Journal of Applied Mathematics and Physics, 2025, 13(2), 454-464
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Finite Time Domain Dynamics of Scalar Fields

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How to cite this paper: Thrapsaniotis, E.G. (2025) Finite Time Domain Dynamics of Scalar Fields. *Journal of Applied Mathematics and Physics*, **13**, 454-464.
https://doi.org/10.4236/jamp.2025.132024

Received: January 13, 2025
Accepted: February 16, 2025
Published: February 19, 2025

Abstract

In the present paper, we study the finite time domain dynamics of a scalar field interacting with external sources. We expand both the scalar field and the corresponding Hamiltonian in annihilation and creation operators and evaluate the relevant path integral. So, we get the Green function within a finite time interval. We apply the solution to the relevant Cauchy problem and further, we study the dynamics of scalar fields coupled with electromagnetic fields via perturbative methods.

Keywords

Scalar Fields, Finite Time Evolution, Electromagnetic Field, Perturbative Methods

1. Introduction

Quantum field theory constitutes a large area of physics with a variety of uses [1]-[5]. It deals with the properties of the various particles and fields. Here, we aim to consider analytic techniques for their finite time evolution as opposed to methods such as the numerical simulations of the dynamic equations [6] [7] or the use of lattice quantum field theory [5]. For previous analytic attempts, see, for example, [8]. Under appropriate preparations, finite time methods may be of value in the prediction of corresponding finite time results and effects in various systems contrasted, for instance, with asymptotic scattering theories.

In the present paper, we consider scalar fields [9]-[11] in a finite time domain and develop path integral methods for their dynamics. We start from the corresponding Lagrangian and Hamiltonian densities, decompose the scalar fields in expressions with annihilation and creation operators and evaluate their relevant path integral [12] [13]. In that way, we can extract the relevant Green function if we consider vacuum-to-vacuum transitions. It describes completely the whole dynamical system. We use it in the solution of the relevant Cauchy problem and further in their finite time propagation in spacetime. We study dimensions two, three and four. We suppose them to be coupled with electromagnetic fields and, more particularly, plane waves. We develop and apply perturbative techniques with respect to those electromagnetic fields. From the derived expressions, we can obtain results on the scalar fields' current densities.

The present paper proceeds as follows. In Section 2, we describe the present scalar field systems, decompose them in annihilation and creation operators, derive the Hamiltonian in a form with annihilation and creation operators and evaluate the relevant path integral in a finite time interval to obtain the transition amplitudes between coherent and more particularly vacuum states. Then, in Section 3, we use these results to evaluate the Green functions in an integral or series form for certain spacetime dimensions. In Section 4, we use those Green functions to solve the Cauchy problem for the present finite time domain Klein-Gordon equation. Then, in Section 5, we consider scalar fields, obeying the Klein-Gordon equation within a finite time interval and coupled with weak classical electromagnetic fields. We derive their time evolution via using, on the one hand, the obtained Green function and, on the other, perturbation theory with respect to the electromagnetic field. Finally, in Section 6, we give our conclusions.

Throughout the paper we use a Minkowski metric with signature $(1,\underbrace{-1,\cdots,-1}_{d-1})$, where d is the spacetime dimension. Moreover, we set $c=\hbar=1$. Then, $e^2=\alpha\simeq\frac{1}{137}$, where α is the fine structure constant. In our expressions, we maintain the mass symbol m for clarity. The scalar particle's rest energy is mc^2 and the units of length, time and energy are $\frac{\hbar^2}{me^2}$, $\frac{\hbar^3}{me^4}$ and $\frac{me^4}{\hbar^2}$ respectively.

2. System Hamiltonian and Path Integration

We intend to study a real scalar field φ of mass m coupled to a current J in a d-dimensional spacetime. Let it have Lagrangian

$$L=\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2+J\varphi \tag{1}$$

and corresponding classical action

$$I_0(\varphi,J)=\int\mathrm{d}\tau\int\mathrm{d}^{d-1}x\left[\frac{1}{2}(\partial\varphi)^2-\frac{m^2}{2}\varphi^2+J\varphi\right] \tag{2}$$

Then, according to variational considerations, it obeys the equation

$$\left(\Box+m^2\right)\varphi=J \tag{3}$$

where $\Box$ is the d'Alembert operator. The corresponding homogeneous equation is the Klein-Gordon equation. We intend to confront the whole problem via path integral methods. Its quantum Hamiltonian has the form

$$H=\int\mathrm{d}^{d-1}x\left[\frac{1}{2}\pi^2+\frac{1}{2}(\nabla\varphi)^2+\frac{m^2}{2}\varphi^2-J\varphi\right] \tag{4}$$

where π is the conjugate momentum. It describes an assembly of quantum oscillators coupled to external forces.

π and φ must satisfy equal-time canonical commutation relations,

$$\left[\varphi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)\right]=i\delta^{(d-1)}(\boldsymbol{x}-\boldsymbol{y}) \tag{5}$$

$$\left[\pi(\boldsymbol{x},t),\pi(\boldsymbol{y},t)\right]=\left[\varphi(\boldsymbol{x},t),\varphi(\boldsymbol{y},t)\right]=0 \tag{6}$$

We decompose the fields in annihilation and creation operator components

$$\varphi(\boldsymbol{x})=\int \mathrm{d}\tilde{k}\left[a(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}+a^{+}(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \tag{7}$$

$$\pi(\boldsymbol{x})=-i\int \mathrm{d}\tilde{k}\,\omega_{\boldsymbol{k}}\left[a(\boldsymbol{k})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}-a^{+}(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \tag{8}$$

where

$$\mathrm{d}\tilde{k}=\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega_{\boldsymbol{k}}} \tag{9}$$

$$\omega_{\boldsymbol{k}}=\sqrt{m^2+\boldsymbol{k}^2} \tag{10}$$

In order commutation relations (5) and (6) to be obeyed, a and a^{+} must obey

$$\left[a(\boldsymbol{k}),a^{+}(\boldsymbol{k}')\right]=(2\pi)^{d-1}2\omega_{\boldsymbol{k}}\delta^{(d-1)}(\boldsymbol{k}-\boldsymbol{k}') \tag{11}$$

$$\left[a(\boldsymbol{k}),a(\boldsymbol{k}')\right]=\left[a^{+}(\boldsymbol{k}),a^{+}(\boldsymbol{k}')\right]=0 \tag{12}$$

Now, on substituting expressions (7) and (8) in the Hamiltonian (4), we obtain the following diagonal expression for the present system's Hamiltonian, up to a constant which in fact is cancelled in Equation (23)

$$H=\int \mathrm{d}\tilde{k}\left[\omega_{\tilde{k}}a^{+}(\boldsymbol{k})a(\boldsymbol{k})-f(\boldsymbol{k},t)a^{+}(\boldsymbol{k})-f^{*}(\boldsymbol{k},t)a(\boldsymbol{k})\right] \tag{13}$$

We have set

$$f(\boldsymbol{k},t)=\int \mathrm{d}^{d-1}x\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}J(\boldsymbol{x},t) \tag{14}$$

The Hamiltonian (13) is diagonal with respect $\boldsymbol{k}$ and therefore on path integrating its diagonal terms, we easily obtain the coherent states propagator

$$\begin{aligned}
&U\left(z_f^{*}(\boldsymbol{k}),t_f;z_i(\boldsymbol{k}),t_i;J\right)\\
&=\exp\left\{\int \mathrm{d}\tilde{k}\left[z_f^{*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t_i)}z_i(\boldsymbol{k})\right.\right.\\
&\quad+i\int_{t_i}^{t_f}\mathrm{d}t\left(z_f^{*}(\boldsymbol{k})\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t)}f(\boldsymbol{k},t)+f^{*}(\boldsymbol{k},t)\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t_i)}z_i(\boldsymbol{k})\right)\\
&\quad\left.\left.-\frac{1}{2}\int_{t_i}^{t_f}\int_{t_i}^{t_f}\mathrm{d}t\mathrm{d}t'f^{*}(\boldsymbol{k},t')\mathrm{e}^{-i\omega_{\boldsymbol{k}}|t-t'|}f(\boldsymbol{k},t)\right]\right\}
\end{aligned} \tag{15}$$

To get the final form of the coherent state's evolution amplitude, we integrate diagonally [12] [13], *i.e.*

$$
\begin{aligned}
&U_0\left(\zeta^*(\boldsymbol{k}),t_f;\eta(\boldsymbol{k}),t_i;J\right)\\
&=\prod_{\tilde{k}}\left[\int\frac{\mathrm{d}^2 z(\boldsymbol{k})}{\pi}\right]\prod_{\boldsymbol{k}}\left[\langle\zeta(\boldsymbol{k})|z(\boldsymbol{k})\rangle\right]U\left(z^*(\boldsymbol{k}),t_f;z(\boldsymbol{k}),t_i;J\right)\prod_{\boldsymbol{k}}\left[\langle z(\boldsymbol{k})|\eta(\boldsymbol{k})\rangle\right]
\end{aligned}
\tag{16}
$$

So, since

$$
\langle\zeta(\boldsymbol{k})|z(\boldsymbol{k})\rangle=\exp\left[-\frac{1}{2}|\zeta(\boldsymbol{k})|^2-\frac{1}{2}|z(\boldsymbol{k})|^2|+\zeta^*(\boldsymbol{k})z(\boldsymbol{k})\right]
\tag{17}
$$

and

$$
\langle z(\boldsymbol{k})|\eta(\boldsymbol{k})\rangle=\exp\left[-\frac{1}{2}|z(\boldsymbol{k})|^2-\frac{1}{2}|\eta(\boldsymbol{k})|^2+z^*(\boldsymbol{k})\eta(\boldsymbol{k})\right]
\tag{18}
$$

we get

$$
\begin{aligned}
&U_0\left(\zeta^*(\boldsymbol{k}),t_f;\eta(\boldsymbol{k}),t_i;J\right)\\
&=\exp\left\{\int\mathrm{d}\tilde{k}\left[-\frac{1}{2}|\zeta(\boldsymbol{k})|^2-\frac{1}{2}|\eta(\boldsymbol{k})|^2+\frac{\zeta^*(\boldsymbol{k})\eta(\boldsymbol{k})}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t_i)}}-\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t_i)}\right)\right.\right.\\
&\quad+i\int_{t_i}^{t_f}\mathrm{d}t\left(-\zeta^*(\boldsymbol{k})\frac{\mathrm{e}^{i\omega_{\boldsymbol{k}}t}}{\mathrm{e}^{i\omega_{\boldsymbol{k}}t_i}-\mathrm{e}^{i\omega_{\boldsymbol{k}}t_f}}f(\boldsymbol{k},t)+f^*(\boldsymbol{k},t)\frac{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}}{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t_i}-\mathrm{e}^{-i\omega_{\boldsymbol{k}}t_f}}\eta(\boldsymbol{k})\right)\\
&\quad\left.\left.-\int_{t_i}^{t_f}\int_{t_i}^{t}f^*(\boldsymbol{k},t)\xi\left(\boldsymbol{k},t_f,t_i,t,t'\right)f(\boldsymbol{k},t')\mathrm{d}t'\mathrm{d}t\right]\right\}
\end{aligned}
\tag{19}
$$

We have set

$$
\begin{aligned}
\xi\left(\boldsymbol{k},t_f,t_i,t,t'\right)&=\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t-t')}+\frac{2\cos\left(\omega_{\boldsymbol{k}}(t-t')\right)}{1-\mathrm{e}^{i\omega_{\boldsymbol{k}}(t_f-t_i)}}\\
&=i\frac{\cos\left(\omega_{\boldsymbol{k}}(t-t')-\dfrac{\omega_{\boldsymbol{k}}(t_f-t_i)}{2}\right)}{\sin\left(\dfrac{\omega_{\boldsymbol{k}}(t_f-t_i)}{2}\right)}
\end{aligned}
\tag{20}
$$

Finally,

$$
\begin{aligned}
&U_0\left(\zeta^*(\boldsymbol{k}),t_f;\eta(\boldsymbol{k}),t_i;J\right)\\
&=\exp\left\{\int\mathrm{d}\tilde{k}\left[-\frac{1}{2}|\zeta(\boldsymbol{k})|^2-\frac{1}{2}|\eta(\boldsymbol{k})|^2+\frac{\zeta^*(\boldsymbol{k})\eta(\boldsymbol{k})}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t_i)}}-\ln\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}(t_f-t_i)}\right)\right.\right.\\
&\quad+i\int_{t_i}^{t_f}\mathrm{d}t\left(-\zeta^*(\boldsymbol{k})\frac{\mathrm{e}^{i\omega_{\boldsymbol{k}}t}}{\mathrm{e}^{i\omega_{\boldsymbol{k}}t_i}-\mathrm{e}^{i\omega_{\boldsymbol{k}}t_f}}f(\boldsymbol{k},t)+f^*(\boldsymbol{k},t)\frac{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t}}{\mathrm{e}^{-i\omega_{\boldsymbol{k}}t_i}-\mathrm{e}^{-i\omega_{\boldsymbol{k}}t_f}}\eta(\boldsymbol{k})\right)\\
&\quad\left.\left.-i\frac{1}{2}\int_{t_i}^{t_f}\int_{t_i}^{t_f}\mathrm{d}t\mathrm{d}t'f^*(\boldsymbol{k},t)\frac{\cos\left(\omega_{\boldsymbol{k}}|t-t'|-\dfrac{\omega_{\boldsymbol{k}}(t_f-t_i)}{2}\right)}{\sin\left(\dfrac{\omega_{\boldsymbol{k}}(t_f-t_i)}{2}\right)}f(\boldsymbol{k},t')\right]\right\}
\end{aligned}
\tag{21}
$$

We can use the above considerations to obtain the finite time interval Green function of scalar fields. We do it in the next section.

3. Green Function Evaluation

Now, we intend to derive Green functions for the present system. Green functions are useful as they constitute the functional inverses of the dynamic operators, giving the evolution of the various dynamic systems. They can be combined with perturbation theory giving a powerful tool. We apply that mathematical structure here.

According to the results of the previous section, the correlation functions' generating functional of the present system has the form

$$Z(J) = U_0\left(0, t_f; 0, t_i; J\right) \tag{22}$$

Then, the d dimensional Green function obeys the relations

$$\begin{aligned} G^{(d)}\left(\boldsymbol{x}-\boldsymbol{x}', t-t'; T\right) &= \left\langle 0 \right| \mathrm{T}\varphi\left(t', \boldsymbol{x}'\right)\varphi\left(t, \boldsymbol{x}\right)\left| 0 \right\rangle \\ &= -\frac{1}{Z(0)} \frac{\delta^2}{\delta J\left(t'\right)\delta J(t)} Z(J)\bigg|_{J=0} \end{aligned} \tag{23}$$

where T is the time ordering operator.

Upon performing the functional derivations according to Equations (21)-(23), we get the form

$$G^{(d)}\left(\boldsymbol{x}-\boldsymbol{x}', t-t'; T\right) = i\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{2\omega_{\boldsymbol{k}}\sin\left(\frac{\omega_{\boldsymbol{k}}T}{2}\right)} \tag{24}$$

where $T = t_f - t_i$ and $0 \le \left|t-t'\right| \le T$.

On writing the sine function in terms of exponentials and performing certain manipulations in Equation (24), we can take the form

$$G^{(d)}\left(\boldsymbol{x}-\boldsymbol{x}', t-t'; T\right) = -\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{1}{\omega_{\boldsymbol{k}}} \frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{1-\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}} \mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \tag{25}$$

So, after an appropriate geometric series expansion, we get

$$\begin{aligned} G^{(d)}\left(\boldsymbol{x}, \tau; T\right) &= -\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega_{\boldsymbol{k}}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \cos\left(\omega_{\boldsymbol{k}}\left|\tau\right| - \frac{\omega_{\boldsymbol{k}}T}{2}\right) \mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \sum_{n=0}^{\infty} \mathrm{e}^{-in\omega_{\boldsymbol{k}}T} \\ &= -\int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\omega_{\boldsymbol{k}}} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|} \sum_{n=0}^{\infty} \mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T} + \mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|} \sum_{n=0}^{\infty} \mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \end{aligned} \tag{26}$$

We notice that the poles appearing in Equation (25) can be bypassed by introducing a parameter μ that is smaller but close to one and using the equation

$$G^{(d)}\left(\boldsymbol{x}-\boldsymbol{x}', t-t'; T\right) = -\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{1}{\omega_{\boldsymbol{k}}} \frac{\cos\left(\omega_{\boldsymbol{k}}\left|t-t'\right| - \frac{\omega_{\boldsymbol{k}}T}{2}\right)}{1-\mu\mathrm{e}^{-i\omega_{\boldsymbol{k}}T}} \mathrm{e}^{-i\frac{\omega_{\boldsymbol{k}}T}{2}} \tag{27}$$

We have applied such an approach in [14].

Alternatively, we can perform the integration on $\boldsymbol{k}$ in Equation (26) analytically. If $d=2$, we obtain

$$\begin{aligned} & G^{(2)}(x,\tau;T) \\ & =-\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\frac{1}{\omega_k}\cos(kx)\left[\mathrm{e}^{i\omega_k|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_k T}+\mathrm{e}^{-i\omega_k|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_k T}\right] \\ & =-\frac{1}{2\pi}\sum_{n=0}^{\infty}K_0\left[im\sqrt{\left((n+1)T-|\tau|\right)^2-x^2}\right]-\frac{1}{2\pi}\sum_{n=0}^{\infty}K_0\left[im\sqrt{\left(nT+|\tau|\right)^2-x^2}\right] \\ & =-\frac{1}{2\pi}\sum_{n=-\infty}^{-1}K_0\left[im\sqrt{\left(nT+|\tau|\right)^2-x^2}\right]-\frac{1}{2\pi}\sum_{n=0}^{\infty}K_0\left[im\sqrt{\left(nT+|\tau|\right)^2-x^2}\right] \\ & =-\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}K_0\left[im\sqrt{\left(nT+|\tau|\right)^2-x^2}\right]=\frac{i}{4}\sum_{n=-\infty}^{\infty}H_0^{(2)}\left[m\sqrt{\left(nT+|\tau|\right)^2-x^2}\right] \end{aligned} \tag{28}$$

In the first sum from the second to the third line, we have replaced $n+1\to n$ and then we have replaced $n\to -n$. K_0 is a modified Bessel function of the third kind. $H_0^{(2)}$ is a Hankel function of the second kind.

If $d=3$ on expanding the factor $\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ in terms of exponentials and Bessel functions [15] and integrating angularly, the Green function (26) takes the form

$$\begin{aligned} G^{(3)}(\boldsymbol{x},\tau;T) &= -\frac{1}{4\pi}\int_0^\infty \mathrm{d}kJ_0(k|\boldsymbol{x}|)\frac{k}{\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \\ &= i\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}}\exp\left[-im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right] \end{aligned} \tag{29}$$

In the last equality, we have evaluated the integrals.

Similarly, if $d\geq 4$ on expanding the factor $\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ in terms of Gegenbauer polynomials and Bessel functions [15] and integrating angularly, the Green function (26) takes the form

$$G^{(d)}(\boldsymbol{x},\tau;T)=-\frac{1}{2(2\pi)^{\frac{d-1}{2}}}\int_0^\infty \mathrm{d}k\frac{J_\nu(k|\boldsymbol{x}|)}{(k|\boldsymbol{x}|)^\nu}\frac{k^{d-2}}{\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \tag{30}$$

where $\nu=\frac{d-3}{2}\neq 0,-1,-2,\cdots$. J_ν is a Bessel function of the first kind. So, the whole problem has been reduced to the evaluation of the k-integrals in Equation (30).

So, if $d=4$ on writing the Bessel function $J_{\frac{1}{2}}$ in terms of the sine function [15], the corresponding Green function takes the form

$$\begin{aligned} G^{(4)}(\boldsymbol{x},\tau;T) &= -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k\frac{\sin(k|\boldsymbol{x}|)}{|\boldsymbol{x}|}\frac{k}{\omega_{\boldsymbol{k}}}\left[\mathrm{e}^{i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-i(n+1)\omega_{\boldsymbol{k}}T}+\mathrm{e}^{-i\omega_{\boldsymbol{k}}|\tau|}\sum_{n=0}^{\infty}\mathrm{e}^{-in\omega_{\boldsymbol{k}}T}\right] \\ &= i\frac{m}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}}K_1\left(im\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right) \\ &= i\frac{m}{8\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}}H_1^{(2)}\left(m\sqrt{\left(nT+|\tau|\right)^2-|\boldsymbol{x}|^2}\right) \end{aligned} \tag{31}$$

K_1 is a modified Bessel function of the third kind. $H_1^{(2)}$ is a Hankel function of the second kind.

We can apply the above considerations in the study of the propagation of scalar fields in spacetime.

4. Boundary Value Problem

Now, we apply the above expressions of the Green functions to the solution of the Cauchy problem and the propagation of the present scalar fields obeying the Klein-Gordon equation.

We consider a spacetime region Ω bounded by the space-like surfaces σ_0 and σ. Let $\varphi(\boldsymbol{x},t)$ be a solution of the field equations such that on σ_0, it takes the initial values $\varphi_0(\boldsymbol{x},t)$ and $n^\mu \partial_\mu \varphi_0(\boldsymbol{x},t)$ for its derivative, where n^μ is normal on σ_0. $\boldsymbol{x}$ is a $d-1$ dimensional vector. Further, we suppose $F_\mu(\boldsymbol{x},t)$ be a function that vanishes as $|\boldsymbol{x}| \to \infty$ of the form

$$F_\mu(\boldsymbol{x}',t') = G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',t-t';T)\partial'_\mu \varphi(\boldsymbol{x}',t') - \partial'_\mu G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',t-t';T)\varphi(\boldsymbol{x}',t') \quad (32)$$

$G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',t-t';T)$ and $\varphi(\boldsymbol{x},t)$ are both solutions of the Klein-Gordon equation. So,

$$\partial^\mu F_\mu(\boldsymbol{x},t) = 0 \quad (33)$$

in Ω and therefore

$$\int_\sigma F_\mu(\boldsymbol{x}',t')\mathrm{d}\sigma'^\mu - \int_{\sigma_0} F_\mu(\boldsymbol{x}',t')\mathrm{d}\sigma'^\mu = \int_\Omega \partial'^\mu F_\mu(\boldsymbol{x}',t')\mathrm{d}^d x' = 0 \quad (34)$$

Now, we observe that if we use Equation (24)

$$\frac{\partial}{\partial \tau} G^{(d)}(\boldsymbol{x},\tau;T)\bigg|_{\tau=0^+} = \frac{i}{2}\delta^{(d-1)}(\boldsymbol{x}) \quad (35)$$

So, if we let σ_0 to be a hyperplane with $t=t'$, $n^\mu = (1,\underbrace{0,\cdots,0}_{d-1})$ and use the above equations, we obtain

$$\begin{aligned}\varphi(\boldsymbol{x},t) = 2i\int\Big\{&\Big[G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',t-t';T) - G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',0;T)\Big]\partial'_0\varphi_0(\boldsymbol{x}',t') \\ &-\partial'_0 G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',t-t';T)\varphi_0(\boldsymbol{x}',t')\Big\}\mathrm{d}^{d-1}\boldsymbol{x}'\end{aligned} \quad (36)$$

where $t \ge t'$.

Equation (36) solves the boundary value problem relevant to the present field.

5. Application to Charged Scalar Fields

Now, we proceed to an application of the above theory. We intend to study the propagation of a charged scalar particle. We start from the Lagrangian of a free-charged scalar field

$$L_0 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi = \sum_{i=1}^{2}\left[\frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{m^2}{2}\varphi_i^2\right] \quad (37)$$

where

$$\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \tag{38}$$

$$\phi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2) \tag{39}$$

where $\varphi_i,\ i = 1,2$ are real scalar fields (see Equation (1)).

We suppose that the charged scalar field interacts with an electromagnetic field of potential A_μ in the Lorentz gauge and performs the minimal substitution $\partial_\mu \phi \to (\partial_\mu + ieA_\mu)\phi$. Then, the action becomes

$$I(\varphi, A) = \int \mathrm{d}\tau \int \mathrm{d}^{d-1}x \left[D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi \right] \tag{40}$$

where

$$D_\mu \phi = \partial_\mu \phi + ieA_\mu \phi \tag{41}$$

We can apply the variational principle for the field ϕ to derive the equation

$$\left(D_\mu D^\mu + m^2 \right)\phi = 0 \tag{42}$$

or equivalently

$$\left(\partial_\mu \partial^\mu + m^2 \right)\phi = -2ieA_\mu \partial^\mu \phi + e^2 A_\mu A^\mu \phi \tag{43}$$

The above equation describes the present scalar field interacting with an electromagnetic field. It can be confronted for instance with numerical methods. Alternatively, we can use the present finite-time analytic methods combined with perturbation theory. Then, proceeding with the Green function $G^{(d)}(\vec{x}, t; T)$, derived in the previous sections, obeys the equation

$$\left(\partial_\mu \partial^\mu + m^2 \right) G^{(d)}(\boldsymbol{x}, t; T) = i\delta^{(d-1)}(\boldsymbol{x})\delta(t) \tag{44}$$

Under perturbative considerations, we set

$$\phi(\boldsymbol{x}, \sigma) = \phi^{(0)}(\boldsymbol{x}, \sigma) + \phi^{(1)}(\boldsymbol{x}, \sigma) + \phi^{(2)}(\boldsymbol{x}, \sigma) + \cdots \tag{45}$$

and suppose that initially both $\phi(\boldsymbol{x}, \sigma)$ and $\phi^{(0)}(\boldsymbol{x}, \sigma)$ obey boundary values $\phi_0(\boldsymbol{x}', 0)$ and $\partial_0' \phi_0(\boldsymbol{x}', 0)$ for their derivative, while the other terms obey homogeneous boundary conditions (here and below, we set $t' = 0$). Then, according to Equation (36), the zeroth order term of the field $\varphi(\boldsymbol{x}, \sigma)$ at time σ has the form

$$\begin{aligned} \phi^{(0)}(\boldsymbol{x}, \sigma) = 2i \int \Big\{ & \left[G^{(d)}(\boldsymbol{x} - \boldsymbol{x}', \sigma; T) - G^{(d)}(\boldsymbol{x} - \boldsymbol{x}', 0; T) \right] \partial_0' \varphi_0(\boldsymbol{x}', 0) \\ & - \partial_0' G^{(d)}(\boldsymbol{x} - \boldsymbol{x}', \sigma; T) \varphi_0(\boldsymbol{x}', 0) \Big\} \mathrm{d}^{d-1}\boldsymbol{x}' \end{aligned} \tag{46}$$

Now, we suppose that A_μ is small enough. That point is essential in order Equation (45) to converge. Further, the magnitude of the electromagnetic field must be small in order for the stability of the present whole system to be ensured against, for example, the generation of other particles.

Then, perturbatively to first order, we get

$$\phi^{(1)}(\boldsymbol{x}, \sigma) = -2e \int \mathrm{d}^{d-1}\boldsymbol{x}' \int_0^T \mathrm{d}t\, G^{(d)}(\boldsymbol{x} - \boldsymbol{x}', \sigma - t; T) A_\mu(\boldsymbol{x}', t) \partial^\mu \phi^{(0)}(\boldsymbol{x}', t) \tag{47}$$

and further in general at order r > 1, if we group in powers of e of the same order, the scalar field satisfies the recurrence relation

$$\begin{aligned}\phi^{(r)}(\boldsymbol{x},\sigma)=&\int \mathrm{d}^{d-1}\boldsymbol{x}'\int_0^T \mathrm{d}t\,G^{(d)}(\boldsymbol{x}-\boldsymbol{x}',\sigma-t;T)\\ &\times\Big[-ie^2A_\mu(\boldsymbol{x}',t)A^\mu(\boldsymbol{x}',t)\phi^{(r-2)}(\boldsymbol{x}',t)\\ &-2eA_\mu(\boldsymbol{x}',t)\partial^\mu\phi^{(r-1)}(\boldsymbol{x}',t)\Big]\end{aligned}\tag{48}$$

The current density j^μ is

$$j^\mu=\phi_b^* i\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\phi_a-2eA^\mu\phi_b^*\phi_a\tag{49}$$

where a and b are the quantum numbers of any two solutions. On using the present results, we can derive its finite time evolution and therefore predict possible outcomes in various systems and experiments.

Now, we let A_μ be the potential of a plane electromagnetic wave with polarization vector $\boldsymbol{\varepsilon}$, wavevector $\boldsymbol{k}_{ph}$, frequency ω_{ph}, and amplitude Δ_0. It has the form

$$\boldsymbol{A}(\boldsymbol{x},\tau)=\Delta_0\left(\boldsymbol{\varepsilon}\mathrm{e}^{i(\omega_{ph}\tau-\boldsymbol{k}_{ph}\cdot\boldsymbol{x})}+\boldsymbol{\varepsilon}\mathrm{e}^{-i(\omega_{ph}\tau-\boldsymbol{k}_{ph}\cdot\boldsymbol{x})}\right)\tag{50}$$

and

$$A_0(\boldsymbol{x},\tau)=0\tag{51}$$

$e\Delta_0$ must have dimension of energy (see also the discussion at the end of the introduction). This is the parameter that has to be small in order convergence to be ensured within the whole present perturbation theory.

Now, we consider the scalar field evolution. So, we suppose that initially

$$\phi(\boldsymbol{x}',0)=\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}'}\tag{52}$$

$$\partial_0'\phi(\boldsymbol{x}',0)=0\tag{53}$$

$\boldsymbol{p}$ is the $d-1$ dimensional initial momentum. Then, from Equation (46), we can extract $\phi^{(0)}$. It is

$$\phi^{(0)}(\boldsymbol{x},\sigma)=\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}\frac{\sin\left(\frac{E_{\boldsymbol{p}}T}{2}-E_{\boldsymbol{p}}\sigma\right)}{\sin\left(\frac{E_{\boldsymbol{p}}T}{2}\right)}\tag{54}$$

The corresponding energy appearing in Equation (54) has the form $E_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2+m^2}$.

If we let $T\to\infty$, we obtain the standard result:

$$\phi^{(0)}_{T=\infty}(\boldsymbol{x},\sigma)=\mathrm{e}^{i(\boldsymbol{p}\cdot\boldsymbol{x}-E_{\boldsymbol{p}}\sigma)}\tag{55}$$

Further, we can insert the solution (54) in Equation (47) to get $\phi^{(1)}$. At $\sigma=T$, it has the form

$$
\begin{aligned}
\phi^{(1)}\left(\boldsymbol{x},T\right) = & -e\Delta_0 \boldsymbol{\varepsilon}\cdot\boldsymbol{p}\frac{1}{1-\mathrm{e}^{-iE_{\boldsymbol{p}}T}}\mathrm{e}^{i\left(\boldsymbol{p}-\boldsymbol{k}_{ph}\right)\cdot\boldsymbol{x}}\frac{1}{\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}T}\right)} \\
& \times\left(\frac{\mathrm{e}^{i\left(\omega_{ph}-E_{\boldsymbol{p}}-\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}\right)T}-1}{\omega_{ph}-E_{\vec{p}}-\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}}+\frac{\mathrm{e}^{i\left(\omega_{ph}-E_{\boldsymbol{p}}\right)T}-\mathrm{e}^{-i\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}T}}{\omega_{ph}-E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}}\right. \\
& \left.-\frac{\mathrm{e}^{i\left(\omega_{ph}-\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}\right)T}-\mathrm{e}^{-iE_{\boldsymbol{p}}T}}{\omega_{ph}+E_{\boldsymbol{p}}-\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}}-\frac{\mathrm{e}^{i\omega_{ph}T}-\mathrm{e}^{-i\left(\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}+E_{\boldsymbol{p}}\right)T}}{\omega_{ph}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}-\boldsymbol{k}_{ph}}}\right) \\
& -e\Delta_0 \boldsymbol{\varepsilon}\cdot\boldsymbol{p}\frac{1}{1-e^{-iE_{\boldsymbol{p}}T}}\mathrm{e}^{i\left(\boldsymbol{p}+\boldsymbol{k}_{ph}\right)\cdot\boldsymbol{x}}\frac{1}{\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}\left(1-\mathrm{e}^{-i\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}T}\right)} \\
& \times\left(-\frac{\mathrm{e}^{-i\left(\omega_{ph}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}\right)T}-1}{\omega_{ph}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}}+\frac{\mathrm{e}^{-i\left(\omega_{ph}+E_{\boldsymbol{p}}\right)T}-\mathrm{e}^{-i\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}T}}{-\omega_{ph}-E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}}\right. \\
& \left.-\frac{\mathrm{e}^{-i\left(\omega_{ph}+\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}\right)T}-\mathrm{e}^{-iE_{\boldsymbol{p}}T}}{-\omega_{ph}+E_{\boldsymbol{p}}-\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}}-\frac{\mathrm{e}^{-i\omega_{ph}T}-\mathrm{e}^{-i\left(\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}+E_{\boldsymbol{p}}\right)T}}{-\omega_{ph}+E_{\boldsymbol{p}}+\omega_{\boldsymbol{p}+\boldsymbol{k}_{ph}}}\right)
\end{aligned}
\tag{56}
$$

Then, we can proceed to higher-order terms by applying Equation (48) recursively. We observe that $\phi^{(r)}$ is of order $\left(e\Delta_0\right)^r$ (see Equation (51)).

We conclude from the above considerations that the scalar particle is scattered by the photonic system and emits or absorbs photons. Moreover, according to Equation (49), there appear currents along the various wavevectors and the polarization of the photonic system, *i.e.* along the directions $\boldsymbol{\varepsilon}$, $\boldsymbol{p}$, $\boldsymbol{p}+\boldsymbol{k}_{ph}$, $\boldsymbol{p}-\boldsymbol{k}_{ph}$ and so on for higher order terms.

6. Conclusions

Here, we considered the dynamics of scalar fields in a finite time domain. We expanded them into annihilation and creation operators and integrated them functionally in the case of dimensions equal to two, three and four. We extracted their Green function in both an integral and a series form and applied that Green function to the perturbative study of the dynamics of scalar plane waves supposed to be coupled with weak electromagnetic fields. From the derived expressions, the scalar fields' current density can be obtained. So, under appropriate preparations, finite-time predictions and possible outcomes can be extracted.

The present method appears as an alternative to the direct numerical integration of the dynamic equations within finite time intervals or the use of lattice quantum field theoretic techniques for the study of quantum field systems.

In conclusion, the present approach is applicable to the dynamics of various field systems and can give interesting results within finite time domains. In subsequent papers, we aim to study the case of other such free or interacting quantum fields and give their dynamics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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