mathcal{D}^{1,2}_{loc}(B_{4\rho}(x_0)) = H^1_{loc}(B_{4\rho}(x_0))$ is still nonnegative with a minimum point x_0. Whence, it follows from the weak Harnack inequality [[28], Lemma 2.113] (in the case of $p = 2$ and $q = a = c = 1$) that

$$0 \le \frac{1}{vol(B_\rho)} \int_{B_\rho(x_0)} \omega(x_0) \mathrm{d}x \le C(N) \inf_{B_\rho(x_0)} \omega = C(N)\omega(x_0) = 0,$$

which implies that $\omega(x) \equiv 0$ in $B_\rho(x_0)$ and thus $B_\rho(x_0) \subset \Sigma'_\lambda$. That is, x_0 is a interior point of Σ'_Σ. Consequently, Σ'_λ is also open. Concluding all above, $\Sigma'_\lambda = \Sigma_\lambda \setminus \{0_\lambda\}$ and thereby $\omega \equiv 0$ in $\Sigma_\lambda \setminus \{0_\lambda\}$. This accomplishes the proof of the lemma.

With crucial lemmas all above at hand, next, we take up proving Theorem 1 with moving the hyperplane T_λ from negative infinity along the x_1-axis.

Proof of Theorem 1. Now, we take up the proof of Theorem 1. And it is split into three steps.

Step1: We have $u \le u_\lambda$ in Σ_λ for $\lambda < 0$ with $|\lambda|$ large enough, which contributes to Lemma 5.

Thus, define

$$\lambda_0 := \sup\Lambda,$$

where

$$\Lambda := \left\{\lambda < 0 \mid u \le u_t \text{ in } \Sigma_t \setminus \{0_t\} \text{ for all } t \in (-\infty, \lambda]\right\}.$$

Step2: We assert that $\lambda_0 = 0$. Arguing by contradiction and assume that $\lambda_0 < 0$. By the continuity, we see that $u \le u_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus \{0_{\lambda_0}\}$. Note that

$$v(x_{\lambda_0}) - v(x) = \int_{\Sigma_{\lambda_0}} \left(u_{\lambda_0}^p(y) - u^p(y)\right)\left(\frac{1}{|x_{\lambda_0} - y|^{N-\alpha}} - \frac{1}{|x-y|^{N-\alpha}}\right)\mathrm{d}y \ge 0$$

and

$$\begin{aligned}\int_{\Sigma_{\lambda_0}} \nabla(u - u_{\lambda_0})\nabla\varphi \mathrm{d}x = \mu\int_{\Sigma_{\lambda_0}} &\left(\frac{u_{\lambda_0}}{|x_{\lambda_0}|^2} - \frac{u}{|x|^2}\right)\varphi \mathrm{d}x \\ &+ \int_{\Sigma_{\lambda_0}} \left[v(x_{\lambda_0}) - v(x)\right]u_{\lambda_0}^q(x)\mathrm{d}x \\ &+ \int_{\Sigma_{\lambda_0}} v(x)\left(u_{\lambda_0}^q(x) - u^q(x)\right)\varphi(x)\mathrm{d}x \\ &\ge 0\end{aligned}$$

for any nonnegative $\varphi \in C_c^\infty\left(\Sigma_{\lambda_0} \setminus \{0_{\lambda_0}\}\right)$. It suffices to take $\omega = u_{\lambda_0} - u$, via Lemma 6, then we immediately deduce that $u < u_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus \{0_{\lambda_0}\}$ as $u \not\equiv u_{\lambda_0}$. Indeed, u would be undefined at $0_{\lambda_0} \in \Sigma_{\lambda_0}$ as long as $u \equiv u_{\lambda_0}$, which is a contradiction. Now, we manage to obtain the contradictory conclusion resulting from the fact that there exists some small $\overline{\tau} > 0$ such that, for any $\tau \in [\lambda_0, \lambda_0 + \overline{\tau}]$, it holds $u < u_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau} \setminus \{0_{\lambda_0}\}$ whenever $\lambda_0 < 0$. Actually, for arbitrary $\delta > 0$, there are $\tau_1 = \tau_1(\gamma, \lambda_0) > 0$ and a compact set K (depending on γ and λ_0) so that $\int_{\Sigma_{\lambda_0 \setminus K}} u^{2^*} \mathrm{d}x < \frac{\gamma}{2}$ and $K \subset \Sigma_\lambda \setminus \{0_\lambda\}$ for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Define $g(x, \lambda) := u(x) - u(x_\lambda)$ on the compact $K \times [\lambda_0, \lambda + \tau_1]$. So it follows that $K \subset \Sigma_{\lambda_0+\tau} \setminus \{0_{\lambda_0+\tau}\}$ and $u < u_{\lambda_0+\tau}$ for any $\tau \in [0, \tau_2]$ with some $\tau_2 = \tau_2(\gamma, \lambda_0) \in (0, \tau_1)$. Also, we can assume that $\tau_2 < \frac{|\lambda_0|}{8}$. Since $u \in L^{2^*}\left(\Sigma_{\lambda_0 + \frac{|\lambda_0|}{8}}\right)$ and $\int_{\Sigma_{\lambda_0} \setminus K} u^{2^*} \mathrm{d}x < \frac{\gamma}{2}$, we assert that there is $\overline{\tau} \in (0, \tau_2)$ such that

$$\int_{\lambda_0 + \overline{\tau} \setminus \Sigma_{\lambda_0}} u^{2^*} \mathrm{d}x < \frac{\gamma}{2},$$

and thereby $\int_{\Sigma_\lambda \setminus K} u^{2^*} \mathrm{d}x < \gamma$ for all $\lambda \in [\lambda_0, \lambda_0 + \overline{\tau}]$. Next, we repeat the proof of Lemma 5 but using test function

$$\varphi := \left(u - u_{\lambda_0+\tau}\right)^+ \varphi_\varepsilon^2 \varphi_R^2 \chi_{\Sigma_{\lambda_0+\tau}}.$$

Hence, after recovering the first and last inequalities in (14), for any $0 \le \tau < \overline{\tau}$, we have

$$\int_{\Sigma_{\lambda_0+\tau}\setminus K}\left|\nabla\left(u-u_{\lambda_0+\tau}\right)^+\right|^2 \mathrm{d}x$$
$$\le 2C(\mu,\alpha,N)S^2\left[(2p+q)\|u\|^p_{L^{2^*}(\Sigma_\lambda)}+q\|u\|^p_{L^{2^*}(\mathbb{R}^N)}\right] \tag{15}$$
$$\times\|u\|^{q-1}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}\int_{\Sigma_{\lambda_0+\tau}\setminus K}\left|\nabla\left(u-u_{\lambda_0+\tau}\right)^+\right|^2 \mathrm{d}x.$$

Since $\left(u-u_{\lambda_0+\tau}\right)^+$ and $\nabla\left(u-u_{\lambda_0+\tau}\right)^+$ both vanish in a neighborhood of K, via the previous construction. Now, fix

$$\gamma<\left[8(p+q)C(\mu,\alpha,N)S^2\|u\|^p_{L^{2^*}(\mathbb{R}^N)}\right]^{\frac{1}{1-q}},$$

then

$$4C(\mu,\alpha,N)S^2(p+q)\|u\|^p_{L^{2^*}(\mathbb{R}^N)}\|u\|^{q-1}_{L^{2^*}(\Sigma_{\lambda_0+\tau}\setminus K)}<\frac{1}{2},\ \forall 0\le\tau<\overline{\tau}$$

combining with (15), we deduce $\int_{\Sigma_{\lambda_0+\tau}\setminus K}\left|\nabla\left(u-u_{\lambda_0+\tau}\right)^+\right|^2 \mathrm{d}x=0$ for every $0\le\tau<\overline{\tau}$. Hence, $\int_{\Sigma_{\lambda_0+\tau}}\left|\nabla\left(u-u_{\lambda_0+\tau}\right)^+\right|^2 \mathrm{d}x=0$ for every $\tau\in[0,\overline{\tau})$, and it means that $u\le u_{\lambda_0+\tau}$ in $\Sigma_{\lambda_0+\tau}\setminus\left\{0_{\lambda_0+\tau}\right\}$, which demonstrates the claim of step 2.

Step3. Conclusion. Similar to the step 1 and step 2, we may define

$$\tilde{\Lambda}:=\left\{\tilde{\lambda}>0\,|\,u\ge u_t \text{ for any } t\in\left[\tilde{\lambda},+\infty\right)\right\}$$

and

$$\tilde{\lambda}_0=\inf\tilde{\Lambda}.$$

Simultaneously, we consider another test function $\psi:=\left(u_{\tilde{\lambda}}-u\right)^+\varphi_\varepsilon^2\varphi_R^2\chi_{\tilde{\Sigma}_{\tilde{\lambda}}}$, where $\tilde{\Sigma}_{\tilde{\lambda}}:=\left\{x_1>\tilde{\lambda}\right\}$. Continuing the proceeding procedure, we see which u is increasing along x_1-direction in $\{x_1<0\}$ and symmetry with respect to $\{x_1=0\}$. This completes the proof of Theorem 1.■

Remark. *Actually, since in* $\mathbb{R}^N$ *the* x_1*-axis is a direction that can be taken arbitrarily in the proof, we may get the radial symmetry of solutions.*

4. Conclusions and Suggestions

The symmetry and monotonicity with respect to the plane $\left\{(x_1,x_2,\cdots,x_N):x_1=0\right\}$ of solutions to a class of Choquard with Hardy potential is derived in this paper via applying the moving plane method. The key is to compare the size between solution u and its reflection u_λ with respect to hyperplane T_λ. By moving T_λ to the limit position to get $u=u_\lambda$, we prove λ must be zero as the limit position is reached, *i.e.* T_λ must stop at the position of T_0. We can get the opposite monotonicity of the solution towards to opposite direction along the according coordinate axis, thereby resulting in the symmetry of the solution. That is the idea of the moving plane technique. Of cause, there are two many mathematical techniques to deal with the problem in this paper which are partly distinct from others' works introduced in the introduction.

As mentioned in the introduction, if there exists a solution of this class of Choquard equation within the range of parameters assumed in this paper, it must be symmetric about some hyperplane throughout the oringin point. And there are still some interesting open problems for further investigations in Mathematics or Physics. For instance, as for the case that potentials V is nonnegative and $q = p - 1$, the physicial explanations can be found in [29]. However, is there any physical significance about (p, q) in such Choquard-type equations with Hardy potential ? In Mathematics, can the symmetry results be generalized to the system coupled by over two Choquard-type equations with Hardy potentials in different domains including $\mathbb{R}^N$? When (p, q) does not belong to the set mentioned in the introduction, is there any symmetric positive solution to the equation? Namely, more difficultly, what is the sharp range of (p, q) for such problems? Thess points are quite fascinating in both Mathematics and Physics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Moroz, V. and Van Schaftingen, J. (2013) Nonexistence and Optimal Decay of Supersolutions to Choquard Equations in Exterior Domains. *Journal of Differential Equations*, **254**, 3089-3145. https://doi.org/10.1016/j.jde.2012.12.019

[2] Le, P. (2019) Symmetry and Classification of Solutions to an Integral Equation of the Choquard Type. *Comptes Rendus. Mathématique*, **357**, 878-888. https://doi.org/10.1016/j.crma.2019.11.005

[3] Zhu, R. and Tang, X. (2022) Existence and Asymptotic Behavior of Solutions for Choquard Equations with Singular Potential under Berestycki-Lions Type Conditions. *Asymptotic Analysis*, **132**, 427-450. https://doi.org/10.3233/asy-221798

[4] Guo, T., Tang, X. and Zhang, Q. (2022) Ground State Solutions for Choquard Equations with Hardy Potentials and Critical Nonlinearity. *Complex Variables and Elliptic Equations*, **67**, 1579-1597. https://doi.org/10.1080/17476933.2021.1885387

[5] Qin, D., Lai, L., Tang, X. and Wu, Q. (2022) Existence and Asymptotic Behavior of Ground States for Choquard-Pekar Equations with Hardy Potential and Critical Reaction. *The Journal of Geometric Analysis*, **32**, Article No. 158. https://doi.org/10.1007/s12220-022-00892-5

[6] Guo, T., Tang, X. and Gui, S. (2024) Ground State Solutions of Nehari-Pankov Type for a Indefinite Choquard Equation with the Hardy Potential and Critical Nonlinearity. *Journal of Mathematical Analysis and Applications*, **540**, Article 128639. https://doi.org/10.1016/j.jmaa.2024.128639

[7] Terracini, S. (1996) On Positive Entire Solutions to a Class of Equations with a Singular Coefficient and Critical Exponent. *Advances in Differential Equations*, **1**, 241-264. https://doi.org/10.57262/ade/1366896239

[8] Felli, V. and Pistoia, A. (2006) Existence of Blowing-Up Solutions for a Nonlinear Elliptic Equation with Hardy Potential and Critical Growth. *Communications in Partial Differential Equations*, **31**, 21-56. https://doi.org/10.1080/03605300500358145

[9] Cîrstea, F. and Fărcăşeanu, M. (2021) Sharp Existence and Classification Results for Nonlinear Elliptic Equations in $R^N \setminus \{0\}$ with Hardy Potential. *Journal of Differential Equations*, **292**, 461-500. https://doi.org/10.1016/j.jde.2021.05.005

[10] Chen, Z. and Zou, W. (2015) Existence and Symmetry of Positive Ground States for a Doubly Critical Schrödinger System. *Transactions of the American Mathematical Society*, **367**, 3599-3646. https://doi.org/10.1090/s0002-9947-2014-06237-5

[11] Kang, D. (2024) Existence and Properties of Radial Solutions to Critical Elliptic Systems Involving Strongly Coupled Hardy Terms. *Journal of Mathematical Analysis and Applications*, **536**, Article 128252. https://doi.org/10.1016/j.jmaa.2024.128252

[12] Esposito, F., López-Soriano, R. and Sciunzi, B. (2024) Classification of Solutions to Hardy-Sobolev Doubly Critical Systems. *Journal de Mathématiques Pures et Appliquées*, **189**, Article 103595. https://doi.org/10.1016/j.matpur.2024.06.010

[13] Li, H. and Zou, W. (2023) Normalized Ground State for the Sobolev Critical Schrödinger Equation Involving Hardy Term with Combined Nonlinearities. *Mathematische Nachrichten*, **296**, 2440-2466. https://doi.org/10.1002/mana.202000481

[14] Li, G., Li, Y. and Tang, C. (2021) Existence and Asymptotic Behavior of Ground State Solutions for Schrödinger Equations with Hardy Potential and Berestycki-Lions Type Conditions. *Journal of Differential Equations*, **275**, 77-115. https://doi.org/10.1016/j.jde.2020.12.007

[15] Xue, Y., Gu, W. and Han, J. (2024) Bound State Solutions for Quasilinear Schrödinger Equations with Hardy Potential. *Results in Applied Mathematics*, **24**, Article 100499. https://doi.org/10.1016/j.rinam.2024.100499

[16] Zhou, S. (2024) Infinitely Many Solutions for Schrödinger Equations with Hardy Potential and Berestycki-Lions Conditions. *Open Mathematics*, **22**, Article 20230175. https://doi.org/10.1515/math-2023-0175

[17] Felli, V. (2009) On the Existence of Ground State Solutions to Nonlinear Schrödinger Equations with Multisingular Inverse-Square Anisotropic Potentials. *Journal d'Analyse Mathématique*, **108**, 189-217. https://doi.org/10.1007/s11854-009-0023-2

[18] Franca, M. and Sfecci, A. (2018) Entire Solutions of Superlinear Problems with Indefinite Weights and Hardy Potentials. *Journal of Dynamics and Differential Equations*, **30**, 1081-1118. https://doi.org/10.1007/s10884-017-9589-z

[19] Alexandrov, A.D. (1962) A Characteristic Property of Spheres. *Annali di Matematica Pura ed Applicata*, **58**, 303-315. https://doi.org/10.1007/bf02413056

[20] Serrin, J. (1971) A Symmetry Problem in Potential Theory. *Archive for Rational Mechanics and Analysis*, **43**, 304-318. https://doi.org/10.1007/bf00250468

[21] Sciunzi, B. (2017) On the Moving Plane Method for Singular Solutions to Semilinear Elliptic Equations. *Journal de Mathématiques Pures et Appliquées*, **108**, 111-123. https://doi.org/10.1016/j.matpur.2016.10.012

[22] Chen, W. and Li, C. (1991) Classification of Solutions of Some Nonlinear Elliptic Equations. *Duke Mathematical Journal*, **63**, 615-622. https://doi.org/10.1215/s0012-7094-91-06325-8

[23] García Azorero, J.P. and Peral Alonso, I. (1998) Hardy Inequalities and Some Critical Elliptic and Parabolic Problems. *Journal of Differential Equations*, **144**, 441-476. https://doi.org/10.1006/jdeq.1997.3375

[24] Stein, E. and Weiss, G. (1958) Fractional Integrals on n-Dimensional Euclidean Space. *Journal of Mathematics and Mechanics*, **7**, 503-514. https://doi.org/10.1512/iumj.1958.7.57030

[25] Chen, W. and Li, C. (2008) The Best Constant in a Weighted Hardy-Littlewood-Sobolev Inequality. *Proceedings of the American Mathematical Society*, **136**, 955-962. https://doi.org/10.1090/s0002-9939-07-09232-5

[26] Biagi, S., Esposito, F. and Vecchi, E. (2021) Symmetry and Monotonicity of Singular Solutions of Double Phase Problems. *Journal of Differential Equations*, **280**, 435-463. https://doi.org/10.1016/j.jde.2021.01.029

[27] Esposito, F., Farina, A. and Sciunzi, B. (2018) Qualitative Properties of Singular Solutions to Semilinear Elliptic Problems. *Journal of Differential Equations*, **265**, 1962-1983. https://doi.org/10.1016/j.jde.2018.04.030

[28] Malý, J. and Ziemer, W. (1997) Fine Regularity of Solutions of Elliptic Partial Differential Equations. American Mathematical Society. https://doi.org/10.1090/surv/051

[29] Moroz, V. and Van Schaftingen, J. (2017) A Guide to the Choquard Equation. *Journal of Fixed Point Theory and Applications*, **19**, 773-813. https://doi.org/10.1007/s11784-016-0373-1