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An Initial-Boundary Value Problem for a Modified Transitional Korteweg-de Vries Equation

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Abstract

We study the following modified transitional Korteweg-de Vries equation $u_t + f(t)u^p u_x + u_{xxx} = 0$, $(x,t) \in \mathbf{R}^+ \times \mathbf{R}^+$, ($p \geq 2$ is an even integer) with initial value $u(x,0) = g(x) \in H^4(\mathbf{R}^+)$ and inhomogeneous boundary value $u(0,t) = Q(t) \in C^2([0,\infty))$. Under the conditions either (i) $f(t) \leq 0$, $f'(t) \geq 0$ or (ii) $f(t) \leq -\alpha$ where $\alpha > 0$, we prove the existence of a unique global classical solution.

Keywords

Modified Transitional KdV Equation, Initial-Boundary Value Problem, Semi-Group, Local and Global Existence

1. Introduction

The Korteweg-de Vries equation (KdV) equation below

$$u_t + uu_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{1.1}$$

is a famous equation in mathematical physics. It was derived as a model for undirectional propagation of small-amplitude long waves in a number of physical systems, such as the evolution of shallow water waves, ion acoustic waves, long waves in shear flows [1]-[5]. The KdV equation is a soliton equation with Hamiltonian structures and an infinitely number of independent motion constants in involution [6]-[8]. The existence of a unique global solution and well-posedness for the KdV equation with smooth initial data can be found in [9]-[14].

The Korteweg-de Vries equation (KdV) equation has a cousin, namely the modified KdV equation (mKdV)

$$u_t - 6\sigma u^2 u_x + u_{xxx} = 0 \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{1.2}$$

also has been a subject of prolific study [15]-[23]. This equation has infinitely many conserved quantities. The famous Miura transformation establishes the connection between KdV and mKdV equations: namely, a solution of the mKdV equation φ yields a solution of the KdV equation $\varphi^2 + \delta\varphi_x$. For the Cauchy problem of the mKdV Equation (1.2), there exists a unique global classical solution. For the following generalized mKdV equation

$$u_t - 6\sigma u^p u_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{1.3}$$

where p is a positive integer, it is found to be integrable only in two cases: $p = 1$ (KdV) and $p = 2$ (mKdV) [24].

We notice that most of these studies are focused on pure initial value problems. However, in many cases of physical interest, the mathematical model leads precisely to a mixed initial-boundary value problem. For example, the KdV equation can describe long waves. In order to assess the performance of the KdV equation as a model for waves in a particular system, sometimes it is not quite convenient to consider the pure initial value problem since there may be difficulty associated with determing the entire wave profile accurately at a given time. One way to obtain undirectional waves to test the appurtenance of KdV, is to generate waves at one end of a homogeneous stretch of the medium in question and to allow them to propagate into the initial undisturbed medium beyond the wavemaker (see [25]-[27] for details). This leads to the following inhomogeneous initial-boundary value problem in which global existence and well-posedness were established [28] [29]:

$$\begin{gathered} u_t - 6\sigma u u_x + u_{xxx} = 0, \quad 0 \le x, t < \infty. \\ u(x,0) = g(x), \quad u(0,t) = Q(t). \end{gathered} \tag{1.4}$$

Meanwhile, for the famous n-dimensional nonlinear Schrödinger equation (NLS) ($g > 0, p > 1$)

$$\begin{gathered} i\partial_t u = \Delta u - g|p|^{p-1} u, \quad x \in \Omega \subset \mathbf{R}^n, \\ u(x,0) = \varphi(x), \quad u(x,t) = Q(x,t) \quad \text{for } x \in \partial\Omega \end{gathered} \tag{1.5}$$

under initial and inhomogeneous boundary conditions $\varphi(x) \in H^1(\Omega)$, $Q \in C^3(\partial\Omega \times (-\infty,\infty))$ (with compact support), there exists a global solution $u \in L^\infty_{\text{loc}}((-\infty,\infty); H^1(\Omega) \cap L^{p+1}(\Omega))$ for $t \in \mathbf{R}$ [30]. The PDE is understood in the sense of distribution while the boundary condition is understood as $u(\cdot,t) - Q(\cdot,t) \in H^1_0(\Omega)$ for a.e. t. Furthermore, if $1 < p < 1 + \frac{4}{n-2}$, this solution is unique.

Another example is the following initial and inhomogeneous boundary value problem for an n-dimensional Ginzburg-Landau equation which describes nonlinear amplitude evolution of wave propagation:

$$u_t = (a + i\alpha)\Delta u - (b + i\beta)\|u\|^2 u, \quad x \in \Omega \subset \mathbf{R}^n \tag{1.6}$$

$$u(x,0)=h(x)\ \text{ for } x\in\Omega,\quad u(x,t)=Q(x,t)\ \text{ on } \partial\Omega$$

where $a,b>0$ and h,Q are smooth functions. If $h\in H^1\cap L^4(\Omega)$, $a>0$, $Q\in C^3(\partial\Omega\times(-\infty,\infty))$, $Q(\cdot,0)=h$ (compatibility condition), then (1.6) has an unique global solution in $u\in C([0,T^*),H^1\cap L^4)\cap L^2([0,T^*),H^2\cap L^6)$, for some $T^*>0$, $u(\cdot,t)-Q(\cdot,t)\in H_0^1$, for a.e. $t\in[0,T^*)$ [31].

For the following initial and inhomogeneous boundary value problem of a modified KdV:

$$u_t-6\sigma u^p u_x+u_{xxx}=0,\quad 0\le x,t<\infty \tag{1.7}$$

$$u(x,t)=g(x),\quad u(0,t)=Q(t)$$

where $p\ge 2$ is an even integer, $\sigma>0$. The existence of a unique global classical solution $u\in C^0([0,\infty),H_0^3(\mathbf{R}^+))\cap C^1((0,\infty),L^2(\mathbf{R}^+))$ is proved in [32] provided that $g(x)\in H^4(\mathbf{R}^+)$, $Q(t)\in C^2([0,\infty))$.

As we know, transitional KdV equation arises in the study of long solitary waves in lakes and estuaries. It propagates on the thermocline separating two layers of fluids of almost equal densities (see [33] for example). Global well-posedness for the Cauchy problem of the following transitional KdV equation was obtained in [34]:

$$\partial_t u+\partial_x^3 u+f(t)u\partial_x u=0,\quad u(x,0)=\varphi(x) \tag{1.8}$$

where $x,t\in\mathbf{R}$, $f\in C(\mathbf{R})$, $f'\in L^1_{\text{loc}}(\mathbf{R})$.

This research is the continuation of an earlier paper [35] about the following modified transitional KdV equation posed in the quarter plane ($f\in C^1([0,\infty))$):

$$u_t+f(t)u^2u_x+u_{xxx}=0,\quad (x,t)\in\mathbf{R}^+\times\mathbf{R}^+, \tag{1.9}$$

$$u(x,0)=g(x)\in H^4(\mathbf{R}^+),\quad u(0,t)=Q(t)\in C^2([0,\infty)).$$

We proved a unique global classical solution $u\in C^0([0,\infty),H^3(\mathbf{R}^+))\cap C^1([0,\infty),L^2(\mathbf{R}))$ for (1.9) under the conditions either (i) $f(t)\le 0$, $f'(t)\ge 0$ or (ii) $f(t)\le -\alpha$ where $\alpha>0$.

In this paper, we will study a more generalized version of modified transitional KdV equation under initial and inhomogeneous boundary conditions

$$u_t+f(t)u^p u_x+u_{xxx}=0,\quad (x,t)\in\mathbf{R}^+\times\mathbf{R}^+, \tag{1.10}$$

$$u(x,0)=g(x)\in H^4(\mathbf{R}^+),\quad u(0,t)=Q(t)\in C^2([0,\infty))$$

where $p\ge 2$ is an even integer. We prove the global existence and uniqueness theorem under similar conditions.

2. Local and Global Existence-Uniqueness Theorems

We first define $u=v+Q(t)\mathrm{e}^{-x}$ and subsitute this in (1.10) to get

$$\begin{aligned} &v_t+f(t)\left(v+Q(t)\mathrm{e}^{-x}\right)^p v_x+v_{xxx}\\ &=\left(Q'(t)-Q(t)\right)\mathrm{e}^{-x}+f(t)\left(v+Q(t)\mathrm{e}^{-x}\right)^p Q(t)\mathrm{e}^{-x}\end{aligned} \tag{2.1}$$

where $v(x,0)=h(x)=g(x)-Q(0)\mathrm{e}^{-x}$, $v(0,t)=0$. Let $H_0^2(\mathbf{R}^+)$ be a subspace of $H^2(\mathbf{R}^+)$ with standard Sobolev norm, then (2.1) is converted to a quasi-linear equation of evolution

$$\frac{\mathrm{d}v}{\mathrm{d}t}+A(t,v)v=B(t,v), \tag{2.2}$$

$$v(x,0)=h(x)=g(x)-Q(0)\mathrm{e}^{-x},\quad v(0,t)=0,$$

where

$$A(t,v)v=f(t)\left(v+Q(t)\mathrm{e}^{-x}\right)^p v_x+v_{xxx}, \tag{2.3}$$

$$B(t,v)=\left(Q'(t)-Q(t)\right)\mathrm{e}^{-x}+f(t)\left(v+Q(t)\mathrm{e}^{-x}\right)^p Q(t)\mathrm{e}^{-x}.$$

Let $S=\left(1+D^2\right)^{s/2}$, $s\ge 3$, $Y=H_0^3(\mathbf{R}^+)$, $X=L_0^2(\mathbf{R}^+)$, then Y is continuously and densely embedded in X with usual norms. Since $A(t,v)=A(v)=D^3+b(t,v)D$ where $b(t,v)=f(t)\left(v+Q(t)\mathrm{e}^{-x}\right)^p$, the leading term D^3 in $A(v)$ is the generator of a contraction semi-group in X, skew-adjoint with $H_0^3(\mathbf{R}^+)$. The perturbing term $b(t,v)D$ is quasi-accretive and relatively bounded with respect to D^3. We consider the solution for (2.2) on any time interval $[0,T]$. Since $\partial_t b(t,v)\in C^1$ as $Q(t)\in C^2$, $A(v)$ is a first-order differential operator with a smooth coefficient $b(v)$. We have the following estimate

$$\begin{aligned}\left\|\left(A(v)-A(z)\right)w\right\|_X&=\left\|\left(b(t,v)-b(t,z)\right)w_x\right\|_X\\&\le\left\|b(t,v)-b(t,z)\right\|_X\left\|w_x\right\|_\infty\\&\le\alpha(T)\left\|v-z\right\|_X\left\|w\right\|_Y\end{aligned} \tag{2.4}$$

provided that $u(x,0)\in H^4(\mathbf{R}^+)$, $u(0,t)\in C^2([0,\infty))$ and $f\in C^1([0,\infty))$. Since $Q\in C^2$ and f are locally bounded functions, we see that $t\to B(t,v)$ is X-Lipschitz continuous for each $t\in[0,T]$. Similar to the results on abstract quasi-linear equation of evolution in [36] [37], we have the following existence theorem.

Theorem 2.1. (*Local Existence and Uniqueness*) *For the modified transitional Korteweg-de Vries Equation* (2.2) *posed in the quarter plane, there exists a unique classical solution* $v\in C^0\left([0,T_M),H_0^3(\mathbf{R}^+)\right)\cap C^1\left([0,T_M),L^2(\mathbf{R}^+)\right)$ *for some* $T_M>0$ *if* $v(x,0)\in H^4(\mathbf{R}^+)$. *Thus there is a unique local classical solution* $u\in C^0\left([0,T_M),H^3(\mathbf{R})\right)\cap C^1\left([0,T_M),L^2(\mathbf{R}^+)\right)$ *for* (1.10) *with inhomogeneous boundary data provided that* $u(x,0)\in H^4(\mathbf{R}^+)$, $f\in C^1(\mathbf{R}^+)$ *and* $Q(t)\in C^2(\mathbf{R}^+)$.

To prove global existence, we need to estimate the H^1 norm for any solution $u(x,t)$ and show it's bounded on any finite interval $[0,T_M]$. Write $Q(t)=u(0,t)$, $P(t)=u_x(0,t)$, $R(t)=u_{xx}(0,t)$. Differentiate both $\|u\|_2^2$ and $\|u_x\|_2^2$ with respect to t variable and substitute them in (1.10) to get

$$\begin{aligned}\partial_t\int_0^\infty u^2\mathrm{d}x&=\int_0^\infty 2uu_t\mathrm{d}x=\int_0^\infty 2u\left(-u_{xxx}-f(t)u^p u_x\right)\mathrm{d}x\\&=-2uu_{xx}\Big|_0^\infty+\int_0^\infty 2u_x u_{xx}\mathrm{d}x-\int_0^\infty 2f(t)u^{p+1}u_x\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
&= 2Q(t)R(t) + (u_x)^2\Big|_0^\infty - \frac{2f(t)}{p+2}u^{p+2}\Big|_0^\infty \\
&= 2Q(t)R(t) - P^2(t) + \frac{2f(t)}{p+2}Q^{p+2}.
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\partial_t \int_0^\infty u_x^2 \mathrm{d}x = \int_0^\infty 2u_x u_{xt}\mathrm{d}x &= 2u_x u_t\Big|_0^\infty - \int_0^\infty 2u_{xx}u_t\mathrm{d}x \\
&= -2P(t)Q'(t) - \int_0^\infty 2u_{xx}\left(-u_{xxx} - f(t)u^p u_x\right)\mathrm{d}x \\
&= -2P(t)Q'(t) + \int_0^\infty 2u_{xx}u_{xxx}\mathrm{d}x + \int_0^\infty 2f(t)u^p u_x u_{xx}\mathrm{d}x \\
&= -2P(t)Q'(t) + (u_{xx})^2\Big|_0^\infty + \int_0^\infty 2f(t)u^p u_x u_{xx}\mathrm{d}x \\
&= -2P(t)Q'(t) - R^2(t) + 2f(t)\int_0^\infty u^p u_x u_{xx}\mathrm{d}x.
\end{aligned} \tag{2.6}$$

Now we differentiate $-\frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x$ with respect to t variable to get

$$\begin{aligned}
&\partial_t\left(-\frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x\right) \\
&= -\frac{2f'(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x - \frac{2f(t)}{(p+1)(p+2)}\int_0^\infty (p+2)u^{p+1}u_t\mathrm{d}x \\
&= -\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2} - \frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+1}\left(-u_{xxx} - f(t)u^p u_x\right)\mathrm{d}x \\
&= -\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2} + \frac{2f(t)}{p+1}\int_0^\infty f(t)u^{p+1}u_{xxx}\mathrm{d}x + \frac{2f^2(t)}{p+1}\int_0^\infty u^{2p+1}u_x\mathrm{d}x \\
&= -\frac{2}{(p+1)(p+2)}f'(t)\|u\|_{p+2}^{p+2} + \frac{2f(t)}{p+1}u^{p+1}u_{xx}\Big|_0^\infty \\
&\quad -2f(t)\int_0^\infty u^p u_x u_{xx}\mathrm{d}x + \frac{f^2(t)}{(p+1)^2}u^{2p+2}\Big|_0^\infty \\
&= -\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2} - \frac{2f(t)}{p+1}Q^{p+1}(t)R(t) \\
&\quad -2f(t)\int_0^\infty u^p u_x u_{xx}\mathrm{d}x - \frac{f^2(t)}{(p+1)^2}Q^{2p+2}(t)
\end{aligned} \tag{2.7}$$

By adding (2.5)-(2.7) we obtain:

$$\begin{aligned}
&\partial_t\left(\|u\|_2^2 + \|u_x\|_2^2 - \frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x\right) \\
&= 2Q(t)R(t) - P^2(t) + \frac{2f(t)}{p+2}Q^{p+2} - 2P(t)Q'(t) - R^2(t) \\
&\quad + 2f(t)\int_0^\infty u^p u_x u_{xx}\mathrm{d}x - \frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2} \\
&\quad - \frac{2f(t)}{p+1}Q^{p+1}(t)R(t) - 2f(t)\int_0^\infty u^p u_x u_{xx}\mathrm{d}x - \frac{f^2(t)}{(p+1)^2}Q^{2p+2}(t)
\end{aligned}$$

$$\begin{aligned}&=2Q(t)R(t)-P^2(t)+\frac{2f(t)}{p+2}Q^{p+2}-2P(t)Q'(t)-R^2(t)\\&-\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}-\frac{2f(t)}{p+1}Q^{p+1}(t)R(t)-\frac{f^2(t)}{(p+1)^2}Q^{2p+2}(t).\end{aligned}\tag{2.8}$$

To prove global existence, we need to show that $\|u\|_{H^1}$ is bounded on any $[0,T_M)$. First we consider the case (i) $f(t)\le 0,\ f'(t)\ge 0$. From (2.8) we see that for any $t\in[0,T_M)$

$$\begin{aligned}&\partial_t\left(\|u\|_2^2+\|u_x\|_2^2-\frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x\right)\\&=2Q(t)R(t)-P^2(t)+\frac{2f(t)}{p+2}Q^{p+2}-2P(t)Q'(t)-R^2(t)\\&-\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}-\frac{2f(t)}{p+1}Q^{p+1}(t)R(t)-\frac{f^2(t)}{(p+1)^2}Q^{2p+2}(t)\\&\le\frac{2f(t)}{p+2}Q^{p+2}-2P(t)Q'(t)-P^2(t)+\left(2Q(t)-\frac{2f(t)}{p+1}Q^{p+1}(t)\right)R(t)-R^2(t)\\&\le c_0+c_1P(t)-P^2(t)+c_2R(t)-R^2(t)\le m\end{aligned}\tag{2.9}$$

for some positive number m which depends on c_0, c_1 and c_2 which in turn depend on $g(x), f(t), Q(t), Q'(t)$ and T_M. By integrating (2.9) in t variable and noting that $f(t)\le 0$ we obtain

$$\|u\|_{H^1}^2\le\|u\|_2^2+\|u_x\|_2^2-\frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x\le\int_0^t m\mathrm{d}t\le mT_M\tag{2.10}$$

which implies that $\|u\|_{H^1}$ is bounded on any $[0,T_M)$.

Next we turn to case (ii) $f(t)\le-\alpha$ where $\alpha>0$ with no restriction on $f'(t)$. Again, from (2.8) we get

$$\begin{aligned}&\partial_t\left(\|u\|_2^2+\|u_x\|_2^2-\frac{2f(t)}{(p+1)(p+2)}\int_0^\infty u^{p+2}\mathrm{d}x\right)\\&=2Q(t)R(t)-P^2(t)+\frac{2f(t)}{p+2}Q^{p+2}-2P(t)Q'(t)-R^2(t)\\&-\frac{2f'(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}-\frac{2f(t)}{p+1}Q^{p+1}(t)R(t)-\frac{f^2(t)}{(p+1)^2}Q^{2p+2}(t)\\&\le\frac{2f(t)}{p+2}Q^{p+2}-2P(t)Q'(t)-P^2(t)-R^2(t)\\&+\left(2Q(t)-\frac{2f(t)}{p+1}Q^{p+1}(t)\right)R(t)+\frac{2|f'(t)|}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}\\&\le c_0+c_1P(t)-P^2(t)+c_2R(t)-R^2(t)+c_3\|u\|_{p+2}^{p+2}\le m+c_3\|u\|_{p+2}^{p+2}\end{aligned}\tag{2.11}$$

for some positive m depending on c_0, c_1, c_2 and c_3 which in turn depend on $g(x), f(t), f'(t), Q(t), Q'(t)$ and T_M. By integrating (2.11) in t variable and noting that $f(t)\le-\alpha$ we obtain

$$\begin{aligned}
&\|u\|_2^2+\|u_x\|_2^2+\frac{2\alpha}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}\\
&\le \|u\|_2^2+\|u_x\|_2^2-\frac{2f(t)}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}\\
&\le \int_0^t\left(m+c_3\|u\|_{p+2}^{p+2}\right)\mathrm{d}t\\
&\le mT_M+m_0\int_0^t\left(\|u\|_2^2+\|u_x\|_2^2+\frac{2\alpha}{(p+1)(p+2)}\|u\|_{p+2}^{p+2}\right)\mathrm{d}t
\end{aligned}\tag{2.12}$$

for some positive number m_0 which depends on c_3 and α. By Gronwall's lemma, $\|u\|_2^2+\|u_x\|_2^2+\frac{\alpha}{6}\|u\|_{p+2}^{p+2}$ is bounded on any $[0,T_M)$, so is $\|u\|_{H^1}$. From Gagliardo-Nirenburg estimate [38], we see that $\|u\|_\infty^2\le\lambda\|u\|_2\|u_x\|_2$ for some $\lambda>0$, therefore $\|u\|_\infty$ is bounded on any $[0,T_M)$.

Now consider the Cauchy problem for the linear equation

$$\mathrm{d}u/\mathrm{d}t+A(t)u=B(t),\quad 0\le t\le T,\quad u(0)=g(x)\tag{2.13}$$

in a Banach space X and if one assumes that $-A(t)$ generates an analytical semigroup then the solution of (1.10) can be written as

$$u(t)=U(t,0)g+\int_0^t U(t,s)B(s)\mathrm{d}s\tag{2.14}$$

where $U(t,s)=\mathrm{e}^{-(t-s)A}$ is defined as the family of operators such that $u(t)=U(t,s)g$ is the solution of the homogeneous differential equation $\mathrm{d}u/\mathrm{d}t+A(t)u=0$ with the initial value $u(s)=g$. For the nonlinear case in a Banach space X:

$$\mathrm{d}u/\mathrm{d}t+A(t,u)=B(t,u),\quad 0\le t\le T,\quad u(0)=g\tag{2.15}$$

we consider the linear equation $\mathrm{d}u/\mathrm{d}t+A(t,v(t))u=B(t,v(t))$, $u(0)=g$ for certain functions $t\to v(t)\in X$. If this equation has a solution $u=u(t)$ then define a mapping $v\to u=G(v)$ and seek a fixed point of G which will be a solution of (2.15). We note that (2.15) is similar to (2.2) as we take boundary data $Q(t)$ into consideration and switch v and u variables. We now can adopt arguments in [39], thinking $X=L_0^2$, and write the following as the solution to (2.2)

$$v(t)=U(t,0)g+\int_0^t U(t,s)f(t,v)\mathrm{d}s\tag{2.16}$$

where U is continuous and bounded operator. Recall from (2.10) and (2.12) that $u=v+Q(t)\mathrm{e}^{-x}$ is bounded under H^1 and L^∞ norms, thus v is also bounded under H^1 and L^∞ norms on any given interval of time $[0,T]$. Take $Y=H_0^3$ norm on both side of (2.16) one obtains the following inequality

$$\|v\|_Y\le c_0+\left\|\int_0^t U(t,s)B(t,v)\mathrm{d}s\right\|_Y\le c_0+c_1\int_0^t\|v\|_Y\,\mathrm{d}s\tag{2.17}$$

Apply the Grownwall lemma on (2.17) one conclude that v is bounded under Y norm on any given interval of time $[0,T]$. Therefore, u is a global classical solution to the inhomogeneous initial-boundary value problem for the modified transitional KdV Equation (1.10). Therefore we have proved the following global existence theorem.

Theorem 2.2. (*Global Existence*) *For the initial-boundary value problem of modified transitional KdV* (1.10), $g(x)\in H^4(\mathbf{R}^+)$, $Q(t)\in C^2([0,\infty))$, *there exists a unique global classical solution* $u\in C^0([0,\infty),H^3(\mathbf{R}^+))\cap C^1([0,\infty),L^2(\mathbf{R}^+))$ *under the conditions either* (*i*) $f(t)\le 0$, $f'(t)\ge 0$ *or* (*ii*) $f(t)\le -\alpha$ *where* $\alpha>0$.

3. Conclusion

The inhomogeneous boundary value problems for the modified transitional KdV are relatively new, and this paper is our initial attempt to study the existence and uniqueness of the global solution. A lot more could be done. For example, the well-posedness of the problem (*i.e.* continuous dependency on initial and boundary data), decay estimates, and numerical simulations. We plan to continue our study on mTKdV in the near future.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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