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Zeros of the Eta and Zeta Functions

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Abstract

The eta function is examined over the critical strip $0<\sigma<1$ and there is an investigation of the statement that all zeros of the zeta function must lie on the critical line $Re(s)=1/2$. A further investigation is made into the claim that there are no other zeros of the zeta or eta functions within the critical strip.

Keywords

Riemann Hypothesis, Zeta Function Zeros, Dirichlet Eta Function Zeros

1. Introduction

The Swiss mathematician Leonhard Euler (1707-1783) introduced the zeta function

$$\zeta(\sigma)=\sum_{n=1}^{\infty}\frac{1}{n^\sigma}=\frac{1}{1^\sigma}+\frac{1}{2^\sigma}+\frac{1}{3^\sigma}+\cdots,\ \sigma \text{ real and } \sigma>1 \tag{1}$$

and showed that the zeta function can also be expressed in terms of prime numbers having the form

$$\zeta(\sigma)=\prod_p\left(1-\frac{1}{p^\sigma}\right)^{-1},\ \sigma>1 \tag{2}$$

where the product runs through all the primes $p=2,3,5,7,\cdots$. The equation (2) is known as the Euler product formula for the zeta function. The derivation of Equation (2) can be found in reference [1].

Bernhard Riemann (1826-1866) a famous German mathematician studied the zeta function having the complex form

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots,\ s=\sigma+it,\ Re(s)>1,\ -\infty<t<\infty \tag{3}$$

Using the definite integral

$$\int_0^\infty \alpha^{n-1}\mathrm{e}^{-\alpha x}\mathrm{d}\alpha=\frac{\Gamma(n)}{x^n} \tag{4}$$

where $\Gamma(n)$ is the gamma function defined

$$\Gamma(s)=\int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x,\ \ Re(s)>0 \tag{5}$$

one can represent the Riemann zeta function in the integral form

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}}{e^t-1}\mathrm{d}t,\ \ Re(s)>1 \tag{6}$$

as demonstrated in the reference [2] (25.5.1).

The Riemann zeta function arises in many areas of physics and engineering. In mathematics it is studied in number theory, probability and statistics. The zeros of the zeta function are closely related to the spacing and occurrence of prime numbers.

In Section 2, there is a review of known properties of the Riemann zeta function. In Section 3, properties of the Dirichlet eta function are examined. In Section 4, it is shown that all zeros of the eta function contain the zeros of the zeta function. A review of known locations where the eta function is zero is given in Section 4. This is followed by representing eta as a function of the complex variable $s=\sigma+it$ where $i^2=-1$. The claim that all nontrivial zeros of the zeta function must lie along the line $\sigma=1/2$ is examined in detail.

2. The Zeta Function at Integer Values

It can be demonstrated that at the even integers $2n$, for $n=1,2,3,\cdots$

$$\zeta(2n)=\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n} \tag{7}$$

where B_{2n} are the Bernoulli numbers, references [1]-[3]. The zeta function evaluated at the odd integers $2n+1$, for $n=1,2,3,\cdots$ can be represented

$$\zeta(2n+1)=\frac{(-4)^n\pi^{2n+1}E_{2n}-2\psi^{(2n)}(3/4)}{2^{2n+1}(2^{2n+1}-1)(2n)!} \tag{8}$$

where E_{2n} are the Euler numbers and $\psi^{(2n)}(3/4)$ are polygamma functions, reference [4].

Riemann also provided a functional equation for the zeta function which can be expressed in either of the forms

$$\zeta(1-s)=2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s),\ \ Re(s)>0 \tag{9}$$

or

$$\zeta(s)=2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),\ \ Re(s)<1 \tag{10}$$

where $\Gamma(s)$ is the gamma function from equation (5), references [1] [2].

The Riemann reflection formula for the zeta function is given by

$$\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)=\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)=\xi(1-s) \tag{11}$$

Make note that the gamma function in the above relations is never zero.

The functional equations (9) and (10) can be used to extend the definition of the zeta function to the region $Re(s)<1$, references [1] [5]. Observe that in the special case $s=1$ the series becomes the harmonic series which slowly diverges. Also in the complex plane where $s=\sigma+it$ it can be shown that

$$\zeta(s)=\frac{1}{1-s}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(s-1)^n \tag{12}$$

where γ_n are the Stieljes constants. This equation shows that the zeta function has a simple pole at $s=1$ with residue 1. Reference [2] (25.2.4).

3. The Dirichlet Eta Function

The Dirichlet eta function is defined

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots,\ s=\sigma+it,\ Re(s)=\sigma>0,\ -\infty<t<\infty \tag{13}$$

The eta function is related to the zeta function by

$$\eta(s)=\left(1-2^{1-s}\right)\zeta(s),\ s=\sigma+it,\ Re(s)=\sigma>0 \tag{14}$$

references [6] [7]. The eta function is a converging alternating series and is sometimes referred to as the alternating zeta function. Note that the zeros of the eta function will include all the zeros the Riemann zeta function.

Using the integral (4) the eta function can be expressed in the integral form

$$\eta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{\mathrm{e}^x+1}\mathrm{d}x,\ Re(s)>0 \tag{15}$$

see reference [3].

The equation (14) shows the Riemann zeta function can be represented in terms of the Dirichlet eta function by

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s},\ Re(s)>0,\ s\neq 1 \tag{16}$$

references [6] [7]. An Euler transformation, references [6], can be applied to the Dirichlet eta function on the right-hand side of equation (16) to represent the Riemann zeta function by a series which is globally convergent. This representation of the Riemann zeta function has the form

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{17}$$

which converges for all $s\neq 1$.

Some special values of the Dirichlet eta function are

$$\eta(0)=\frac{1}{2},\ \eta(1)=\log(2),\ \eta(2n)=(-1)^{n+1}\frac{B_{2n}\pi^{2n}\left(2^{2n-1}-1\right)}{(2n)!},\ n=1,2,3,\cdots$$

reference [3], where B_{2n} are the Bernoulli numbers.

4. Zeros of the Zeta Function

The Euler product formula (2) with σ replaced by $s=\sigma+it$ implies that in the region $Re(s)>1$ the zeta function is never zero. In the region $Re(s)<0$ the functional equation (10) can be employed to show that at the negative even integers $-2k$ for $k=1,2,3,\cdots$ the zeta function is zero since

$$\zeta(-2k)=2(2\pi)^{-2k-1}\sin\left(\frac{\pi(-2k)}{2}\right)\Gamma(2k+1)\zeta(2k+1)=0 \tag{18}$$

This is because the sine function in this equation has the value zero. In general, for nonpositive integers one can show $\zeta(-k)=(-1)^k\frac{B_{k+1}}{k+1}$ where B_{k+1} are Bernoulli numbers. Reference [3]. Any zeros of the zeta function determined by the equation (18) are called the trivial zeros of the zeta function. All other zeros are called nontrivial zeros.

There are no nontrival zeros of the zeta function for $Re(s)<0$. There are no zeros on the line $Re(s)=1$ and by the reflection formula there are no zeros on the line $Re(s)=0$. If other zeros exist they must lie in the region $0<Re(s)<1$ and $-\infty<t<\infty$. This region is called the critical region and any zeros in this region are called nontrivial zeros. The Riemann hypothesis is that all nontrivial zeros of the zeta function must lie on the line $Re(s)=\frac{1}{2}$. This line is called the critical line.

The equation (14) can be used to show that whenever $\zeta(s)=0$, then $\eta(s)=0$. The factor $\left(1-2^{1-s}\right)$ in equation (14) is zero at the points $s=1+\frac{i2\pi n}{\log(2)}$, for all nonzero integer values for n and represents additional zeros for the eta function.

5. The Zeta Function in the Complex Domain

For $s=\sigma+it$ one finds $\eta(s)=\eta(\sigma+it)=u(\sigma,t)+iv(\sigma,t)$ and for s in the critical region where $0<\sigma<1$ one can demonstrate that the real and imaginary parts of the equation (13) can be expressed

$$\begin{aligned} u(\sigma,t)&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cos(\theta_n(t)) \\ v(\sigma,t)&=-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\sin(\theta_n(t)) \end{aligned} \tag{19}$$

where $\theta_n(t)=t\log(n)$, for $n=1,2,3,\cdots$. These series are convergent for $\sigma>0$ and t in the critical strip.

Observe that $\frac{\partial u}{\partial\sigma}=\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial\sigma}=-\frac{\partial u}{\partial t}$ so the Cauchy-Riemann equations are satisfied. This shows $\eta(s)$ is a holomorphic function which satisfies $\eta(\bar{s})=\overline{\eta(s)}$. This implies that if $\eta(s)=0$ for some value of s, then its conjugate $\bar{s}$ satisfies $\eta(\bar{s})=0$. Hence, the zeros of the zeta function are symmetric about the σ -axis. The equation $\eta(s)=0$ is satisfied if both the real part $u(\sigma,t)$

and imaginary part $v(\sigma,t)$ of the eta function are zero simultaneously.

6. Graphical Display of Zeta Function Roots

Instead of examining graphs of each the functions $u(\sigma,t)$ and $v(\sigma,t)$ to find points where they are both zero simultaneously, we undertake an examination of a function associated with the sum of squares with the real and imaginary parts of the eta function.

Define the function

$$f(\sigma,t)=\alpha-\alpha\big/\left(1+u^2(\sigma,t)+v^2(\sigma,t)\right) \tag{20}$$

where α is a positive scale constant. Observe that the function $f(\sigma,t)$ has the value of 0 when the sum of squares is zero. Whenever the sum of squares is greater then zero, the function $f(\sigma,t)$ satisfies $0<f(\sigma,t)<\alpha$.

Figure 1 illustrates the equation (20) in the special case $\alpha=1$ and $\sigma=1/2$.

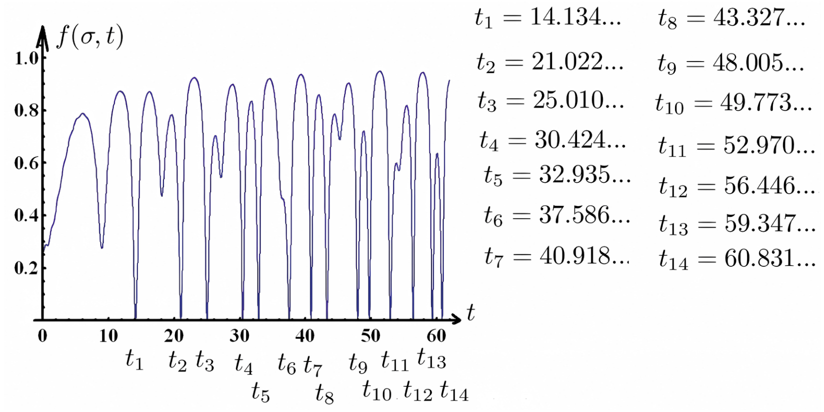


Figure 1. Plot of $f(\sigma,t)$ vs t, for $0\le t<62$, with $\sigma=1/2$ and $\alpha=1$.

Observe in **Figure 1** that for a fixed value of $\sigma=1/2$ the zeros of the zeta function are displayed at values of t where the sum of squares $u^2(\sigma,t)+v^2(\sigma,t)$ equals zero. This occurs whenever $u(\sigma,t)$ and $v(\sigma,t)$ are zero simultaneously.

Some additional examples of graphs of the function $f(\sigma,t)$ with scale $\alpha=1$ and selected values of σ and t are illustrated in **Figures 2-4**.

Examine **Figure 2** and **Figure 4** where $\sigma=0.4$ and $\sigma=0.6$ and make note that $f(\sigma,t)>0$ for the range of t selected. In **Figure 3**, the function $f(0.5,t)$ has many values of t where $f(0.5,t)=0$. These figures suggest the following might be true in the critical region $0<\sigma<1$ and $-\infty<t<\infty$

(i) $u^2+v^2>0$ for all t while $\sigma\ne 1/2$ and

(ii) $u^2+v^2\ge 0$ for all t while $\sigma=1/2$

If the above is true then the zeros of the Riemann zeta function within the critical region must lie on the critical line $\sigma=1/2$.

Another illustration to support the above observation is **Figure 5** which illustrates a close up of the function $f(\sigma,t)$ around $t=21$ for σ values of 0.4, 0.5, 0.6 and scale factor $\alpha = 0.1$. Observe that $f(0.5,t)$ has a zero while $f(0.4,t)$ and $f(0.6,t)$ are not zero indicating the sum of squares are positive for these values of σ and t.

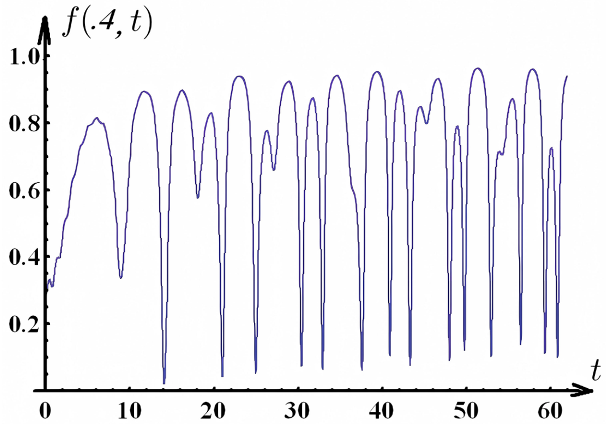


Figure 2. Plot of $f(0.4,t)$ for $0 < t < 60$.

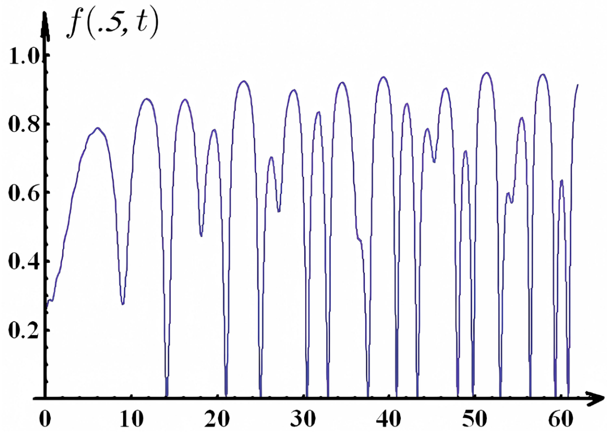


Figure 3. Plot of $f(0.5,t)$ for $0 < t < 60$.

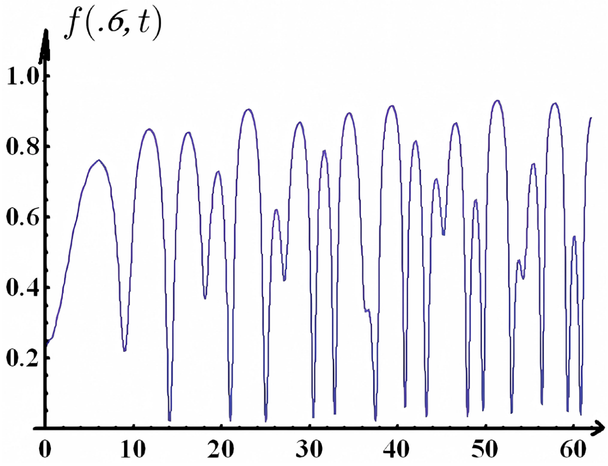


Figure 4. Plot of $f(0.6,t)$ for $0 < t < 60$.

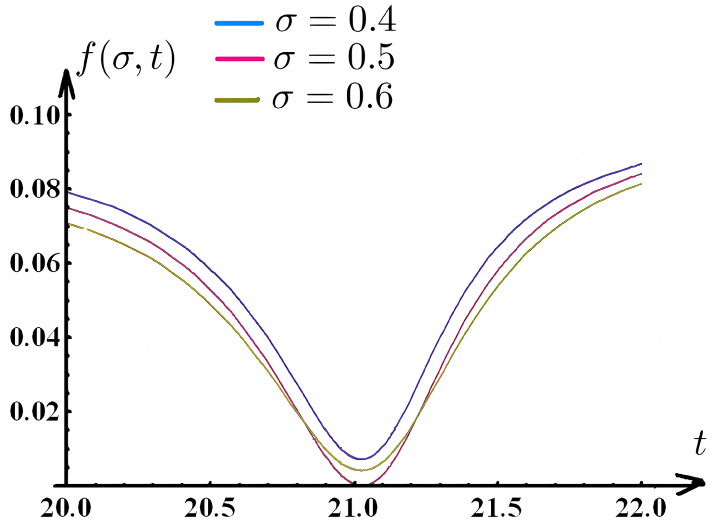


Figure 5. Plot of $f(\sigma,t)$ for $20 < t < 22$.

The remainder of this paper will present evidence to support the above observation by developing the sum of squares into a form amenable for analysis.

7. Introduction to the Series $u(\sigma,t)+v(\sigma,t)$ and $u(\sigma,t)-v(\sigma,t)$

Observe that by addition of the equations (19) one can express $u+v$ in the form

$$\begin{aligned} u(\sigma,t)+v(\sigma,t) &= 1+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\sqrt{2}\left(\frac{\cos(\theta_n(t))}{\sqrt{2}}-\frac{\sin(\theta_n(t))}{\sqrt{2}}\right) \\ &= 1+\sqrt{2}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cos\left(\theta_n(t)+\frac{\pi}{4}\right) \qquad (21) \\ &= 1-\frac{\sqrt{2}}{2^{\sigma}}\cos\left(\theta_2(t)+\frac{\pi}{4}\right)+\sqrt{2}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cos\left(\theta_n+\frac{\pi}{4}\right) \end{aligned}$$

In a similar fashion one can show by subtraction of the equations (19) and after simplification there results

$$u(\sigma,t)-v(\sigma,t)=1-\frac{\sqrt{2}}{2^{\sigma}}\cos\left(\theta_2(t)-\frac{\pi}{4}\right)+\sqrt{2}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}}\cos\left(\theta_n(t)-\frac{\pi}{4}\right) \qquad (22)$$

Using the identity $\cos(x)=1-2\sin^2(x/2)$ the Equations (21) and (22) can be expressed in the form

$$\begin{aligned} u(\sigma,t)+v(\sigma,t) &= A(\sigma)+B_1(\sigma,t) \\ u(\sigma,t)-v(\sigma,t) &= A(\sigma)+B_2(\sigma,t) \end{aligned} \qquad (23)$$

where

$$A(\sigma)=1-\frac{\sqrt{2}}{2^{\sigma}} \qquad (24)$$

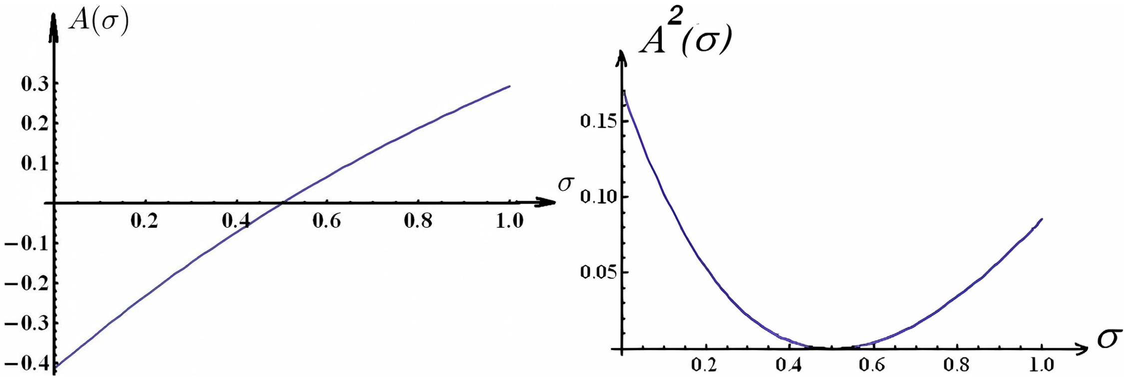
and

$$\begin{aligned} B_1(\sigma,t) &= \frac{2\sqrt{2}}{2^\sigma}\sin^2\left(\frac{\theta_2(t)}{2}+\frac{\pi}{8}\right)+\sqrt{2}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cos\left(\theta_n(t)+\frac{\pi}{4}\right) \\ B_2(\sigma,t) &= \frac{2\sqrt{2}}{2^\sigma}\sin^2\left(\frac{\theta_2(t)}{2}-\frac{\pi}{8}\right)+\sqrt{2}\sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{n^\sigma}\cos\left(\theta_n(t)-\frac{\pi}{4}\right) \end{aligned} \tag{25}$$

The Equations (23) can be added and subtracted to obtain the relations

$$\begin{aligned} u(\sigma,t) &= A(\sigma)+\frac{B_1(\sigma,t)+B_2(\sigma,t)}{2} \\ v(\sigma,t) &= \frac{B_1(\sigma,t)-B_2(\sigma,t)}{2} \end{aligned} \tag{26}$$

The following **Figure 6** & **Figure 7** are sketches of the functions $A(\sigma), A^2(\sigma), B_1(\sigma,t), B_2(\sigma,t)$ for selected values of σ and t.



(a) (b)

Figure 6. (a) $A(\sigma)$ for $0<\sigma<1$; (b) $A^2(\sigma)$ for $0<\sigma<1$.

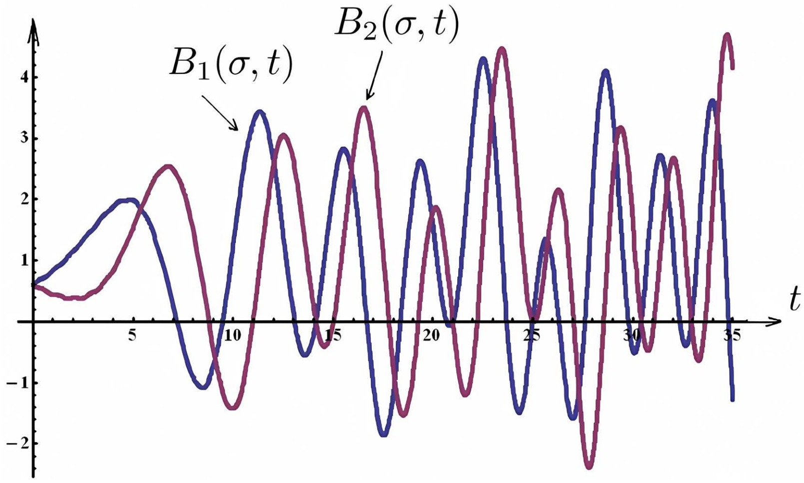


Figure 7. Plot of $B_1(\sigma,t)$ and $B_2(\sigma,t)$ for $\sigma=0.4$ and $0<t<35$.

8. Summary

Examing the zeta function in the complex plane $s=\sigma+it$ we have found

(i) There are no zeros in the region $Re(s)\geq 1$

(ii) There are trivial zeros at $-2k$, $k = 1,2,3,\cdots$

(iii) There are no nontrivial zeros in the region $Re(s) \le 0$

(iv) There are no zeros on the lines $Re(s) = 0$ and $Re(s) = 1$

(v) The critical region $0 < Re(s) < 1$ remains to be investigated. In the next section let us examine the statements that in the critical region

$$\begin{aligned} &(\text{i})\ u^2 + v^2 > 0 \text{ for all } t \text{ while } \sigma \ne 1/2 \text{ and} \\ &(\text{ii})\ u^2 + v^2 \ge 0 \text{ for all } t \text{ while } \sigma = 1/2 \end{aligned} \tag{27}$$

because if they are true then the zeros of the Riemann zeta function within the critical region must lie on the critical line $\sigma = 1/2$.

9. The Sum of Squares Analysis

The representations of $u(\sigma,t)$ and $v(\sigma,t)$ in the Equations (26) allows one to obtain the expression

$$\left(u(\sigma,t) - A(\sigma)\right)^2 + v(\sigma,t)^2 = \frac{B_1^2(\sigma,t) + B_2^2(\sigma,t)}{2} \tag{28}$$

which we will employ to analyze the equations (27). Consider the following cases.

Case 1: $A(\sigma) = 0$ which implies $\sigma = 1/2$ and $u^2(\sigma,t) + v^2(\sigma,t) \ge 0$ for all $t \ge 0$

When $A(\sigma) = 0$ the equation (28) reduces to

$$u^2(1/2,t) + v^2(1/2,t) = \frac{B_1^2(1/2,t) + B_2^2(1/2,t)}{2} \ge 0 \tag{29}$$

Make note that the functions $B_1(1/2,t)$ and $B_2(1/2,t)$ are oscillating functions so that when these functions are squared, there results a spike in the graph of the right-hand side when these function intersect and are zero simultaneously. The equation (29) is the equation which defines the values for t where the right-hand side is zero. If t_i represents a zero of the right-hand side when $\sigma = 1/2$ then a zero of the zeta function has the form $s = \frac{1}{2} + it_i$. G.H. Hardy and J.E. Littlewood, reference [8], showed there are an infinite number of values for t_i which make the right-hand side of equation (29) equal to zero.

In this case the t-values producing the zeros of the zeta function must lie on the critical line $\sigma = 1/2$ where t must satisfy the equation

$$u^2(1/2,t) + v^2(1/2,t) = \frac{B_1^2(1/2,t) + B_2^2(1/2,t)}{2} = 0 \tag{30}$$

For values of t that do not satisfy the Equation (29) we obtain the equation

$$u^2(1/2,t) + v^2(1/2,t) = \frac{B_1^2(1/2,t) + B_2^2(1/2,t)}{2} > 0 \tag{31}$$

since all the terms are squared.

If $u(1/2,t)$ and $v(1/2,t)$ are simultaneously zero then the equations (26) become

$$u(1/2,t)=\frac{B_1(1/2,t)+B_2(1/2,t)}{2}=0$$
$$v(1/2,t)=\frac{B_1(1/2,t)-B_2(1/2,t)}{2}=0$$

which implies that the values of t which satisfy the equations

$$B_1(1/2,t)=0 \text{ and } B_2(1/2,t)=0$$

simultaneously will also provide the roots of the zeta function on the critical line.

Case 2: $A(\sigma)\neq 0$ with σ and t lying in the critical strip, $\sigma\neq 1/2$

Consider next the case where the right-hand side of equation (28) is zero for some value of σ and t, then the equation (28) becomes

$$\left(u(\sigma,t)-A(\sigma)\right)^2+v(\sigma,t)^2=0$$

which implies $u(\sigma,t)=A(\sigma)$ and $v(\sigma,t)=0$ and consequently one finds that

$$u^2(\sigma,t)+v^2(\sigma,t)=A^2(\sigma)>0$$

which represents a sum of squares different from zero for those values of σ and t in the critical strip which create a zero for the right-hand side of equation (28).

If the right-hand side of equation (28) is greater than zero for some value of σ and t in the critical strip, say

$$\left(u(\sigma,t)-A(\sigma)\right)^2+v(\sigma,t)^2=\frac{B_1^2(\sigma,t)+B_2^2(\sigma,t)}{2}=k\neq 0 \tag{32}$$

where k is a positive constant, then it can be demonstrated that

$$u^2(\sigma,t)+v^2(\sigma,t)>0$$

For example, let C_1,C_2,C_3,C_4 denote nonzero constants and consider the following cases

1) If $u-A=C_1$ and $v=0$, then $C_1^2=k$ and $u^2+v^2=(C_1+A)^2$

2) If $u-A=0$ and $v=C_2$, then $C_2^2=k$ and $u^2+v^2=A^2+C_2^2$

3) If $u-A=C_3$ and $v=C_4$, then $C_3^2+C_4^2=k$ and $u^2+v^2=(A+C_3)^2+C_4^2$

so that in all cases the sum of squares is some constant which is greater than zero for all values of σ and t producing a nonzero right-hand side k.

Therefore the sum of squares $u^2(\sigma,t)+v^2(\sigma,t)$ is always greater then zero for all values of σ and t in the critical region when $\sigma\neq 1/2$.

10. Conclusions

Instead of examining the functions $u(\sigma,t)$ and $v(\sigma,t)$ we have elected to examine the sum of squares $u^2(\sigma,t)+v^2(\sigma,t)$ using graphics associated with the function

$$f(\sigma,t)=\alpha-\alpha\big/\left(1+u^2(\sigma,t)+v^2(\sigma,t)\right)$$

where α is a scale factor. We have demonstrated that for $s=\sigma+it$ in the critical strip

(i) $u^2(\sigma,t)+v^2(\sigma,t)>0$ for all t while $\sigma\neq 1/2$ and

$$\text{(ii)}\quad u^2(\sigma,t)+v^2(\sigma,t)\geq 0 \quad \text{for all } t \quad \text{while} \quad \sigma = 1/2$$

so that the zeros of the Riemann zeta function within the critical region must lie on the critical line $\sigma = 1/2$. The solutions t_i obtained by solving the equation

$$u^2(1/2,t)+v^2(1/2,t)=\frac{B_1^2(1/2,t)+B_2^2(1/2,t)}{2}=0 \tag{33}$$

produce the zeta function roots $s=\frac{1}{2}+it_i$. The right-hand side of the above equation (33) represents oscillations and so one can get an understanding of the infinite number of roots predicted by G.H. Hardy.

The above analysis is suggested as an alternative way to study the Riemann hypothesis that all zeros of the zeta function must have the form $s=\frac{1}{2}+it_j$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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