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Note on “Sharp Isolated Toughness Bound for Fractional (k,m)-Deleted Graphs”

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Abstract

As an appendix of [Gao *et al.* Sharp isolated toughness bound for fractional (k,m)-deleted graphs, Acta Mathematicae Applicatae Sinica, English Series, 2025, 41(1): 252-269], the detailed proof of Theorem 4.1 in this work is presented.

Keywords

Graph, Isolated Toughness Variant, Fractional (k,m)-Deleted Graph

1. Introduction

Let $k\in\mathbb{N}$ and $h:E(G)\to[0,1]$ be an indicator function defined on the edge set. A *fractional* k*-factor* is a spanning subgraph induced by $E_h=\{e\in E(G)\mid h(e)>0\}$ if $d_G^h(v)=\sum_{v'\in N(v)}h(vv')=k$ for each vertex v. For $m\in\mathbb{N}$, we say that G is a fractional (k,m)-deleted graph if removing any m edges from G, the resulting subgraph still admits a fractional k-factor.

The isolated toughness variant was defined by Ma and Liu [1] to measure the vulnerability of the network, which is formalized by:

$$I'(G)=\min\left\{\frac{|S|}{i(G-S)-1}\mid S\subset V(G),i(G-S)\ge 2\right\}$$

if G is a non-complete graph. Moreover, $I'(G)=+\infty$ if G is a complete graph. In [2], Gao *et al.* provide the tight $I'(G)$ bound for a graph G to be fractional (k,m)-deleted. Theorem 4.1 in [2] revealed the isolated toughness variant bound of fractional (k,m)-deleted graphs, but the detailed proof was not provided because its proof trick is similar to that of the main theorem in [2]. As an appendix of Gao *et al.* [2], the aim of this paper is to present the detailed proof of

Theorem 4.1 in [2]. It is stated that all symbols and notations are consistent with reference [2].

2. Proof of Theorem 4.1 in [2]

If G is complete, then Theorem 4.1 is obvious in light of $\delta(G) \ge k+m$. We always assume that G is not complete in the subsequent discussion. Suppose that G meets all the conditions of Theorem 4.1, but is not fractional (k,m)-deleted. In what follows, $k \ge 2$ and $m \ge 1$ are two positive integers, $\left\lfloor \frac{2m}{k} \right\rfloor^*$, $l_{k,m}^L$, $l_{k,m}^U$ and $\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^*$ are denoted as the statements in proof of Theorem 1.1. Also, it is akin to the definition of $\lfloor \cdot \rfloor^*$ during the subsequent proof process.

In view of Lemma 2.2 in [3] and Lemma 2.2 in [4], there exist disjoint subsets S and T of $V(G)$ satisfying

$$\begin{aligned} &k|S| - k|T| + \sum_{x\in T} d_{G-S}(x) \\ &= k|S| + \sum_{x\in T}\left(d_{G-S}(x) - k\right) \\ &\le \sum_{x\in T} d_H(x) - e_H(S,T) - 1 \le 2m-1. \end{aligned} \tag{1}$$

We select S and T such that $|T|$ is minimum. Thus, we immediately yield $T \ne \varnothing$, and $d_{G-S}(x) \le k-1$ for any $x \in T$.

Due to $T \ne \varnothing$ and $\delta(G) \ge k+m$, we have the following observation.

Observation 1. $|S| \ge m+1$.

Let l be the number of the components of $G[T]$ which are isomorphic to K_k and let $T_0 = \{x \in V(G[T]) \mid d_{G-S}(x) = 0\}$. Let G' be the subgraph inferred from $G[T] - T_0$ by deleting those l components isomorphic to K_k. Let S' be a set of vertices that contains exactly $k-1$ vertices in each component of K_k in $G[T]$.

Claim 1. $|V(G')| > 0$.

Proof of Claim 1. If $|V(G')| = 0$, then from (1) and Observation 1, we obtain $m+1 \le |S| \le |T_0| + l + \left\lfloor \frac{2m}{k} \right\rfloor^*$ since $|S|$ is an integer. We verify $|T_0| + l \ge m+1-\left\lfloor \frac{2m}{k} \right\rfloor^* \ge 2$. Hence, $i(G - S \cup S') = |T_0| + l \ge 2$ and we discuss the following subcases.

- If $l \ge l_{k,m}^U = l_{k,m}^L + 1$, then H can be selected with m edges in $G[T]$. We need to compare the value of $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$ and $l_{k,m}^U$.

If $k \ge 3$, we check that

$$m+1-\left\lfloor \frac{2m}{k} \right\rfloor^* \ge \frac{2m}{k(k-1)}, \tag{2}$$

which implies $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^* \ge l_{k,m}^U$ by the definition of $l_{k,m}^U$, and hence we use $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$ as the lower bound of $|T_0|+l$. If $k=2$, then $l_{k,m}^U = m$ and $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^* = 2$. Hence, if $k=2$ and $m=1$ or 2 then $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^* \ge l_{k,m}^U$ still holds. Only when $k=2$ and $m \ge 3$, then $l_{k,m}^U > m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$ and $l_{k,m}^U$ is acted as the lower bound of $|T_0|+l$.

Collectively, if $k=2$ and $m \ge 3$, then

$$
\begin{aligned}
I''(G) &\le \frac{|S \cup S'|}{i(G-S-S')-1} \le \frac{|T_0|+l+\left\lfloor \frac{2m}{k} \right\rfloor^* + l(k-1)}{|T_0|+l-1} \\
&\le k + \frac{k+\left\lfloor \frac{2m}{k} \right\rfloor^*}{|T_0|+l-1} \le k + \frac{k+\left\lfloor \frac{2m}{k} \right\rfloor^*}{l_{k,m}^U - 1} \\
&= 2 + \frac{2+m-1}{m-1} = 3 + \frac{2}{m-1},
\end{aligned}
$$

which contradicts to $I'(G) > 3 + \frac{2}{m-1}$. For other combination of (k,m), we get:

$$
\begin{aligned}
I'(G) &\le \frac{|S \cup S'|}{i(G-S-S')-1} \le \frac{|T_0|+l+\left\lfloor \frac{2m}{k} \right\rfloor^* + l(k-1)}{|T_0|+l-1} \\
&\le k + \frac{k+\left\lfloor \frac{2m}{k} \right\rfloor^*}{|T_0|+l-1} \le k + \frac{k+\left\lfloor \frac{2m}{k} \right\rfloor^*}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*},
\end{aligned}
$$

which contradicts to $I'(G) > k-1+\frac{m+k}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*}$.

- If $l \le l_{k,m}^L$, then the number of edges in $G[T]$ is smaller than m and $\sum_{x\in T} d_H(x) - e_H(S,T) \le k(k-1)l$. We verify

$|S| \le |T_0| + l + \frac{k(k-1)l}{k} - 1 = |T_0| + kl - 1$.

If $|T_0| \ge 1$, then $k+m \le |S| \le |T_0|+kl-1$, $|T_0| \ge k+m+1-kl$ and

$$
\begin{aligned}
I'(G) &\le \frac{|S \cup S'|}{i(G-S-S')-1} \le \frac{|T_0|+lk-1+l(k-1)}{|T_0|+l-1} \\
&= 1 + \frac{2(k-1)l}{|T_0|+l-1} \le 1 + \frac{2(k-1)l}{k+m-kl+l}.
\end{aligned}
$$

If $l=0$, then $I'(G) \le 1$, which contradicts to the hypothesis of $I'(G)$. Suppose $l \ge 1$, then

$$I'(G) \le -1 + \frac{2k+2m}{k+m-(k-1)l} \le -1 + \frac{2k+2m}{k+m-(k-1)l^L_{k,m}}.$$

If $k=2$, then $-1 + \frac{2k+2m}{k+m-(k-1)l^L_{k,m}} = \frac{1+2m}{3}$. Note that $|T_0| \ge k+m+1-kl \ge k+m+1-kl^L_{k,m} = 5-m$ when $k=2$. When $m=3$ (resp. $m=4$) then $I'(G) \le \frac{1+2m}{3} = \frac{7}{3}$ (resp. $I'(G) \le \frac{1+2m}{3} = \frac{9}{3}$), which contradicts to $I'(G) > 3 + \frac{4}{m+1} = 4$ (resp. $I'(G) > 3 + \frac{4}{m} = 4$). When $m \ge 5$, we use $|T_0| \ge 1$ instead of $|T_0| \ge 5-m$. Thus, we have:

$$I'(G) \le 1 + \frac{2(k-1)l}{|T_0|+l-1} \le 1 + \frac{2(k-1)l}{l} = 2k-1 = 3,$$

which contradicts to the hypothesis of $I'(G)$. For $(k,m)=(2,1)$ (resp. $(k,m)=(2,2)$), we have $I'(G) \le \frac{1+2m}{3} = \frac{3}{3}$ (resp. $I'(G) \le \frac{1+2m}{3} = \frac{5}{3}$), which contradicts to $I'(G) > k-1+\frac{m+k}{m-\left\lceil \frac{2m}{k} \right\rceil^*} = 4$ (resp. $I'(G) > 5$). For $k \ge 3$, we yield:

$$I'(G) < -1 + \frac{2k+2m}{k+m-(k-1)\frac{2m}{k(k-1)}} = \frac{k+m+\frac{2m}{k}}{k+m-\frac{2m}{k}},$$

which contradicts to $I'(G) > k-1+\frac{k+m}{m-\left\lceil \frac{2m}{k} \right\rceil^*}$.

If $|T_0|=0$, then $m+1 \le |S| \le kl-1$, and $\frac{2m}{k(k-1)} > l^L_{k,m} \ge l \ge \frac{m+2}{k}$ which reveals $k=2$ and $m \ge 3$. Hence, $l^L_{k,m} = l^L_{2,m} = m-1$, $l \ge \frac{m+2}{2}$ if m is even, $l \ge \frac{m+3}{2}$ if m is odd. Hence, if $m \ge 4$ is even, then

$$\begin{aligned} I'(G) &\le \frac{|S \cup S'|}{i(G-S-S')-1} \le \frac{lk-1+l(k-1)}{l-1} = 2k-1+\frac{2k-2}{l-1} \\ &\le 2k-1+\frac{2k-2}{\frac{m+2}{k}-1} = 3+\frac{4}{m}, \end{aligned}$$

which contradicts to $I'(G) > 3 + \frac{4}{m}$.

If $m \ge 3$ is odd, then

$$\begin{aligned} I'(G) &\le \frac{|S \cup S'|}{i(G-S-S')-1} \le \frac{lk-1+l(k-1)}{l-1} = 2k-1+\frac{2k-2}{l-1} \\ &\le 2k-1+\frac{2k-2}{\frac{m+3}{k}-1} = 3+\frac{4}{m+1}, \end{aligned}$$

which contradicts to $I'(G) > 3 + \frac{4}{m+1}$. □

Let $G' = G_1 \cup G_2$ where G_1 is the union of components of G' which satisfies that $d_{G-S}(v) = k-1$ for each vertex $v \in V(G_1)$ and $G_2 = G' - G_1$. In terms of Lemma 2.2 in [5], G_1 has a maximum independent set I_1 and the covering set $C_1 = V(G_1) - I_1$ such that

$$\left|V(G_1)\right| \le \sum_{i=1}^{k}(k-i+1)\left|I^{(i)}\right| - \frac{\left|I^{(1)}\right|}{2} \le \left(k-\frac{1}{2}\right)\left|I_1\right|, \tag{3}$$

and

$$\left|C_1\right| \le \sum_{i=1}^{k}(k-i)\left|I^{(i)}\right| - \frac{\left|I^{(1)}\right|}{2}, \tag{4}$$

where $I^{(i)} = \left\{v \in I_1, d_{G_1}(v) = k-i\right\}$ for $1 \le i \le k$ and $\sum_{i=1}^{k}\left|I^{(i)}\right| = \left|I_1\right|$. Using the definition of G' and G_2, we verify that each component of G_2 has a vertex of degree at most $k-2$ in $G-S$. We have the following observation by the definition of G_2 and $\delta(G) \ge k+m$.

Observation 2. *If* $G_2 \ne \varnothing$, *then* $k \ge 3$ *and* $|S| \ge m+2$.

Denote I_2 by an independent set of G_2 which is selected by the procedure described in Gao *et al.* [2]. Set $W = V(G) - S - T$ and
$U = S \cup S' \cup C_1 \cup \left(N_G(I_1) \cap W\right) \cup N_{G-S}(I_2) = S \cup S' \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$.

The following discussion is divided into two cases by means of whether $|T_0| + l = 0$.

Case 1. $|T_0| + l \ge 1$.

The two claims below consider two extreme cases respectively.

Claim 2. *If* $|T_0| + l \ge 1$, *then* $|I_2| \ne 0$.

Proof of Claim 2. Suppose $|I_2| = 0$, then $|I_1| \ne 0$ by $|V(G')| > 0$.

In view of (1) and (3), we obtain $|T| = |V(G_1)| + |T_0| + lk$,

$$\begin{aligned} k|S| &\le k|T| - d_{G-S}(T) + \left(\sum_{x\in T} d_H(x) - e_H(S,T)\right) - 1 \\ &\le \left(k-\frac{1}{2}\right)|I_1| + k|T_0| + lk + \left(\sum_{x\in T} d_H(x) - e_H(S,T)\right) - 1 \end{aligned}$$

and

$$\begin{aligned} |S| &\le \left(1-\frac{1}{2k}\right)|I_1| - \frac{1}{k} + |T_0| + l + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \\ &\le |I_1| + |T_0| + l + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* \\ &\le |I_1| + |T_0| + l + \left\lfloor \frac{2m}{k} \right\rfloor^*. \end{aligned}$$

Then, we infer:

$$m+1 \le |S| \le |I_1| + |T_0| + l + \left\lfloor \frac{2m}{k} \right\rfloor^*,$$

which implies $|T_0|+l+|I_1| \ge m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$. Therefore,

$$I'(G) \le \frac{|S \cup S' \cup N_{G-S}(I_1)|}{i(G-S \cup S' \cup N_{G-S}(I_1))-1}$$

$$\le \frac{|I_1| + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + |T_0| + l + l(k-1) + |I_1|(k-1)}{|I_1|+l+|T_0|-1}$$

$$\le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1|+l+|T_0|-1}.$$

- If $G[T]$ contains at least m edges, then $\sum_{x\in T} d_H(x) - e_H(S,T) \le 2m$ and $|I_1|+l \ge l_{k,m}^L + 1 = l_{k,m}^U$. It is necessary to compare the value of $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$ and $l_{k,m}^U$, hence to determine the lower bound of $|T_0|+l+|I_1|$. Using the same fashion in Claim 1, we know that if $k=2$ and $m \ge 3$, then $l_{k,m}^U > m+1-\left\lfloor \frac{2m}{k} \right\rfloor^*$, and otherwise the opposite is true.

Hence, if $k \ge 3$ or $(k,m)=(2,1)$ or $(k,m)=(2,2)$, then

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1|+l+|T_0|-1} \le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^* + k}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*},$$

which contradicts to $I'(G) > k-1+\frac{m+k}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*}$. If $k=2$ and $m \ge 3$, then

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1|+l+|T_0|-1}$$

$$\le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^* + k}{l_{k,m}^U - 1} = 2 + \frac{m+1}{m-1} = 3 + \frac{2}{m-1},$$

which contradicts to the hypothesis of $I'(G)$.

- If $|E(G[T])| < m$, then $\sum_{x\in T} d_H(x) - e_H(S,T) \le 2m-1$. We get:

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1|+l+|T_0|-1} \le k + \frac{\left\lfloor \frac{2m-1}{k} \right\rfloor^* + k}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*},$$

a contradiction to $I'(G) > k-1+\frac{m+k}{m-\left\lfloor \frac{2m}{k} \right\rfloor^*}$ if $k \ge 3$ or $(k,m)=(2,1)$ or

$(k,m)=(2,2)$.

Suppose $k=2$ and $m\ge 3$. Then, $l\le m-1$, $|T|=|I_1|+|T_0|+2l$,

$$2|S|\le 2|T|-d_{G-S}(T)+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1$$
$$\le |I_1|+2|T_0|+2l+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1,$$
$$|S|\le \frac{|I_1|}{2}+|T_0|+l+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2},$$

and

$$I'(G)\le \frac{|S\cup S'\cup N_{G-S}(I_1)|}{i(G-S\cup S'\cup N_{G-S}(I_1))-1}$$
$$\le \frac{\frac{|I_1|}{2}+|T_0|+l+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2}+l+|I_1|}{|I_1|+|T_0|+l-1}$$
$$=2+\frac{\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2}-|T_0|-\frac{1}{2}|I_1|+2}{l+|I_1|+|T_0|-1}.$$

If $\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2}-|T_0|-\frac{1}{2}|I_1|+2<0$, then $I'(G)<2$, a contradiction. Thus, $\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2}-|T_0|-\frac{1}{2}|I_1|+2\ge 0$.

To maximize $\sum_{x\in T}d_H(x)-e_H(S,T)$, we first take l edges from lK_2 into H (each edge contributes 2 to $\sum_{x\in T}d_H(x)-e_H(S,T)$), then take $\min\{m-l,|I_1|\}$ edges between I_1 and W into H (each edge contributes 1 to $\sum_{x\in T}d_H(x)-e_H(S,T)$), and finally randomly select $m-l-|I_1|$ edges in the rest subgraphs if $m-l-|I_1|\ge 1$ (each edge contributes zero to $\sum_{x\in T}d_H(x)-e_H(S,T)$). We have:

$$I'(G)\le 2+\frac{\frac{2l+\min\{m-l,|I_1|\}-1}{2}-|T_0|-\frac{1}{2}|I_1|+2}{|I_1|+|T_0|+l-1}.$$

Suppose $|T_0|\ge 1$. If $\min\{m-l,|I_1|\}=m-l$, then $m-l\le |I_1|$, $\max\left\{\frac{m+l-|I_1|}{2}\right\}=m-1$ and it reaches the maximum value when $l=m-1$ and $|I_1|=1$, and hence

$$I'(G)\le 2+\frac{\frac{m+l-1}{2}-|T_0|-\frac{1}{2}|I_1|+2}{|I_1|+|T_0|+l-1}\le 2+\frac{m-1+\frac{3}{2}-1}{|I_1|+1+l-1}$$
$$=2+\frac{m-\frac{1}{2}}{|I_1|+l}<2+\frac{m}{m}=3,$$

a contradiction to the hypothesis of $I'(G)$. If $\min\{m-l,|I_1|\}=|I_1|$, then $m-l\ge|I_1|$ and

$$I'(G)\le 2+\frac{\frac{2l+|I_1|-1}{2}-|T_0|-\frac{1}{2}|I_1|+2}{|I_1|+|T_0|+l-1}$$
$$\le 2+\frac{\frac{2l+|I_1|-1}{2}-1-\frac{1}{2}|I_1|+2}{|I_1|+1+l-1}$$
$$=2+\frac{l+\frac{1}{2}}{|I_1|+l}<2+\frac{l+1}{1+l}=3,$$

a contradiction to the hypothesis of $I'(G)$.

Considering $|T_0|=0$, hence $l\ge 1$ and

$$I'(G)\le 2+\frac{\frac{2l+\min\{m-l,|I_1|\}-1}{2}-\frac{1}{2}|I_1|+2}{|I_1|+l-1}.$$

If $\min\{m-l,|I_1|\}=m-l$, then $m-l\le|I_1|$, $\max\left\{\frac{m+l-|I_1|}{2}\right\}=m-1$ and it reaches the maximum value when $l=m-1$ and $|I_1|=1$, and hence

$$I'(G)\le 2+\frac{\frac{m+l-1}{2}-\frac{1}{2}|I_1|+2}{|I_1|+l-1}\le 3+\frac{3}{2(m-1)},$$

a contradiction to the hypothesis of $I'(G)$. If $\min\{m-l,|I_1|\}=|I_1|$, then $m-l\ge|I_1|$ and

$$I'(G)\le 2+\frac{\frac{2l+|I_1|-1}{2}-\frac{1}{2}|I_1|+2}{|I_1|+l-1}.$$

Since the numerator part refer to $|S\cup S'\cup N_{G-S}(I_1)|$ which is an integer, we infer:

$$I'(G)\le 2+\frac{l+1}{|I_1|+l-1}\le 2+\frac{l+1}{1+l-1}=3+\frac{1}{l}.$$

Note that when $|I_1|=1$, according to

$$m+1\le|S|\le\frac{|I_1|}{2}+|T_0|+l+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{2}$$
$$\le\frac{1}{2}+l+\frac{2l+\min\{m-l,|I_1|\}-1}{2}$$
$$=\frac{1}{2}+2l,$$

we have $2l\ge m+1$ (*i.e.* $l\ge\frac{m+1}{2}$). Hence $I'(G)\le 3+\frac{1}{l}\le 3+\frac{2}{m+1}$, a contradiction to the hypothesis of $I'(G)$. □

Claim 3. *If* $|T_0|+l\ge 1$, *then* $|I_1|\ne 0$.

Proof of Claim 3. Suppose $|I_1|=0$. We yield $|I_2|\ne 0$ by $|V(G')|>0$, and hence $k\ge 3$ (thus we don't consider the case in $k=2$).

Let $v_1,v_2,\cdots,v_{|I_2|}$ be vertices in I_2 such that $d_{G-S}(v_1)\le k-2$ and $d_{G-S}(v_1)\le d_{G-S}(v_2)\le\cdots\le d_{G-S}(v_{|I_2|})$. Then $|T|=|V(G_2)|+|T_0|+lk$,

$$\begin{aligned}k|S| &\le k|T|-d_{G-S}(T)+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1\\ &\le k|T_0|+lk+\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)\\ &\quad+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1\end{aligned}$$

and

$$\begin{aligned}|S| &\le |T_0|+l+\frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)}{k}\\ &\quad+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{1}{k}.\end{aligned}$$

We acquire $i(G-U)\ge 2$ where $U=S\cup S'\cup N_{G-S}(I_2)$ and

$$\begin{aligned}|U| &\le |S|+|S'|+|N_{G-S}(I_2)|\\ &\le |T_0|+l+\frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)}{k}\\ &\quad+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{1}{k}+l(k-1)+\sum_{i=1}^{|I_2|}d_{G-S}(v_i)\\ &=|T_0|+lk+\sum_{i=1}^{|I_2|}\left(-\frac{d_{G-S}^2(v_i)}{k}+\left(2-\frac{1}{k}\right)d_{G-S}(v_i)+1\right)\\ &\quad+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{1}{k}\\ &\le |T_0|+lk+\left(|I_2|-1\right)\left(-\frac{(k-1)^2}{k}+\left(2-\frac{1}{k}\right)(k-1)+1\right)\\ &\quad+\left(-\frac{(k-2)^2}{k}+\left(2-\frac{1}{k}\right)(k-2)+1\right)\\ &\quad+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{1}{k}\\ &=|T_0|+lk+k|I_2|+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{3}{k}.\end{aligned}$$

Hence, $|U|\le |T_0|+lk+k|I_2|+\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}\right\rfloor^{*}$.

It ensures that to maximize $|U|$, only one vertex in I_2 has degree $k-2$ in $G-S$, and others have degree $k-1$ in $G-S$. Hence, $|S|$ can be re-bounded by:

$$\begin{aligned}|S| &\le |T_0| + l + \frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)}{k} \\ &\quad + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} - \frac{1}{k} \\ &\le \frac{\left(|I_2|-1\right)\left((k-1)+1\right)\left(k-(k-1)\right)+\left((k-2)+1\right)\left(k-(k-2)\right)}{k} \\ &\quad |T_0| + l + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} - \frac{1}{k} \\ &= |T_0| + l + |I_2| + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} + 1 - \frac{3}{k},\end{aligned}$$

which reveals $m+2 \le |S| \le |T_0| + l + |I_2| + 1 + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T) - 3}{k} \right\rfloor$ and

$$|T_0| + l + |I_2| \ge m + 1 - \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T) - 3}{k} \right\rfloor \ge m + 1 - \left\lfloor \frac{2m}{k} \right\rfloor^{*}.$$

Furthermore, we get:

$$\begin{aligned} I'(G) &\le \frac{|U|}{i(G-U)-1} \le \frac{|T_0| + lk + k|I_2| + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^{*}}{|I_2| + |T_0| + l - 1} \\ &\le \frac{k\left(|T_0| + l + |I_2|\right) + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^{*}}{|I_2| + |T_0| + l - 1} \\ &= k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^{*} + k}{|I_2| + |T_0| + l - 1} \\ &\le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^{*} + k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^{*}},\end{aligned}$$

which contradicts to the hypothesis of $I'(G)$. □

From Claim 2 and Claim 3, we have $|I_1| \ge 1$, $|I_2| \ge 1$ and $k \ge 3$ (hence we don't consider the special circumstances of $k = 2$).

Denote $v_1, v_2, \cdots, v_{|I_2|}$ as vertices in I_2 as defined in Claim 3. We obtain:

$$|T| = |V(G_1)| + |V(G_2)| + |T_0| + lk,$$

$$\begin{aligned} k|S| &\le k\left(|T_0| + l\right) + \left(k - \frac{1}{2}\right)|I_1| + \sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right) \\ &\quad + \left(\sum_{x\in T} d_H(x) - e_H(S,T)\right) - 1 \end{aligned}$$

and

$$|S| \le |T_0| + l + \left(1 - \frac{1}{2k}\right)|I_1| + \frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k - d_{G-S}(v_i)\right)}{k} + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} - \frac{1}{k}.$$

We confirm that $i(G-U) \ge 3$ where
$U = S \cup S' \cup C_1 \cup \left(N_G(I_1) \cap W\right) \cup N_{G-S}(I_2)$ and
$|U| \le |T_0| + lk + k|I_1| + k|I_2| + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^*$ since $|U|$ is an integer.

By maximizing $|U|$, we can see that in the extreme value setting, only one vertex in I_2 has degree $k-2$ in $G-S$, and the others have degree $k-1$ in $G-S$. Thus, $|S|$ can be re-bounded by:

$$|S| \le \frac{\left(|I_2|-1\right)\left((k-1)+1\right)\left(k-(k-1)\right) + \left((k-2)+1\right)\left(k-(k-2)\right)}{k}$$
$$|T_0| + l + \left(1 - \frac{1}{2k}\right)|I_1| + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} - \frac{1}{k}$$
$$= |T_0| + l + |I_1| + |I_2| + \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} + 1 - \frac{3}{k},$$

which reveals $m + 2 \le |S| \le |T_0| + l + |I_1| + |I_2| + 1 + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T) - 3}{k} \right\rfloor$ by

Observation 2. We further infer $|T_0| + l + |I_1| + |I_2| \ge m + 1 - \left\lfloor \frac{2m}{k} \right\rfloor^*$.

Therefore, we infer:

$$I'(G) \le \frac{|U|}{i(G-U)-1}$$
$$\le \frac{|T_0| + lk + k|I_1| + k|I_2| + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^*}{|I_1| + |I_2| + |T_0| + l - 1}$$
$$\le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1| + |I_2| + |T_0| + l - 1} \le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^* + k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^*},$$

a contradiction.

Case 2. $|T_0| + l = 0$.

Again, we consider two extreme cases in Claim 4 and Claim 5 respectively.

Claim 4. *If* $|T_0| + l = 0$, *then* $|I_2| \ne 0$.

Proof of Claim 4. Suppose $|I_2| = 0$, then we infer $|I_1| \ne 0$, $|T| = |V(G_1)|$ and
$k|S| \le k|T| - d_{G-S}(T) + \left(\sum_{x\in T} d_H(x) - e_H(S,T)\right) - 1 = |T| + \sum_{x\in T} d_H(x) - e_H(S,T) - 1$.

If $|I_1| = 1$, then $|T| \le k-1$ and

$$|S| \le \frac{|T| + \left(\sum_{x\in T} d_H(x) - e_H(S,T)\right) - 1}{k} \le \frac{\sum_{x\in T} d_H(x) - e_H(S,T) - 2}{k} + 1.$$ Thus,

$k+m \le \delta(G) \le |S|+(k-1) \le k+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)-2}{k} \le k+\frac{2m-2}{k}$, a contradiction. Hence, $|I_1| \ge 2$.

We acquire $|T| \le \left(k-\frac{1}{2}\right)|I_1|$,

$|S| \le \frac{|T|+\left(\sum_{x\in T} d_H(x)-e_H(S,T)\right)-1}{k} \le \left(1-\frac{1}{2k}\right)|I_1|+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}-\frac{1}{k}$,

i.e. $m+1 \le |S| \le |I_1|+\left\lfloor \frac{\sum_{x\in T} d_H(x)-e_H(S,T)-1}{k} \right\rfloor$. Therefore,

$$
\begin{aligned}
|I_1| &\ge m+1-\left\lfloor \frac{\sum_{x\in T} d_H(x)-e_H(S,T)-1}{k} \right\rfloor \\
&\ge m+1-\left\lfloor \frac{2m}{k} \right\rfloor^{*}.
\end{aligned}
$$

Set $U=S\cup C_1\cup\left(N_G(I_1)\cap W\right)$, we have $i(G-U)=|I_1|\ge 2$. According to (4), we yield:

$$
\begin{aligned}
|U| &\le |S|+|C_1|+\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| \\
&\le \left(1-\frac{1}{2k}\right)|I_1|+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k} \\
&\quad -\frac{1}{k}+\sum_{i=1}^{k}(k-i)\left|I^{(i)}\right|-\frac{\left|I^{(1)}\right|}{2}+\sum_{i=1}^{k}(i-1)\left|I^{(i)}\right| \\
&\le k|I_1|+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}-\frac{|I_1|}{2k}-\frac{1}{k}.
\end{aligned}
$$

Hence,

$$
|U| \le k|I_1|+\left\lfloor \frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k} \right\rfloor^{*}
$$

due to the integer property of $|U|$, and

$$
\begin{aligned}
I'(G) &\le \frac{|U|}{i(G-U)-1} \le \frac{k|I_1|+\left\lfloor \frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k} \right\rfloor^{*}}{|I_1|-1} \\
&= k+\frac{\left\lfloor \frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k} \right\rfloor^{*}+k}{|I_1|-1}.
\end{aligned}
$$

- If $|E(G[T])| \ge m$, then $\sum_{x\in T} d_H(x)-e_H(S,T) \le 2m$ and $|I_1| \ge l_{k,m}^{U}$. If $k\ge 3$ or $(k,m)=(2,1)$ or $(k,m)=(2,2)$, then $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^{*} > l_{k,m}^{U}$, hence, the lower bound of $|I_1|$ takes $m+1-\left\lfloor \frac{2m}{k} \right\rfloor^{*}$ and

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1| - 1} \le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^* + k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^*},$$

a contradiction to $I'(G) > k - 1 + \frac{m+k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^*}$. If $k = 2$ and $m \ge 3$, then

$l_{2,m}^U = m > 2 = m + 1 - \left\lfloor \frac{2m}{k} \right\rfloor^*$ and

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1| - 1}$$

$$\le k + \frac{\left\lfloor \frac{2m}{k} \right\rfloor^* + k}{l_{k,m}^U - 1} = 2 + \frac{m - 1 + 2}{m - 1} = 3 + \frac{2}{m - 1},$$

a contradiction.

- If $|E(G[T])| < m$, then $\sum_{x\in T} d_H(x) - e_H(S,T) \le 2m - 1$. If $k \ge 3$ or $(k,m) = (2,1)$ or $(k,m) = (2,2)$, then

$$I'(G) \le k + \frac{\left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T)}{k} \right\rfloor^* + k}{|I_1| - 1} \le k + \frac{\left\lfloor \frac{2m-1}{k} \right\rfloor^* + k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^*},$$

a contradiction to $I'(G) > k - 1 + \frac{m+k}{m - \left\lfloor \frac{2m}{k} \right\rfloor^*}$.

Suppose $k = 2$ and $m \ge 3$. Then, $\sum_{x\in T} d_H(x) - e_H(S,T) \le \min\{m, |I_1|\}$. If $|I_1| \le m$, then

$$m + 1 \le |S| \le |I_1| + \left\lfloor \frac{\sum_{x\in T} d_H(x) - e_H(S,T) - 1}{k} \right\rfloor$$

$$\le |I_1| + \left\lfloor \frac{|I_1| - 1}{2} \right\rfloor \le \frac{3|I_1|}{2} - \frac{1}{2},$$

$|I_1| \ge \frac{2m}{3} + 1$, and

$$I'(G) \le \frac{|S \cup N_{G-S}(I_1)|}{i(G - S \cup N_{G-S}(I_1)) - 1}$$

$$\le \frac{\frac{3|I_1|}{2} - \frac{1}{2} + |I_1|}{|I_1| - 1} = \frac{5}{2} + \frac{2}{|I_1| - 1}$$

$$\le \frac{5}{2} + \frac{2}{\frac{2}{3}m} = \frac{5}{2} + \frac{3}{m},$$

a contradiction to the hypothesis of $I'(G)$.

If $m<|I_1|$, then

$$|S|\le|I_1|+\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-1}{k}\right\rfloor$$
$$\le|I_1|+\left\lfloor\frac{m-1}{2}\right\rfloor\le|I_1|+\frac{m-1}{2},$$

and thus

$$I'(G)\le\frac{|S\cup N_{G-S}(I_1)|}{i(G-S\cup N_{G-S}(I_1))-1}\le\frac{|I_1|+\frac{m-1}{2}+|I_1|}{|I_1|-1}$$
$$=2+\frac{m+3}{2(|I_1|-1)}\le2+\frac{m+3}{2(m+1-1)}=\frac{5}{2}+\frac{3}{2m},$$

a contradiction. □

Claim 5. *If* $|T_0|+l=0$, *then* $|I_1|\ne0$.

Proof of Claim 5. Suppose $|I_1|=0$, then $|I_2|\ne0$ in terms of $|V(G')|>0$, and hence $k\ge3$.

If $|I_2|=1$, then we set $d_{\min}=\min\{d_{G-S}(v)\mid v\in G_2\}$ and $z\in V(G_2)$ such that $d_{G-S}(z)=d_{\min}$, thus $d_{\min}\in\{1,\cdots,k-2\}$. Hence, we deduce:

$$k|S|\le k|T|-d_{G-S}(T)+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1$$
$$\le|T|(k-d_{\min})+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1,$$
$$|S|\le\frac{|T|(k-d_{\min})+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1}{k}$$
$$\le\frac{(k-1)(k-d_{\min})-1}{k}+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}$$

and

$$k+m\le\delta(G)\le d_{\min}+|S|$$
$$\le d_{\min}+\frac{(k-1)(k-d_{\min})-1}{k}+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}$$
$$=k-1+\frac{d_{\min}}{k}-\frac{1}{k}+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}$$
$$\le k-1+\frac{k-2}{k}-\frac{1}{k}+\frac{2m}{k}$$
$$=k+\frac{2m-3}{k},$$

a contradiction.

Hence, we get $|I_2|\ge2$. Let $v_1,v_2,\cdots,v_{|I_2|}$ be vertices in I_2 as defined in Claim 3, thus $d_{G-S}(v_1)\le k-2$ and $d_{G-S}(v_1)\le d_{G-S}(v_2)\le\cdots\le d_{G-S}(v_{|I_2|})$. We have $|T|=|V(G_2)|$,

$$k|S|\le\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(k-d_{G-S}(v_i))+\left(\sum_{x\in T}d_H(x)-e_H(S,T)\right)-1$$

and

$$|S| \le \frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)}{k} + \frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k} - \frac{1}{k}.$$

In terms of $i(G-U)=|I_2|\ge 2$ where $U=S\cup N_{G-S}(I_2)$ and $|U|\le k|I_2|+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}-\frac{3}{k}$ which implies $|U|\le k|I_2|+\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}\right\rfloor^*$. We yield:

$$I'(G) \le \frac{k|I_2|+\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}\right\rfloor^*}{|I_2|-1}$$
$$= k+\frac{\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}\right\rfloor^*+k}{|I_2|-1}.$$

It is clear to see that to maximize $|U|$, only one vertex in $|I_2|$ has degree $k-2$ in $G-S$, and others have degree $k-1$ in $G-S$. Using the same fashion as Claim 3, $|S|$ can be formulated by:

$$m+2\le |S| \le |I_2|+1+\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)-3}{k}\right\rfloor$$

and hence

$$|I_2| \ge m+1-\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)-3}{k}\right\rfloor \ge m+1-\left\lfloor\frac{2m}{k}\right\rfloor^*. \quad (5)$$

Hence, we have:

$$I'(G) \le k+\frac{\left\lfloor\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}\right\rfloor^*+k}{|I_2|-1} \le k+\frac{\left\lfloor\frac{2m}{k}\right\rfloor^*+k}{m-\left\lfloor\frac{2m}{k}\right\rfloor^*},$$

a contradiction. □

From Claim 4 and Claim 5, we ensure $|I_1|\ge 1$ and $|I_2|\ge 1$, and thus $k\ge 3$.

Denote $v_1, v_2, \cdots, v_{|I_2|}$ by vertices in I_2 as defined in Claim 3. We get $|T|=|V(G_1)|+|V(G_2)|$,

$$k|S| \le \left(k-\frac{1}{2}\right)|I_1|+\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)$$
$$+\left(\sum_{x\in T} d_H(x)-e_H(S,T)\right)-1,$$

$$|S| \le \left(1-\frac{1}{2k}\right)|I_1|+\frac{\sum_{i=1}^{|I_2|}\left(d_{G-S}(v_i)+1\right)\left(k-d_{G-S}(v_i)\right)}{k}$$
$$+\frac{\sum_{x\in T} d_H(x)-e_H(S,T)}{k}-\frac{1}{k},$$

$i(G-U)\geq 2$ where $U=S\cup C_1\cup(N_G(I_1)\cap W)\cup N_{G-S}(I_2)$ and

$$|U|\leq\left(k-\frac{1}{2k}\right)|I_1|+k|I_2|+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}-\frac{3}{k}.$$

Hence, $|U|\leq k|I_1|+k|I_2|+\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}\right\rfloor^*$ since $|U|$ is an integer.

By maximizing $|U|$, we can see that in the extreme value setting, only one vertex in I_2 has degree $k-2$ in $G-S$, and the others have degree $k-1$ in $G-S$. Thus, in terms of the similar discussion in Case 1, $|S|$ can be re-bounded by:

$$|S|\leq|I_1|+|I_2|+\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}+1-\frac{3}{k},$$

which reveals $m+2\leq|S|\leq|I_1|+|I_2|+1+\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-3}{k}\right\rfloor$ and

$$|I_1|+|I_2|\geq m+1-\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)-3}{k}\right\rfloor\geq m+1-\left\lfloor\frac{2m}{k}\right\rfloor^*. \tag{6}$$

Collectively, we infer:

$$I'(G)\leq\frac{|U|}{i(G-U)-1}\leq k+\frac{\left\lfloor\frac{\sum_{x\in T}d_H(x)-e_H(S,T)}{k}\right\rfloor^*+k}{|I_1|+|I_2|-1}\leq k+\frac{\left\lfloor\frac{2m}{k}\right\rfloor^*+k}{m-\left\lfloor\frac{2m}{k}\right\rfloor^*},$$

a contradiction.

In all, the proof of the desired theorem is completed. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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