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Research on Nilpotent Groups Based on Fuzzy Hypergroups

Yi Yan[1,2], Yan Yan[1,2*], Xinru Shang[1,2]

[1]College of Science, North China University of Science and Technology, Tangshan, China
[2]Hebei Key Laboratory of Data Science and Application, Tangshan, China
Email: *yanhblg@163.com

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Abstract

In this paper, by constructing the smallest equivalence relation θ^* on a finite fuzzy hypergroup H, the quotient group (the set of equivalence classes) H/θ^* is a nilpotent group, and the nilpotent group is characterized by the strong fuzzy regularity of the equivalence relation. Finally, the concept of θ-part of fuzzy hypergroup is introduced to determine the necessary and sufficient condition for the equivalence relation θ to be transitive.

Keywords

Fuzzy Hypergroup, Equivalence Relation, Strong Fuzzy Regular Relation, Nilpotent Group

1. Introduction

The theory of hyperstructures originated from the eighth Congress of Scandinavian Mathematicians in 1934 [1]. Marty introduced the concept of hypergroups as a generalization of groups at the conference. He first used hypergroups to solve some problems of groups, algebraic functions, and rational fractions. Fuzzy subset was introduced by Zadeh in 1965 as an extension of the classical concept of set [2]. In 1971, Rosenfeld introduced fuzzy sets in the context of group theory and proposed the concept of fuzzy subgroups of groups [3]. In 2001, Pelea proposed the equivalence relation on multiple algebras and gave the general theory of constructing basic algebras [4].

Equivalence relation is one of the most important and interesting concepts in hyperstructures. Ordinary algebraic structures are derived from hyperstructures. In 2013, Aghabozorgi *et al.* constructed an equivalence relation v^* on a hypergroup such that the quotient group H/v^* is a nilpotent group [5]. In 2014, Ameri *et al.* constructed an equivalence relation ξ^* on a hypergroup such that the quotient group H/ξ^* is an Engel group [6]. In 2015, Mohammadzadeh *et al.* constructed an equivalence relation ζ^* on fuzzy hypergroups such that the quotient group H/ζ^* is a solvable group [7]. In 2016, Jafarpour *et al.* constructed an equivalence relation τ^* on a hypergroup such that the quotient group H/τ^* is a solvable group [8]. In 2016, Nozari studied the basic relation β^* on fuzzy hypersemigroups, and studied the smallest strongly regular equivalence relation γ^* on fuzzy hypersemigroups H, such that H/γ^* is a commutative semigroup [9].

Equivalence relations play a crucial role in fuzzy hyperstructures, as they allow us to capture the fuzziness and uncertainty inherent in fuzzy concepts. By defining equivalence relations, we can partition the elements of a fuzzy set into different equivalence classes, thereby revealing the similarities and differences between them. This paper focuses on studying the equivalence relation θ^* of a finite fuzzy hypergroup, with the goal of ensuring that the quotient group H/θ^* is nilpotent, and exploring the necessary and sufficient conditions for the equivalence relation to be transitive.

2. Preparation

For a nonempty set H, a fuzzy subset μ of H is a function from H to the real number interval $[0,1]$. We use $\mathcal{P}^*(H)$ to denote the set of all nonzero fuzzy subsets of H [6]. At the same time, for two fuzzy subsets μ_1 and μ_2 of H, if μ_1 is smaller than μ_2, it is denoted by $\mu_1 \le \mu_2$, and for $\forall x \in H$, there is $\mu_1(x) \le \mu_2(x)$. Thus, we define:

$$(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\} \tag{1}$$

$$(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} \tag{2}$$

A fuzzy hyperoperation on H is a mapping $\circ: H \times H \mapsto \mathcal{P}^*(H)$ denote by:

$$(a,b) \mapsto a \circ b = ab \tag{3}$$

the structure $(H,\circ)$ is called a fuzzy hypergroup.

Definition 2.1 [10]. A fuzzy hypergroup H is called a fuzzy hypersemigroup if the following conditions are satisfied. For $\forall a,b,c,\in H$, $(a\circ b)\circ c = a\circ(b\circ c)$, where for any fuzzy subset μ of H, for $\forall r \in H$, we have:

$$(a\circ\mu)(r) = \begin{cases} \bigvee_{t\in H}\left((a\circ t)(r)\wedge\mu(t)\right), & \mu \ne 0 \\ 0, & \mu = 0 \end{cases} \tag{4}$$

$$(\mu\circ a)(r) = \begin{cases} \bigvee_{t\in H}\left(\mu(t)\wedge(t\circ a)(r)\right), & \mu \ne 0 \\ 0, & \mu = 0 \end{cases} \tag{5}$$

Definition 2.2 [10]. Let μ, ν be two fuzzy subsets of a fuzzy hypergroup $(H,\circ)$, then for any $t \in H$, we define $\mu\circ\nu$ as
$(\mu\circ\nu)(t) = \bigvee_{t\in H}\left(\mu(p)\wedge(p\circ q)(t)\wedge\nu(q)\right)$.

Definition 2.3 [7]. If for $\forall x \in H$, $x \circ H = H \circ x = \chi_H$, then a fuzzy hypersemigroup $(H, \circ)$ is called a fuzzy hypergroup, where χ_H is the characteristic function of H.

Theorem 2.4 [7]. Let $(H, \circ)$ be a fuzzy hypersemigroup, then for all $a, b \in H$, $\chi_a \circ \chi_b = a \circ b$.

Definition 2.5 [4]. For any non-empty subsets A and B, we define, for $\forall a \in A$, $b \in B$,

$$A \overline{\overline{\rho}} B \Leftrightarrow a \rho b. \tag{6}$$

where ρ is an equivalent relation on a fuzzy hypersemigroup $(H, \circ)$, and $\rho \subseteq H \times H$.

Definition 2.6 [11]. An equivalence relation ρ of a fuzzy hypersemigroup $(H, \circ)$ is called strongly fuzzy regular on the left (on the right), if
$x \rho y \Rightarrow a \circ x \overline{\overline{\rho}} a \circ y$ ($x \rho y \Rightarrow x \circ a \overline{\overline{\rho}} y \circ a$). If ρ is strongly fuzzy regular on the left and on the right, then ρ is called strongly fuzzy regular.

If ρ is an equivalence relation on a fuzzy hypersemigroup $(H, \circ)$, then we consider the following hyperoperation on a quotient structure H/ρ for all $a\rho$, $b\rho \in H/\rho$, we have,

$$a\rho \circ b\rho = \{c\rho : (a' \circ b')(c) > 0, a\rho a', b\rho b'\}. \tag{7}$$

Theorem 2.7 [7]. Let $(H, \circ)$ be a fuzzy hypergroup and ρ be an equivalence relation on H. Then

(i) If $(H/\rho, \circ)$ is a semigroup, then the relation ρ is fuzzy regular on $(H, \circ)$.

(ii) If $(H/\rho, \circ)$ is a group, then the relation ρ is strongly fuzzy regular on $(H, \circ)$.

3. Strong Regular Relation θ_n^* on Fuzzy Hypergroups

In this section, we will construct and analyse the equivalence relation θ_n^* defined on a fuzzy hypergroup and prove that its strong fuzzy regularity makes the quotient group H/θ_n^* is a nilpotent group.

Definition 3.1 [7]. Let $(H, \circ)$ be a fuzzy hypergroup. We define, for every $k \geq 0$,

(i) $L_0(H) = H$;

(ii) $L_{k+1}(H) = \{t \in H | (xy)(r) > 0, (tyx)(r) > 0, \text{in which } x, y \in L_k(H), r \in H\}$.

Let $\forall n \in \mathbb{N}$, and $\theta_n = \bigcup_{m>1} \theta_{m,n}$, where $\theta_{1,n}$ is the diagonal relation and every $m \geq 1$, $\theta_{m,n}$ is the relation defined as follows: $x \theta_{m,n} y \Leftrightarrow \exists (z_1, \cdots, z_m) \in H^m$, $\exists \sigma \in \mathbb{S}_m$: if $z_i \notin L_n(H)$, $\sigma(i) = i$, we have:

$$(z_1 \circ \cdots \circ z_m)(x) = \left(\prod_{i=1}^{m} z_i\right)(x) > 0 \tag{8}$$

$$\left(z_{\sigma(1)} \circ \cdots \circ z_{\sigma(m)}\right)(y) = \left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(y) > 0 \tag{9}$$

where $\mathbb{S}_m$ is a symmetric group of order *m*.

Obviously, the relation θ_n is reflexive and symmetric. We call θ_n^* to be transitive closure of θ_n.

Theorem 3.2. Since θ_n^* is a transitive closure of θ_n, for every $n \in \mathbb{N}$, there is

$$\beta^* \subseteq \theta_n^* \subseteq \gamma^* \tag{10}$$

Theorem 3.3. For $\forall n \in \mathbb{N}$, the relation θ_n^* is a strongly fuzzy regular relation.

Proof. Suppose $\forall n \in \mathbb{N}$, θ_n^* is an equivalence relation. If we want to prove that θ_n^* is strongly fuzzy regular, we need to prove that it is strongly fuzzy regular on the left and on the right. So, for $\forall x, y, z \in H$:

$$x\theta_n y \Rightarrow x \circ z \overline{\overline{\theta}}_n^* y \circ z, \quad z \circ x \overline{\overline{\theta}}_n^* z \circ y. \tag{11}$$

If $x\theta_n y$, then $\exists m \in \mathbb{N}(xz)(r) > 0$, such that $x\theta_{m,n}y$, and there $\exists(z_1,\cdots,z_m) \in H^m$, and $\exists \sigma \in \mathbb{S}_m$: if $z_i \notin L_n(H)$, $\sigma(i) = i$, then $\left(\prod_{i=1}^m z_i\right)(x) > 0$, $\left(\prod_{i=1}^m z_{\sigma(i)}\right)(y) > 0$, where $\mathbb{S}_m$ is the symmetric group of order m.

Let $z \in H$, for $\forall r, s$, we have $(xz)(r) > 0$ and $(yz)(s) > 0$, so

$$\left(\left(\prod_{i=1}^m z_i\right)z\right)(r) = \bigvee_p \left\{\left(\prod_{i=1}^m z_i\right)(p) \wedge (pz)(r)\right\} \tag{12}$$

let $p = x$, then $\left(\left(\prod_{i=1}^m z_i\right)z\right)(r) > 0$, if $z_i \notin L_n(H)$, $\sigma(i) = i$, so

$$\left(\left(\prod_{i=1}^m z_{\sigma(i)}\right)(z)\right)(s) = \bigvee_q \left\{\left(\prod_{i=1}^m z_{\sigma(i)}\right)(q) \wedge (qz)(s)\right\} \tag{13}$$

let $q = y$, if $z_i \notin L_n(H)$, $\sigma(i) = i$, then we have $\left(\left(\prod_{i=1}^m z_{\sigma(i)}\right)(z)\right)(s) > 0$. Suppose that $z_{m+1} = z$ and we define $\exists \sigma' \in \mathbb{S}_{m+1} : \sigma'(i) = \begin{cases} \sigma(i), & \forall i \in \{1,2,\cdots,m\} \\ m+1, & i = m+1 \end{cases}$.

Thus for $\forall r, s \in H$, $\left(\prod_{i=1}^m z_i\right)(r) > 0$, $\left(\prod_{i=1}^m z_{\sigma'(i)}\right)(s) > 0$, if $z_i \notin L_n(H)$, $\sigma'(i) = i$.

If $x\theta_n^* y$, then for $\exists k \in \mathbb{N}$ and exists $x = u_0, u_1, \cdots, u_k = y \in H^{k+1}$, such that:

$$x = u_0\theta_n u_1\theta_n \cdots \theta_n u_k = y \tag{14}$$

by the above result, we have $x \circ z = u_0 \circ z\overline{\overline{\theta}}_n^* u_1 \circ z\overline{\overline{\theta}}_n^* \cdots \overline{\overline{\theta}}_n^* u_k \circ z = y \circ z$, and $x \circ z\overline{\overline{\theta}}_n^* y \circ z$. Similarly, we can show that $z \circ x\overline{\overline{\theta}}_n^* z \circ y$.

Therefore, θ_n^* is a strong fuzzy regular relation on fuzzy hypergroup H. □

Theorem 3.4. For $\forall n \in \mathbb{N}$, we have $\theta_{n+1}^* \subseteq \theta_n^*$.

Proof. Let $x\theta_{n+1}y$, then $\exists(z_1,\cdots,z_m) \in H^m$, $\exists \sigma \in \mathbb{S}_m$, if $z_i \notin L_n(H)$, $\sigma(i) = i$, such that $\left(\prod_{i=1}^m z_i\right)(x) > 0$, $\left(\prod_{i=1}^m z_{\sigma(i)}\right)(y) > 0$, let $\sigma_1 = \sigma$, then $L_{n+1}(H) \subseteq L_n(H)$, so $\theta_{n+1}^* \subseteq \theta_n^*$. □

Corollary 3.5. If $(H,\circ)$ is a commutative fuzzy hypergroup, then $\beta^* = \theta_n^* = \gamma^*$.

Definition 3.6 [12]. For any group G, we said:

$$G = Q_1 \geq Q_2 \geq \cdots \geq Q_{s+1} = 1 \tag{15}$$

is a central group series of G. If $[Q_i, G] \leq Q_{i+1}$, $i = 1,\cdots,s$, then said s the length of the central group column. A group with a central sequence is called a nilpotent group. Any term of the central cluster $Q_i \trianglelefteq G$, and $Q_i/Q_{i+1} \le Z(G/Q_{i+1})$.

If for every n, $Z_n(G)/Z_{n-1}(G)$ is the center of $G/Z_{n-1}(G)$, call a group column $1 = Z_0(G) \le Z_1(G) \le \cdots \le Z_n(G) \le \cdots$ is the upper central series of G. Then $Z_n(G)$ is called the n th center of G.

Theorem 3.7 [5]**.** A group G is nilpotent of class n if and only if $\ell_n(G) = \{e\}$, where $\ell_n(G) = \langle L_n(G) \rangle$.

Theorem 3.8 [7]**.** If $(H,\circ)$ is a fuzzy hypergroup, and ε is a strongly fuzzy regular relation on H, then:

$$L_{k+1}(H/\varepsilon) = \langle \overline{t} \,|\, t \in L_k(H) \rangle \tag{16}$$

Theorem 3.9. H/θ_n^* is a nilpotent group of the class at most $n+1$.

Proof. Since θ_n^* is strongly fuzzy regular relation, and H/θ_n^* is a group. According to theorem 3.7 and 3.8, then we have $L_{k+1}(H/\theta_n^*) = \langle \overline{t} \,|\, t \in L_k(H) \rangle$. Using $\ell_n(G) = \langle L_n(G) \rangle$, so we can get $L_{n+1}(H/\theta_n^*) = \{e\}$.

That is, the following only needs to prove $L_{n+1-i}(H/\theta_n^*) \le Z_i$, for $\forall 0 \le i \le n$. For the case of $n = 0$, there is no need to consider.

So let $a \in H/\theta_n^*$ and $b \in L_{n-i}(H/\theta_n^*)$. Hence $aba^{-1}b^{-1} \in L_{n+1-i}(H/\theta_n^*)$, and so $aba^{-1}b^{-1} \in Z_i$.

Thus, according to $L_{n+1-i}(H/\theta_n^*) \le Z_i$ and $b \in L_{n-i}(H/\theta_n^*)$, then we have $b \in Z_{i+1}$. So, we have $a \in L_0(H/\theta_n^*) = Z_{n+1}$, then $H/\theta_n^* = Z_{n+1}$.

In summary, H/θ_n^* is a nilpotent group of class at most $n+1$.

Theorem 3.10. If H/θ_n^* is a nilpotent group of class n, then $H/\theta_n^* \cong H$.

Example 3.11. Let $H = \{e,a,b,c,d\}$ is a fuzzy hypergroup, and $\circ$ is the fuzzy hyperoperation (Table 1).

Table 1. The result after fuzzy hyperoperation.

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	a	$a \circ b$	$a \circ c$	$a \circ d$
b	b	$a \circ b$	b	$b \circ c$	$b \circ d$
c	c	$a \circ c$	$b \circ c$	c	$c \circ d$
d	d	$a \circ d$	$b \circ d$	$c \circ d$	d

Let θ_n^* is a strongly fuzzy regular relation on H, H/θ_n^* is the quotient group. Then exist $Q_i \trianglelefteq H/\theta_n^*$, such that $H/\theta_n^* = Q_1 \ge Q_2 \ge \cdots \ge Q_{s+1} = 1$. Hence H/θ_n^* is a nilpotent group.

4. Nilpotent Groups Based on Finite Fuzzy Hypergroups

In this section, we construct and analyse the smallest equivalence relation θ^* defined on a finite fuzzy hypergroup H, and prove its strong fuzzy regularity such that H/θ^* is a nilpotent group.

Definition 4.1 [7]**.** Let $(H,\circ)$ is a finite hypergroup. Then we define the relation θ^* on the H as follow:

$$\theta^* = \bigcap_{n\geq 1} \theta_n^* \tag{17}$$

Definition 4.2. The relation θ^* is a strongly fuzzy regular relation on a finite hypergroup $(H,\circ)$ such that H/θ^* is a nilpotent group.

Proof. Because $\theta^* = \bigcap_{n\geq 1} \theta_n^*$, then θ_n^* is a strongly fuzzy regular relation. Then θ^* is a strongly fuzzy regular relation on $(H,\circ)$. Then exist $k \in \mathbb{N}$, such that $\theta_{k+1}^* = \theta_k^*$, so $\theta^* = \theta_k^*$. According to the arbitrariness of k, then H/θ_k^* is a nilpotent group. Thus, H/θ^* is a nilpotent group on a finite hypergroup. □

Definition 4.3. The relation θ^* is the smallest strongly fuzzy regular relation on a finite hypergroup $(H,\circ)$ such that H/θ^* is a nilpotent group.

Proof. Let ρ is a strongly fuzzy regular relation on $(H,\circ)$ such that H/ρ become a nilpotent group of class n. Suppose that $x\theta y$. Then $x\theta_n y$, for $\forall n \in \mathbb{N}$ and $\exists m \in \mathbb{N}$ such that $x\theta_{m,n} y \Leftrightarrow \exists (z_1,\cdots,z_m) \in H^m$, $\exists \sigma \in \mathbb{S}_m$: if $z_i \notin L_n(H)$, $\sigma(i) = i$. We have $\left(\prod_{i=1}^{m} z_i\right)(x) > 0$, $\left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(y) > 0$. From this, we can be obtained

$$L_{k+1}(H/\rho) = \langle \rho(t); t \in L_k(H) \rangle = \{e\} \tag{18}$$

then for $\forall z_i \in L_k(H)$, we have $\rho(z_i)\rho(\theta) = \rho(\theta)\rho(z_i)$. Therefore $\rho(x) = \rho(y)$ which implies that $x\rho y$. □

5. Transitivity Condition of Strongly Regular Relation θ^*

In this section we introduce the concept of θ-part of a fuzzy hypergroup and we determine necessary and sufficient conditions for the relation θ to be transitive.

Definition 5.1 [9]. Let X is nonempty subset of $(H,\circ)$. Then, we say that X is a θ-part of H if for $\forall k \in \mathbb{N}$, $(z_1,\cdots,z_m) \in H^m$ and $\forall \sigma \in \mathbb{S}_m$, if $z_i \notin \bigcup_{n\geq 1} L_n(H)$, $\sigma(i) = i$, then for $\forall x \in H$ and $\forall y \in H/X$, we have:

$$\left(\prod_{i=1}^{m} z_i\right)(x) > 0,\ \left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(y) = 0. \tag{19}$$

Theorem 5.2 [7]. Let X is a nonempty subset of a hypergroup $(H,\circ)$. Then, the following conditions are equivalent:

(i) X is a θ-part of H;

(ii) $x \in X$, $x\theta y \Rightarrow y \in X$;

(iii) $x \in X$, $x\theta^* y \Rightarrow y \in X$.

Theorem 5.3. The following conditions are equivalent:

(i) For $\forall x \in H$, $\theta(x)$ is a θ-part of H;

(ii) θ is transitive.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that $x\theta^* y$, then $\exists (z_1,\cdots,z_m) \in H^m$, such that $x = z_0\theta z_1\theta\cdots\theta z_m = y$. For $0 \leq i \leq m$, $\theta(z_i)$ is a θ-part of H. We have $z_i \in \theta(z_{i-1})$, for $i \in [0,m]$. Thus $y \in \theta(x)$ which means $x\theta y$.

((ii) $\Rightarrow$ (i)) Suppose that $x \in H$, $z \in \theta(x)$ and $z\theta y$. By transitivity of θ, we have $y \in \theta(x)$. According to $x \in H$, $y \in \theta(x)$, we have $y \in X$, so $\theta(x)$ is a θ-part of H. □

Definition 5.4 [5]**.** The intersection of all θ-part which contain A is called θ-closure of A in H and it will denoted by $K(A)$.

Follow, we will determine the set $D(A)$, where A is a non-empty subset of H. We set:

(i) $D_1(A) = A$;

(ii) $D_{n+1}(A) = \left\{ x \in H \middle| \exists (z_1, \cdots, z_m) \in H^m, \left(\prod_{i=1}^m z_i\right)(x) > 0,\ \exists \sigma \in \mathbb{S}_m, \text{ if } z_i \notin L_n(H), \text{ such that } \sigma(i) = i, \text{ and } \exists a \in D_n(A), \text{ such that } \left(\prod_{i=1}^m z_i\right)(a) > 0 \right\}$.

We denote $D(A) = \bigcup_{n \ge 1} D_n(A)$.

Theorem 5.5. For any nonempty subset of H which has the following statements:

(i) $D(A) = K(A)$;

(ii) $K(A) = \bigcup_{a \in A} K(a)$.

Proof. (i) Since $K(A)$ is the intersection sets of all θ-parts containing A. We suppose $\left(\prod_{i=1}^m z_i\right)(a) > 0$ and $\exists \sigma \in \mathbb{S}_m$, if $z_i \notin \bigcup_{n \ge 1} L_n(H)$, such that $\sigma(i) = i$. Then for $\exists n \in \mathbb{N}$, $a \in D_n(A)$, we have $\left(\prod_{i=1}^m z_i\right)(a) > 0$. So $\exists t \in H$, such that $\left(\prod_{i=1}^m z_{\sigma(i)}\right)(t) > 0$. From this can be obtained $t \in D_{n+1}(A)$. Because of $D(A) = \bigcup_{n \ge 1} D_n(A)$, so $t \in D(A)$. According to the arbitrariness of n, $D(A)$ is a θ-part. According to the lemma, if $A \subseteq B$, then B is a θ-part, such that $D(A) \subseteq B$. We suppose $D_n(A) \subseteq B$. If $z \in D_{n+1}(A)$, then there $\exists k \in \mathbb{N}$, $(z_1, \cdots, z_m) \in H^m$, such that $\left(\prod_{i=1}^m z_i\right)(z) > 0$ and there $\exists \sigma \in H_m$, if $z_i \notin \bigcup_{t \ge 1} L_t(H)$, such that $\sigma(i) = i$ and there exists $t \in D_n(A)$, such that $\left(\prod_{i=1}^m z_{\sigma(i)}\right)(t) > 0$. According to $D_n(A) \subseteq B$, we have $t \in B$. So B is a θ-part then $z \in B$, therefore $D(A) = K(A)$.

(ii) Know by definition, $\forall a \in A$, $K(a) \subseteq K(A)$. Bu part (i), we have $K(A) = \bigcup_{n \ge 1} D_n(A)$ and $D_1(A) = A = \bigcup_{a \in A} \{a\}$. There for $\forall n \in \mathbb{N}$, $D_n(A) = \bigcup_{a \in A} D_n(a)$. If $z \in D_{n+1}(A)$, then there $\exists k \in \mathbb{N}$, $(z_1, \cdots, z_m) \in H^m$, $\left(\prod_{i=1}^m z_i\right)(z) > 0$ and $\exists \sigma \in H_m$, if $z_i \notin \bigcup_{t \ge 1} L_t(H)$, such that $\sigma(i) = i$ and there $\exists a \in D_n(A)$, such that $\left(\prod_{i=1}^m z_{\sigma(i)}\right)(a) > 0$. According to the arbitrariness of n, there exists $a \in D_n(A) = \bigcup_{b \in A} D_n(b)$. Then for $\forall a' \in W_n(b)$, $b \in A$, we have $\left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(a') > 0$. Therefore $\exists z \in D_{n+1}(b)$, and so $D_{n+1}(A) \subseteq \bigcup_{b \in A} D_{n+1}(b)$. Hence $K(A) = \bigcup_{a \in A} K(a)$. □

Theorem 5.6. The following relation is an equivalence relation on H,

$$xDy \Leftrightarrow x \in D(y)$$

for every $(x, y) \in H^2$, where $D(y) = D(\{y\})$.

Proof. It is easy to see that D is reflexive and transitive. We prove that D is symmetric. To this, we check that:

(i) For all $n \geq 2$, and $x \in H$, we have $D_n(D_2(x)) = D_{n+1}(x)$;

(ii) $x \in D_n(y)$ if and only if $y \in D_n(x)$.

We suppose $a \in D_2(D_2(x))$, such that $\left(\prod_{i=1}^{m} z_i\right)(a) > 0$, then for $\exists \sigma \in \mathbb{S}_m$, if $z_i \notin \bigcup_{m \geq 1} L_m(H)$, $\sigma(i) = i$. And $\exists y \in D_2(x)$, such that $\left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(y) > 0$. Let $(D_2(x)) = D_{n+1}(x)$, then

$D_{n+1}(D_2(x)) = \left\{ a \middle| \exists m \in \mathbb{N}, (z_1, \cdots, z_m) \in H^m, \left(\prod_{i=1}^{m} z_i\right)(a) > 0,\ \exists \sigma \in \mathbb{S}_m, \text{if } z_i \notin \bigcup_{m \geq 1} L_m(H),\ \sigma(i) = i, \text{ then } \exists t \in D_n(D_2(x)),\ \left(\prod_{i=1} z_{\sigma(i)}\right)(t) > 0 \right\}$. Therefore $x \in D_2(y)$ if and only if $y \in D_2(x)$. Suppose $x \in D_n(y)$ if and only if $y \in D_n(x)$. Let $x \in D_{n+1}(y)$, then there exists $m \in \mathbb{N}$, $(z_1, \cdots, z_m) \in H^m$, $\left(\prod_{i=1}^{m} z_{\sigma(i)}\right)(t) > 0$, $\exists \sigma \in \mathbb{S}_m$, if $z_i \notin \bigcup_{m \geq 1} L_m(H)$, such that $\sigma(i) = i$, and $t \in D_2(x)$. Since $t \in D_n(y)$, then by hypotheses of induction $y \in D_n(t)$ and we see that $t \in D_2(x)$, therefore $y \in D_n(D_2(x)) = D_{n+1}(x)$. □

Definition 5.7 [7]**.** Let H is a fuzzy hypergroups, then H/θ is a group. And $\varphi : H \to H/\theta$ is a canonical projection. We denote by 1 the identity of the group H/θ. The set $\varphi^{-1}(1)$ is called the θ-heart of H and it is denoted by ω_θ.

Theorem 5.8 [6]**.** If H is a fuzzy hypergroup and G is a nonempty subset of H, then:

(i) $\varphi^{-1}(\varphi(G)) = \{x \in H : (\omega_\theta G)(x) > 0\} = \{x \in H : (x)(\omega_\theta G) > 0\}$;

(ii) If G is a θ-part of H, then $\varphi^{-1}(\varphi(G)) = G$.

Definition 5.9 [7]**.** Let $(H, \circ)$ is a fuzzy hypergroup, $H_1 \subseteq H$ is called a fuzzy subhypergroup of H if:

(i) $(a \circ b) \circ c = a \circ (b \circ c)$, for $\forall a, b, c \in H$;

(ii) $a \circ H_1 = \chi_{H_1}$, for $\forall a \in H_1$.

Theorem 5.10. ω_θ is the smallest fuzzy subhypergroups of H, which is also a θ-part of H.

Proof. First, we check that ω_θ is a subhypergroup of H. Because $\omega_\theta \subseteq H$

and so $(a\circ b)\circ c=a\circ(b\circ c)$, $\forall a,b,c\in\omega_\theta$. Let $x,y\in\omega_\theta$, then there $\exists z\in H$, such that $(zy)(x)>0$. There $\overline{zy}=\overline{x}$, which implies that $\overline{z}=1$. Thus $z\in\omega_\theta$, consequently $\omega_\theta y=\chi_{\omega_\theta}$ for $\forall y\in H$. Hence ω_θ is a fuzzy hypergroup of H.

Now wo prove ω_θ is a θ-part of H. Let $z\in\varphi^{-1}(\varphi(\{x\}))$ if and only if $\varphi(z)=\varphi(x)$, which means $z\theta x$. Therefore $\theta^*(z)=\theta^*(x)$ which means $z\theta^* x$. From the above, if $z\in\theta^*(z)$, then $z\in\theta^*(z)=\omega(\{x\})=H_1(x)$.

Therefore, if $x\in\omega_\theta$, then $H_1(x)=\omega_\theta$. It shows that ω_θ is a θ-part of H. And ω_θ is the smallest fuzzy subhypergroups of H. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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