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Global Well-Posedness and Large-Time Behavior to the 3D Two-Fluid Model with Degenerate Viscosities

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Abstract

In the paper, we study a compressible two-fluid model in $\mathbb{R}^3$, where $\gamma^{\pm} > 1$. The pressure of the two fluids is equal. Different from previous research, we consider that viscosity coefficient both μ and λ are functions of density. The global well-posedness of the three-dimensional compressible two-phase flow model is an open problem due to its dissipative, nonlinear structure. In the paper, setting $m^{\pm} = M^{\pm}$ and $Z = P - \overline{P}$, by exploiting the dissipation structure, we obtain energy estimates for (Z, w, n) and its derivatives, then we obtain the time decay rates for (Z, w, n). So we derive global well-posedness and large time behavior to the three dimensional compressible two-fluid model.

Keywords

Two-Fluid Model, Degenerate Viscosities, Large-Time Behavior, Global Well-Posedness

1. Introduction

Two-phase flow refers to a flow system consisting of two substances in different phases (at least one of which is a fluid). The flow system may be in different combinations such as gas-liquid and liquid-solid. Two-phase flow is also widely present in nature and other engineering fields, such as the drift of rain, snow, clouds and fog [1] [2], blood circulation in living organisms [3] [4], sediment movement [5] [6], high-speed aerated water flow in hydraulic engineering, and air pollution caused by smoke and dust in environmental engineering. With the passage of time and the development of science, the problem of two-phase flow has received increasing attention. Two-phase flow plays an important role in aerospace [7] [8], biomedical engineering [9]-[13], chemical engineering [14]-[16] and other fields. However, the analysis of two-phase flow problem is relatively complex. It involves knowledge from multiple fields such as fluid mechanics, thermodynamics, and heat transfer. People use different models to study two-phase flow problems. The most common is the compressible two-fluid model in [17] established using the Navier-Stokes equations:

$$\begin{cases} \partial_t\left(\sigma^+\rho\right)+\operatorname{div}\left(\sigma^+\rho w\right)=0, \\ \partial_t\left(\sigma^+\rho w\right)+\operatorname{div}\left(\sigma^+\rho w\otimes w\right)+\sigma^+\nabla P^+\left(\rho\right)=\operatorname{div}\left(\sigma^+\tau_1\right) \\ \partial_t\left(\sigma^-\eta\right)+\operatorname{div}\left(\sigma^-\eta n\right)=0, \\ \partial_t\left(\sigma^-\eta n\right)+\operatorname{div}\left(\sigma^-\eta n\otimes n\right)+\sigma^-\nabla P^-\left(\eta\right)=\operatorname{div}\left(\sigma^-\tau_2\right) \\ P=P^+\left(\rho\right)=P^-\left(\eta\right)=a^+\left(\rho\right)^{\overline{\gamma}^+}=a^-\left(\eta\right)^{\overline{\gamma}^-} \end{cases} \tag{1.1}$$

in which ρ, η are no less than 0, $\overline{\gamma}^{\pm}\geq 1$, σ^+ is the volume fraction of liquid, while σ^- is that of gas, both σ^+ and σ^- are positive constants. $w(x,t)$, $n(x,t)$ represent the velocities of each phase. Let $a^+=a^-=1$. Moreover, τ_1, τ_2 are the viscous stress tensors:

$$\begin{cases} \tau_1:=\mu_1\left(\nabla w+\nabla^t w\right)+\lambda_1\operatorname{div}w\mathrm{Id}, \\ \tau_2:=\mu_2\left(\nabla n+\nabla^t n\right)+\lambda_2\operatorname{div}n\mathrm{Id}, \end{cases} \tag{1.2}$$

where μ_1, μ_2 and λ_1, λ_2 are shear and bulk viscosity coefficients: μ_1, $\mu_2>0$, $2\mu_1+3\lambda_1\geq 0$, $2\mu_2+3\lambda_2\geq 0$, then $\mu_1+\lambda_1>0$, $\mu_2+\lambda_2>0$.

When μ_1, μ_2, λ_1 and λ_2 are constants, there are many studies on the solutions of (1.1). Bresch, Huang and Li obtained global weak solutions of a non-conservative viscous compressible two-phase model in one space dimension in [18]. For the multi-dimensional case, Yao, Zhang and Zhu obtained the existence and asymptotic behavior of global weak solutions for a two-phase model of two-dimensional viscous liquid gas flow by using the idea of studying single-phase Navier-Stokes equations in [19]. Vasseur, Wen and Yu in [20] got the existence of global weak solutions for the model like (1.1), where the pressure contingent upon two variables. Cui, Wang, Yao and Zhu in [21] studied the time-decay rate of a non conservative compressible model by analysing systems and estimating energy. When $P^+=P^-$, Wu and Yao obtained the properties of the classical solutions to the three dimensional space in [22].

With the intensive study of the single phase Navier-Stokes equations, researchers have obtained some important conclusions, one of which is particularly noteworthy: The dynamic viscosity coefficient and volumetric viscosity coefficient both depend on density, see [23]-[27] and the references therein. For the two-fluid model, we make the following assumptions:

$$\begin{cases} \tau_1:=\rho^{\mu_1}\left(\nabla w+\nabla^t w\right)+\rho^{\lambda_1}\operatorname{div}w\mathrm{Id}, \\ \tau_2:=\eta^{\mu_2}\left(\nabla n+\nabla^t n\right)+\eta^{\lambda_2}\operatorname{div}n\mathrm{Id},. \end{cases} \tag{1.3}$$

where μ_1, λ_1, μ_2, $\lambda_2 \geq 1$. The biggest difference between this article and previous research is the viscosity term, in which both μ and λ are functions of density.

When the velocities of the two fluids are the same, Zodji in [28] proved the existence of a unique local-time solution when density and dynamic viscosity coefficients have an infimum. Li, Wang and Zhang in [29] researched a model analogous to (1.1), where the dynamic viscosity coefficient and volumetric viscosity coefficient are functions of density. There are no finite-energy classical solutions to this system for any small time. The assumption made by Bresch in [30] is that the viscosity coefficients of two-phase motion are $\mu_1\rho$ and $\mu_2\eta$ respectively, and the volume viscosity coefficients are both 0, Bresch obtained the global weak solutions, while $1<\gamma^{\pm}<6$. Evje, Wang and Wen in [31] supposed:

$$P^+(\rho)-P^-(\eta)=\rho^{\bar{\gamma}^+}-\eta^{\bar{\gamma}^-}=f\left(\sigma^-\eta\right), \tag{1.4}$$

where f represents capillary pressure, $f\in C^3\left(\left[0,\infty\right)\right)$. If f satisfies

$$-\frac{r_-^2(1,1)}{\sigma^-(1,1)}<f'(1)<0, \tag{1.5}$$

then f decreases strictly monotonically in the vicinity of the equilibrium point. Where $r_+^2:=\frac{\mathrm{d}P^+}{\mathrm{d}\rho}(\rho)=\bar{\gamma}^+\frac{P^+(\rho)}{\rho}$, $r_-^2:=\frac{\mathrm{d}P^-}{\mathrm{d}\eta}(\eta)=\bar{\gamma}^-\frac{P^-(\eta)}{\eta}$ represent the sound speed respectively. They achieved global existence and decay rates of the solutions in [31].

There are currently few research results on viscous stress tensors (1.3). Therefore, we study the properties of solutions when both μ and λ are functions of density.

The research methods and techniques used in this article are derived from [22], which obtains the global goodness and large time behavior of the classical solution of a three-dimensional compressible two-phase flow model with common pressure through energy estimation and continuity induction.

We observed $(1.1)_5$ and obtained the following relationship

$$\mathrm{d}P=r_+^2\mathrm{d}\rho=r_-^2\mathrm{d}\eta, \tag{1.6}$$

We introduce fractional density like in [30]

$$M^+=\sigma^+\rho,\, M^-=\sigma^-\eta, \tag{1.7}$$

Combined with $\sigma^++\sigma^-=1$, it gives

$$\mathrm{d}\rho=\frac{1}{\sigma^+}\left(\mathrm{d}M^+-\rho\mathrm{d}\sigma^+\right),\quad \mathrm{d}\eta=\frac{1}{\sigma^-}\left(\mathrm{d}M^-+\eta\mathrm{d}\sigma^-\right). \tag{1.8}$$

From (1.6) and (1.7), we get

$$\mathrm{d}\sigma^+=\frac{\sigma^-r_+^2}{\sigma^-\rho r_+^2+\sigma^+\eta r_-^2}\mathrm{d}M^+-\frac{\sigma^+r_-^2}{\sigma^-\rho r_+^2+\sigma^+\eta r_-^2}\mathrm{d}M^-. \tag{1.9}$$

Putting (1.9) into (1.8), we derive

$$\mathrm{d}\rho=\frac{\rho\eta r_-^2}{M^-(\rho)^2r_+^2+M^+(\eta)^2r_-^2}\left(\eta\mathrm{d}M^++\rho\mathrm{d}M^-\right),$$

and

$$\mathrm{d}\eta=\frac{\rho\eta r_+^2}{M^-(\rho)^2 r_+^2+M^+(\eta)^2 r_-^2}\left(\eta\mathrm{d}M^++\rho\mathrm{d}M^-\right),$$

We get $\mathrm{d}P$:

$$\mathrm{d}P=\mathcal{L}\left(\eta\mathrm{d}M^++\rho\mathrm{d}M^-\right)$$

where

$$\mathcal{L}:=\frac{r_-^2 r_+^2}{\sigma^-\rho r_+^2+\sigma^+\eta r_-^2}.$$

From $\sigma^++\sigma^-=1$, we get

$$\frac{M^+}{\rho}+\frac{M^-}{\eta}=1,\ \text{thus}\ M^-=\frac{\eta\left(\rho-M^+\right)}{\rho}=\frac{P^{1/\gamma^-}\left(P^{1/\gamma^+}-M^+\right)}{P^{1/\gamma^+}}. \tag{1.10}$$

With $(1.1)_5$, (1.7) and (1.10), we define $\sigma^\pm$

$$\begin{cases}\sigma^+\left(P,M^+\right)=\dfrac{M^+}{P^{1/\gamma^+}},\\ \sigma^-\left(P,M^+\right)=1-\dfrac{M^+}{P^{1/\gamma^+}}.\end{cases} \tag{1.11}$$

Rewrite (1.1) using $\left(M^\pm,P,u,v\right)$:

$$\begin{cases}\partial_t M^++\operatorname{div}\left(M^+w\right)=0,\\ \partial_t M^-+\operatorname{div}\left(M^-n\right)=0,\\ \partial_t P+\mathcal{L}\eta\operatorname{div}\left(M^+w\right)+\mathcal{L}\rho\operatorname{div}\left(M^-n\right)=0,\\ \partial_t\left(M^+w\right)+\operatorname{div}\left(M^+w\otimes w\right)+\sigma^+\nabla P=\operatorname{div}\left\{\sigma^+\left[\rho^{\mu_1}\left(\nabla w+\nabla^t w\right)+\rho^{\lambda_1}\operatorname{div}w\,\mathrm{Id}\right]\right\},\\ \partial_t\left(M^-n\right)+\operatorname{div}\left(M^-n\otimes n\right)+\sigma^-\nabla P=\operatorname{div}\left\{\sigma^-\left[\eta^{\mu_2}\left(\nabla n+\nabla^t n\right)+\eta^{\lambda_2}\operatorname{div}n\,\mathrm{Id}\right]\right\}.\end{cases} \tag{1.12}$$

This paper considers (1.12) in 3D space with initial data.

$$\left(M^\pm,P,u,v\right)(x,0)=\left(M_0^\pm,P_0,u_0,v_0\right)(x)\to\left(\bar{M}^\pm,\bar{P},\vec{0},\vec{0}\right)\ \text{as}\ |x|\to\infty\in\mathbb{R}^3, \tag{1.13}$$

$\bar{M}^\pm$ and $\bar{P}$ are constants for the background doping profile. Let $\bar{M}^\pm=1$, thus $\bar{P}$ depends on $M^-\left(\bar{P},1\right)-1=0$.

Moreover, setting

$$Z=P-\bar{P},\quad m^\pm=M^\pm-1,$$

We restate (1.12)-(1.13) based on the above equation:

$$\begin{cases}\partial_t m^++\operatorname{div}w=-\operatorname{div}\left(m^+w\right),\\ \partial_t m^-+\operatorname{div}n=-\operatorname{div}\left(m^-n\right),\\ \partial_t Z+\beta_1\operatorname{div}w+\beta_2\operatorname{div}n=G_1,\\ \partial_t w+\beta_3\nabla Z-\kappa_1\Delta w-\kappa_2\nabla\operatorname{div}w=H_1+R_1,\\ \partial_t n+Z_4\nabla Z-\varkappa_1\Delta n-\varkappa_2\nabla\operatorname{div}n=H_2+R_2,\end{cases} \tag{1.14}$$

with initial data

$$\left(m^{\pm},Z,w,n\right)(x,0)=\left(m_0^{\pm},Z_0,u_0,v_0\right)(x)\to\left(0,0,\vec{0},\vec{0}\right),\quad \text{as } |x|\to+\infty. \tag{1.15}$$

where $\beta_1=\mathcal{L}\left(1,\bar{P}\right)\eta\left(\bar{P}\right)$, $\beta_2=\mathcal{L}\left(1,\bar{P}\right)\rho\left(\bar{P}\right)$, $\beta_3=\dfrac{1}{\rho\left(\bar{P}\right)}$, $\beta_4=\dfrac{1}{\eta\left(\bar{P}\right)}$,

$\kappa_1=\dfrac{\rho\left(\bar{P}\right)^{\mu_1}}{\rho\left(\bar{P}\right)}$, $\varkappa_1=\dfrac{\eta\left(\bar{P}\right)^{\mu_2}}{\eta\left(\bar{P}\right)}$, $\kappa_2=\dfrac{\rho\left(\bar{P}\right)^{\mu_1}+\rho\left(\bar{P}\right)^{\lambda_1}}{\rho\left(\bar{P}\right)}$, $\varkappa_2=\dfrac{\eta\left(\bar{P}\right)^{\mu_2}+\eta\left(\bar{P}\right)^{\lambda_2}}{\eta\left(\bar{P}\right)}$

and the nonlinear terms are given by

$$\begin{aligned}G_1=&-h_1\left(m^+,Z\right)\mathrm{div}w-h_2\left(m^+,Z\right)\mathrm{div}n-\mathcal{L}\left(m^++1,Z+\bar{P}\right)\eta u\cdot\nabla m^+\\&-\mathcal{L}\left(m^++1,Z+\bar{P}\right)\rho n\cdot\nabla m^-,\end{aligned} \tag{1.16}$$

$$\begin{aligned}H_1=&-w\cdot\nabla w-\left(\frac{1}{\rho}-\beta_3\right)\nabla Z+\left(\frac{\rho^{\mu_1}}{\rho}-\kappa_1\right)\Delta w\\&+\left(\frac{\rho^{\mu_1}+\rho^{\lambda_1}}{\rho}-\kappa_2\right)\nabla\mathrm{div}w,\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}R_1=&\frac{\rho^{\mu_1}\left(\nabla w+\nabla^t w\right)\nabla\sigma^+}{M^+}+\frac{\rho^{\lambda_1}\mathrm{div}w\nabla\sigma^+}{M^+}\\&+\frac{\nabla\left(\rho^{\mu_1}\right)\left(\nabla w+\nabla^t w\right)}{\rho}+\frac{\nabla\left(\rho^{\lambda_1}\right)\mathrm{div}w}{\rho},\end{aligned} \tag{1.18}$$

$$\begin{aligned}H_2=&-n\cdot\nabla n-\left(\frac{1}{\eta}-\beta_4\right)\nabla Z+\left(\frac{\eta^{\mu_2}}{\eta}-\varkappa_1\right)\Delta n\\&+\left(\frac{\eta^{\mu_2}+\eta^{\lambda_2}}{\eta}-\varkappa_2\right)\nabla\mathrm{div}n,\end{aligned} \tag{1.19}$$

$$\begin{aligned}R_2=&\frac{\eta^{\mu_2}\left(\nabla n+\nabla^t n\right)\nabla\sigma^-}{M^-}+\frac{\eta^{\lambda_2}\mathrm{div}n\nabla\sigma^-}{M^-}\\&+\frac{\nabla\left(\eta^{\mu_2}\right)\left(\nabla n+\nabla^t n\right)}{\eta}+\frac{\nabla\left(\eta^{\lambda_2}\right)\mathrm{div}n}{\eta},\end{aligned} \tag{1.20}$$

where $h_i\left(i=1,2\right)$ are given by

$$\begin{cases}h_1\left(m^+,Z\right)=\mathcal{L}\left(m^++1,Z+\bar{P}\right)\eta M^+-\beta_1,\\h_2\left(m^+,Z\right)=\mathcal{L}\left(m^++1,Z+\bar{P}\right)\rho M^--\beta_2.\end{cases} \tag{1.21}$$

Now, we will show the difficulties in the proof and how to fix them.

First of all, we rewrite (1.12) as (1.14) by taking $m^{\pm}=M^{\pm}-1$, $Z=P-\bar{P}$. We obtain the corresponding linear system (1.26) for (1.14). To study time-decay estimates of the linear system (1.26), we must study the properties of the semigroup in detail. We use the Hodge decomposition technique [32] to decompose the linear system into three equations. We can get linear L^2 estimates using the Fourier transform.

Second, similar to [22], we multiply $(1.14)_3$, $(1.14)_4$, $(1.14)_5$ by $\dfrac{\beta_3}{\beta_1}\nabla^k Z$, $\nabla^k w$, and $\nabla^k n$ $\left(k=0,1,2\right)$, respectively, and then integrate them. The difference between this paper and [22] is that when estimating the term on the right, we need to estimate the norms of ρ^{μ_1}, ρ^{λ_1}, η^{μ_2} and η^{λ_2}. We noticed that ρ and η are related to P, making $Z = P - \bar{P}$. Therefore, norm estimates related to ρ^{μ_1}, ρ^{λ_1}, η^{μ_2} and η^{λ_2} can all be calculated using norm estimates related to Z. We deduce the energy of (Z, w, n) and its derivatives.

In the last, we need handle $\mathcal{L}\eta w\cdot\nabla m^+$ and $\mathcal{L}\rho n\cdot\nabla m^-$ in (1.16). because we have chosen P as our research variable. Dealing with items like $\int_{\mathbb{R}^3}\mathcal{L}\eta w\cdot\nabla\nabla^3 m^+\nabla^3 Z dx$ and $\int_{\mathbb{R}^3}\mathcal{L}\rho n\cdot\nabla\nabla^3 m^-\nabla^3 Z dx$ can be more difficult for us, so we have rewritten it as (5.11). At the same time, applying related inequalities, We deduced the time decay rates for (Z, n, w). We have proven Theorem 1.2.

Main results.

Theorem 1.1. If $M_0^{\pm} - 1 \in H^3(\mathbb{R}^3)$, $P_0 - \bar{P}, w_0, n_0 \in H^3(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$, then there is a constant δ_0 such that if

$$N_0 := \left\|\left(M_0^{\pm} - 1\right)\right\|_{H^3} + \left\|\left(P_0 - \bar{P}, w_0, n_0\right)\right\|_{H^3\cap L^1} \le \delta_0, \tag{1.22}$$

then the Cauchy problem (1.9)-(1.10) admits a unique global solution $(M^{\pm}, P, w, n)$ in time, satisfying

$$M^{\pm} - 1, P - \bar{P} \in C^0\left([0,\infty); H^3(\mathbb{R}^3)\right)\cap C^1\left([0,\infty); H^2(\mathbb{R}^3)\right),$$
$$w, n \in C^0\left([0,\infty); H^3(\mathbb{R}^3)\right)\cap C^1\left([0,\infty); H^1(\mathbb{R}^3)\right).$$

After obtaining the global fitness of the model solutions, we further investigate the properties of the solution and study the time decay rate of the solutions.

Theorem 1.2. If $(M^{\pm}, P, w, n)$ satisfies the conditions in Theorem 1.1, for any $t \ge 0$, constant $C_0 > 0$ exists and is independent of t, we can get estimates of the solution $(M^{\pm}, P, w, n)$:

$$\left\|\left(P - \bar{P}, w, n\right)(t)\right\|_{L^2} \le C_0 N_0 (1+t)^{-\frac{3}{4}}, \tag{1.23}$$

$$\left\|\nabla P(t)\right\|_{H^1} + \left\|\nabla(w, n)(t)\right\|_{H^2} \le C_0 N_0 (1+t)^{-\frac{5}{4}}. \tag{1.24}$$

and

$$\left\|\nabla^3 P(t)\right\|_{L^2} + \left\|\left(M^+ - 1, M^- - 1\right)(t)\right\|_{H^3} \le C_0 N_0. \tag{1.25}$$

2. Preliminaries

In this paper, $\|\cdot\|_{H^k}$ is expressed as the norm in Sobolev space $H^k(\mathbb{R}^3)$. $L^p(\mathbb{R}^3)$ is expressed as the norm in L^p space, $1 \le p \le \infty$. The norm $\|(g,h)\|_X$ is short for $\|g\|_X + \|h\|_X$. $x \lesssim y$ means that $x \le Ky$, where positive constant $K > 0$. We represent $\nabla = \partial_x = (\partial_1, \partial_2, \partial_3)$ like [33], let $\partial_x^\ell g = \nabla^\ell g = \nabla(\nabla^{\ell-1} g)$. Λ^r is a pseudo differential operator, defined as

$$\Lambda^r g = \mathfrak{G}^{-1}\left(|\xi|^r \hat{g}\right), \text{ for } r \in \mathbb{R},$$

where $\hat{g}$ and $\mathfrak{G}(g)$ are the Fourier transform of g. $\|g\|_{\dot{H}^r} \triangleq \|\Lambda^r g\|_{L^2}$ represents the norm of homogenous Sobolev space $\dot{H}^r(\mathbb{R}^3)$. The radial function Ψ satisfies

$$\begin{cases} \Psi(\varsigma)=1, & |\varsigma|\le \dfrac{\zeta}{2} \\ \Psi(\varsigma)=0, & |\varsigma|\ge \zeta \end{cases}$$

see Lemma 3.1 for the definition of ζ.

We define

$$g^l=\mathfrak{G}^{-1}\left[\Psi(\varsigma)\hat{g}\right],\quad g^h=\mathfrak{G}^{-1}\left[\left(1-\Psi(\varsigma)\right)\hat{g}\right].$$

where g^l and g^h respectively represent the high-frequency and low-frequency parts of g. If the Fourier transform of g exists, then $g=g^l+g^h$.

Moreover, for the convenience of the reader, here is a summary of the Gagliardo-Nirenberg inequality.

Lemma 2.1. If $0\le l,k\le s$, then

$$\left\|\nabla^l g\right\|_{L^\alpha}\lesssim\left\|\nabla^k g\right\|_{L^\theta}^{1-b}\left\|\nabla^s g\right\|_{L^r}^{b}$$

where b belongs to $\left[\frac{l}{s},1\right]$ and satisfies

$$\frac{l}{3}-\frac{1}{\alpha}=\left(\frac{k}{3}-\frac{1}{\theta}\right)(1-b)+\left(\frac{s}{3}-\frac{1}{r}\right)b.$$

Particularly, if p, q and r are all equal to 2, we get

$$\left\|\nabla^l g\right\|_{L^2}\lesssim\left\|\nabla^k g\right\|_{L^2}^{\frac{s-l}{s-k}}\left\|\nabla^k g\right\|_{L^2}^{\frac{l-k}{s-k}}.$$

Proof. The proof is shown on page 125 of [34]. □

Lemma 2.2. For $s\ge 1$, we have

$$\left\|\nabla^s(gh)\right\|_{L^r}\lesssim\|g\|_{L^{r_1}}\left\|\nabla^s h\right\|_{L^{r_2}}+\left\|\nabla^s g\right\|_{L^{r_3}}\|h\|_{L^{r_4}},$$

and

$$\left\|\nabla^s(gh)-f\nabla^s h\right\|_{L^r}\lesssim\|\nabla g\|_{L^{r_1}}\left\|\nabla^{s-1}h\right\|_{L^{r_2}}+\left\|\nabla^s g\right\|_{L^{r_3}}\|h\|_{L^{r_4}},$$

where $r,r_1,r_2,r_3,r_4\in[1,\infty]$ and

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}.$$

Proof. The lemma has been proved in [35] and [36]. □

3. Spectral Analysis and Linear L^2 Estimates

3.1. Reformulation

The linearized system for Cauchy problem (1.14)-(1.15) is

$$\begin{cases} \partial_t m^+ + \operatorname{div} w=0, \\ \partial_t m^- + \operatorname{div} n=0, \\ \partial_t Z+\beta_1\operatorname{div} w+\beta_2\operatorname{div} n=0, \\ \partial_t w+\beta_3\nabla Z-\kappa_1\Delta w-\kappa_2\nabla\operatorname{div} w=0, \\ \partial_t n+\beta_4\nabla Z-\varkappa_1\Delta n-\varkappa_2\nabla\operatorname{div} n=0, \\ \left(m^+,Z,w,n\right)(x,0)=\left(m_0^+,Z_0,w_0,n_0\right)(x) \end{cases} \tag{3.1}$$

It is straightforward to verify the presence of a zero eigenvalue in the linearized problem (3.1), it adds to the difficulty and complexity of the issue. In order to eliminate the zero eigenvalues, we study the Cauchy problem of (Z,w,n):

$$\begin{cases}\partial_t Z+\beta_1 \operatorname{div} w+\beta_2 \operatorname{div} n=0,\\ \partial_t w+\beta_3\nabla Z-\kappa_1\Delta w-\kappa_2\nabla\operatorname{div} w=0,\\ \partial_t n+\beta_4\nabla Z-\varkappa_1\Delta n-\varkappa_2\nabla\operatorname{div} n=0,\\ (Z,w,n)(x,0)=(Z_0,w_0,n_0)(x).\end{cases} \tag{3.2}$$

According to semigroup theory, (3.2) can be expressed by $\chi=(Z,w,n)^t$ as

$$\begin{cases}\chi_t=\mathcal{T}\chi,\\ \chi|_{t=0}=\chi_0,\end{cases} \tag{3.3}$$

where the operator $\mathcal{T}$ is given by

$$\mathcal{T}=\begin{pmatrix}0 & -\beta_1\operatorname{div} & -\beta_2\operatorname{div}\\ -\beta_3\nabla & \kappa_1\Delta+\kappa_2\nabla\operatorname{div} & 0\\ -\beta_4\nabla & 0 & \varkappa_1\Delta+\varkappa_2\nabla\operatorname{div}\end{pmatrix}.$$

By Fourier transform to the system (3.1), we have

$$\begin{cases}\hat{\chi}_t=\mathcal{K}(\varsigma)\hat{\chi},\\ \hat{\chi}|_{t=0}=\hat{\chi}_0,\end{cases} \tag{3.4}$$

where $\hat{\chi}(\varsigma,t)=\mathfrak{G}(\chi(x,t))$, $\varsigma=(\varsigma^1,\varsigma^2,\varsigma^3)^t$ and $\mathcal{K}(\xi)$ is defined by

$$\mathcal{K}(\varsigma)=\begin{pmatrix}0 & -i\beta_1\varsigma^t & -i\beta_2\varsigma^t\\ -i\beta_3\varsigma & -\kappa_1|\varsigma|^2\mathrm{I}_{3\times3}-\kappa_2\varsigma\otimes\varsigma & 0\\ -i\beta_4\varsigma & 0 & -\varkappa_1|\varsigma|^2\mathrm{I}_{3\times3}-\varkappa_2\varsigma\otimes\varsigma\end{pmatrix}.$$

We study the properties of hemigroup employing the actual methods in [37]-[39].

Let $\Psi^+=\Lambda^{-1}\operatorname{div} w$ and $\Psi^-=\Lambda^{-1}\operatorname{div} n$, they are "compressible part" of w and n, and denote $\Phi^+=\Lambda^{-1}\operatorname{curl} w$ and $\Phi^-=\Lambda^{-1}\operatorname{curl} n$ ($(\operatorname{curl} z)_i^j=\partial_{x_j}z^i-\partial_{x_i}z^j$) by the "incompressible part" of w and n. Rewrite (3.2) as follows:

$$\begin{cases}\partial_t Z+\beta_1\Lambda\Psi^++\beta_2\Lambda\Psi^-=0,\\ \partial_t\Psi^+-\beta_3\Lambda Z+\kappa\Lambda^2\Psi^+=0,\\ \partial_t\Psi^--\beta_4\Lambda Z+\varkappa\Lambda^2\Psi^-=0,\\ (Z,\Psi^+,\Psi^-)\big|_{t=0}=(\theta_0,\Lambda^{-1}\operatorname{div} w_0,\Lambda^{-1}\operatorname{div} n_0)(x),\end{cases} \tag{3.5}$$

and

$$\begin{cases}\partial_t\Phi^++\kappa_1\Lambda^2\Phi^+=0,\\ \partial_t\Phi^-+\varkappa_1\Lambda^2\Phi^-=0,\\ \Phi^+\big|_{t=0}=\Lambda^{-1}\operatorname{curl} w_0(x),\\ \Phi^-\big|_{t=0}=\Lambda^{-1}\operatorname{curl} n_0(x),\end{cases} \tag{3.6}$$

where $\kappa=\kappa_1+\kappa_2$, $\varkappa=\varkappa_1+\varkappa_2$.

3.2. Spectral Analysis for IVP (3.5)

The IVP (3.3) for $\chi'=\left(Z,\Psi^+,\Psi^-\right)^t$ can be represented as

$$\begin{cases}\chi'_t=\mathcal{T}_1\chi',\\ \chi'\big|_{t=0}=\chi'_0,\end{cases} \tag{3.7}$$

where $\mathcal{T}_1$ is the operator,

$$\mathcal{T}_1=\begin{pmatrix}0 & -\beta_1\Lambda & -\beta_2\Lambda\\ \beta_3\Lambda & -\kappa\Lambda^2 & 0\\ \beta_4\Lambda & 0 & -\varkappa\Lambda^2\end{pmatrix}.$$

The Fourier transform of (3.5) gives

$$\begin{cases}\hat{\chi}'_t=\mathcal{K}_1(\varsigma)\hat{\chi}',\\ \hat{\chi}'\big|_{t=0}=\hat{\chi}'_0,\end{cases} \tag{3.8}$$

where $\hat{\chi}'(\varsigma,t)=\mathfrak{G}\left(\chi'(x,t)\right)$ and $\mathcal{K}_1(\varsigma)$ is given by

$$\mathcal{K}_1(\varsigma)=\begin{pmatrix}0 & -\beta_1|\varsigma| & -\beta_2|\varsigma|\\ \beta_3|\varsigma| & -\kappa|\varsigma|^2 & 0\\ \beta_4|\varsigma| & 0 & -\varkappa|\varsigma|^2\end{pmatrix}.$$

We calculate

$$\begin{aligned}&\det\left(PI-\mathcal{K}_1(\varsigma)\right)\\ &=P^3+(\kappa+\varkappa)|\varsigma|^2P^2+\left(\beta_1\beta_3+\beta_2\beta_4+\kappa\varkappa|\varsigma|^2\right)|\varsigma|^2P\\ &\quad+\left(\beta_1\beta_3\varkappa+\beta_2\beta_4\kappa\right)|\varsigma|^4,\end{aligned} \tag{3.9}$$

so we obtain three different eigenvalues:

$$P_1=P_1(|\varsigma|),\quad P_2=P_2(|\varsigma|),\quad P_3=P(|\varsigma|).$$

We have

$$e^{t\mathcal{K}_1(\varsigma)}=\sum_{k=1}^{3}e^{P_kt}Q_k(\varsigma), \tag{3.10}$$

where

$$Q_k(\varsigma)=\prod_{j\neq k}\frac{\mathcal{K}_1(\varsigma)-P_jI}{P_k-P_j},\quad k,j=1,2,3. \tag{3.11}$$

So we get the solution of (3.8):

$$\hat{\chi}'(\varsigma,t)=e^{t\mathcal{K}_1(\varsigma)}\hat{\chi}'_0(\varsigma)=\left(\sum_{k=1}^{3}e^{P_kt}Q_k(\varsigma)\right)\hat{\chi}'_0(\varsigma). \tag{3.12}$$

Using comparable methods in [37]-[39], we get

Lemma 3.1. If ζ is a positive constant and $\zeta\ll1$, the following Taylor series expansion exists for $|\xi|\le\zeta$

$$\begin{cases} P_1 = -\dfrac{\beta_1\beta_3\kappa + \beta_2\beta_4\varkappa}{2(\beta_1\beta_3+\beta_2\beta_4)}|\varsigma|^2 + \sqrt{\beta_1\beta_3+\beta_2\beta_4}\,\mathrm{i}|\varsigma| + \mathcal{O}\left(|\varsigma|^3\right), \\ P_2 = -\dfrac{\beta_1\beta_3\kappa + \beta_2\beta_4\varkappa}{2(\beta_1\beta_3+\beta_2\beta_4)}|\varsigma|^2 - \sqrt{\beta_1\beta_3+\beta_2\beta_4}\,\mathrm{i}|\varsigma| + \mathcal{O}\left(|\varsigma|^3\right), \\ P_3 = -\dfrac{\beta_1\beta_3\varkappa + \beta_2\beta_4\kappa}{\beta_1\beta_3+\beta_2\beta_4}|\varsigma|^2 + \mathcal{O}\left(|\varsigma|^3\right). \end{cases} \tag{3.13}$$

Lemma 3.2. Let $\bar{\nu} = \min\left\{\dfrac{\beta_1\beta_3\kappa+\beta_2\beta_4\varkappa}{2(\beta_1\beta_3+\beta_2\beta_4)}, \dfrac{\beta_1\beta_3\varkappa+\beta_2\beta_4\kappa}{\beta_1\beta_3+\beta_2\beta_4}\right\} > 0$, for any $|\varsigma| \le \eta$, then

$$\left|\hat{Z}\right|, \left|\widehat{\Psi^+}\right|, \left|\widehat{\Psi^-}\right| \lesssim \mathrm{e}^{-\bar{\nu}|\varsigma|^2 t}\left(\left|\widehat{Z_0}\right| + \left|\widehat{\Psi_0^+}\right| + \left|\widehat{\Psi_0^-}\right|\right), \tag{3.14}$$

Proof. Owing to (3.11) and P_k $(k=1,2,3)$ in (3.13), we can obtain Q_k $(k=1,2,3)$:

$$Q_1(\varsigma) = \begin{pmatrix} \frac{1}{2} & \frac{\beta_1}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & \frac{\beta_2}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} \\ -\frac{\beta_3}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & \frac{\beta_1\beta_3}{2(\beta_1\beta_3+\beta_2\beta_4)} & \frac{\beta_2\beta_3}{2(\beta_1\beta_3+\beta_2\beta_4)} \\ -\frac{\beta_4}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & \frac{\beta_1\beta_4}{2(\beta_1\beta_3+\beta_2\beta_4)} & \frac{\beta_2\beta_4}{2(\beta_1\beta_3+\beta_2\beta_4)} \end{pmatrix} + \mathcal{O}\left(|\varsigma|\right), \tag{3.15}$$

$$Q_2(\varsigma) = \begin{pmatrix} \frac{1}{2} & -\frac{\beta_1}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & -\frac{\beta_2}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} \\ \frac{\beta_3}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & \frac{\beta_1\beta_3}{2(\beta_1\beta_3+\beta_2\beta_4)} & \frac{\beta_2\beta_3}{2(\beta_1\beta_3+\beta_2\beta_4)} \\ \frac{\beta_4}{2\sqrt{\beta_1\beta_3+\beta_2\beta_4}}\mathrm{i} & \frac{\beta_1\beta_4}{2(\beta_1\beta_3+\beta_2\beta_4)} & \frac{\beta_2\beta_4}{2(\beta_1\beta_3+\beta_2\beta_4)} \end{pmatrix} + \mathcal{O}\left(|\varsigma|\right), \tag{3.16}$$

and

$$Q_3(\varsigma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\beta_2\beta_4}{\beta_1\beta_3+\beta_2\beta_4} & -\frac{\beta_2\beta_3}{\beta_1\beta_3+\beta_2\beta_4} \\ 0 & -\frac{\beta_1\beta_4}{\beta_1\beta_3+\beta_2\beta_4} & \frac{\beta_1\beta_3}{\beta_1\beta_3+\beta_2\beta_4} \end{pmatrix} + \mathcal{O}\left(|\varsigma|\right), \tag{3.17}$$

for any $|\varsigma| \le \zeta$. Therefore, (3.14) follows from (3.12)-(3.13) and (3.15)-(3.17) immediately. □

From (3.14), we can get Proposition 3.3.

Proposition 3.3 (L^2-theory). If $k \ge 0$, $\forall\ t \ge 0$, then

$$\left\|\nabla^k \mathrm{e}^{t\mathcal{K}_1}\chi''(0)\right\|_{L^2} \lesssim (1+t)^{\frac{3}{4}-\frac{1}{2}\left(k+\frac{3}{p}\right)}\left\|\chi'(0)\right\|_{L^p}, \tag{3.18}$$

where $1 \le p \le 2$.

Proof. Using Plancherel theorem, we have

$$\begin{aligned}\left\|\nabla^k \mathrm{e}^{t\mathcal{T}_1}\chi^l(0)\right\|_{L^2}^2 &= \left\||\varsigma|^k \mathrm{e}^{t\mathcal{K}_1(\varsigma)}\hat{\chi}'^l(0)\right\|_{L^2}^2 \\ &\lesssim \int_{|\varsigma|\le\zeta} \mathrm{e}^{-2\bar{\nu}|\varsigma|^2 t}|\varsigma|^{2k}\left|\hat{\chi}'^l(0)\right|^2 \mathrm{d}\varsigma \\ &\lesssim (1+t)^{\frac{3}{2}-\left(\frac{3}{p}+k\right)}\left\|\hat{\chi}'^l(0)\right\|_{L^q}^2 \\ &\lesssim (1+t)^{\frac{3}{2}-\left(\frac{3}{p}+k\right)}\left\|\chi'(0)\right\|_{L^p}^2,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proposition 3.3. is proved. □

The L^2-convergence rates obtained above are optimum. In fact, there is a lower bound on the rate of convergence, see Proposition 3.3.

Proposition 3.4. Provided that $\left(\hat{Z},\widehat{\Psi^+},\widehat{\Psi^-}\right)\in L^1$, if

$$\widehat{\Psi_0^+}(\xi)=\widehat{\Psi_0^-}(\xi)=0,\quad \left|\widehat{\varsigma_0}(\varsigma)\right|\ge c_0, \tag{3.19}$$

for $\forall\ |\xi|\le\zeta$, t is large enough, then the global solution $\left(Z,\Psi^+,\Psi^-\right)$ of (3.7) satisfies

$$\min\left\{\left\|Z^l\right\|_{L^2},\left\|\Psi^{+,l}\right\|_{L^2},\left\|\Psi^{-,l}\right\|_{L^2}\right\}\gtrsim c_0(1+t)^{-\frac{3}{4}}. \tag{3.20}$$

Proof. Let $\bar{\nu}_1=\frac{\beta_1\beta_3\kappa+\beta_2\beta_4\varkappa}{2(\beta_1\beta_3+\beta_2\beta_4)}>0$. Due to (3.19), it follows from (3.12)-(3.13) and (3.15)-(3.17) that

$$\hat{Z}^l\sim \mathrm{e}^{-\bar{\nu}_1|\varsigma|^2 t}\cos\left(\sqrt{\beta_1\beta_3+\beta_2\beta_4}\,|\varsigma|t+\mathcal{O}\left(|\varsigma|^3\right)t\right)\hat{Z}_0^l$$

Combining the double angle formula and Plancherel theorem, we get

$$\begin{aligned}\left\|Z^l\right\|_{L^2}^2 &= \left\|\hat{Z}^l\right\|_{L^2}^2 \\ &\ge \frac{c_0^2}{2}\int_{|\varsigma|\le\zeta}\mathrm{e}^{-2\bar{\nu}_1|\varsigma|^2 t}\cos^2\left(\sqrt{\beta_1\beta_3+\beta_2\beta_4}\,|\varsigma|t+\mathcal{O}\left(|\varsigma|^3\right)t\right)\mathrm{d}\varsigma \\ &\ge \frac{c_0^2}{4}(1+t)^{-\frac{3}{2}}-Cc_0^2(1+t)^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}} \\ &\gtrsim c_0^2(1+t)^{-\frac{3}{2}},\end{aligned} \tag{3.21}$$

if t is large enough. (3.20) is derived from (3.21) using the same method. Therefore, we proved Proposition 3.4. □

3.3. Spectral Analysis for IVP (3.6)

Proposition 3.5 (L^2-theory). *If* $k\ge 0$, $\forall\ t\ge 0$, $1\le p\le 2$, *then*

$$\begin{aligned}\left\|\nabla^k \mathrm{e}^{-\kappa_1\Lambda^2 t}\Psi^{+,l}(0)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}\left(k+\frac{3}{p}\right)+\frac{3}{4}}\left\|\Psi^+(0)\right\|_{L^p}, \\ \left\|\nabla^k \mathrm{e}^{-\varkappa_1\Lambda^2 t}\Psi^{-,l}(0)\right\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}\left(k+\frac{3}{p}\right)+\frac{3}{4}}\left\|\Psi^-(0)\right\|_{L^p},\end{aligned} \tag{3.22}$$

3.4. L^2 Decay Estimates for IVP (3.3)

By the definition of $\Psi^{\pm}$, $\Phi^{\pm}$, and

$$w = -\wedge^{-1}\nabla\Psi^{+} - \wedge^{-1}\operatorname{div}\Phi^{+}$$

$$n = -\wedge^{-1}\nabla\Psi^{-} - \wedge^{-1}\operatorname{div}\Phi^{-}$$

By integrating Propositions 3.3, 3.4 and 3.5, we obtain semigroup $e^{-t\mathcal{K}}$.

Proposition 3.6. If the initial data $\chi_0 \in L^p(\mathbb{R}^3)$. Assume that $k \ge 0$ and $1 \le p \le 2$, $\forall\ t \ge 0$, the global solution $\chi = (Z, w, n)^t$ of the IVP (3.2) satisfies

$$\left\|\nabla^k e^{t\mathcal{K}}\chi^l(0)\right\|_{L^2} \le C(1+t)^{-\frac{1}{2}\left(k+\frac{3}{p}\right)+\frac{3}{4}}\left\|\chi(0)\right\|_{L^p}. \tag{3.23}$$

If t is large enough and the initial data also satisfy (3.19), then

$$\min\left\{\left\|Z^l(t)\right\|_{L^2}, \left\|w^l(t)\right\|_{L^2}, \left\|n^l(t)\right\|_{L^2}\right\} \ge C_1 c_0 (1+t)^{-\frac{3}{4}}. \tag{3.24}$$

4. The Energy Estimate of (*Z*, *w*, *n*) and Its Derivatives

Due to the fact that local classical solutions can be proven in [37] [38], which omit details. Combining the results of local existence with prior estimates through classical argumentation, the global existence of the solution is obtained. So we suppose a priori that

$$\left\|\left(m^{+}, Z, w, m^{-}, n\right)\right\|_{H^3} \le \delta \ll 1, \tag{4.1}$$

where δ is little sufficient, $\delta \sim \delta_0$. It is in conjunction with Sobolev inequality particularly indicates that

$$\left\|\left(m^{+}, Z, w, m^{-}, n\right)\right\|_{W^{1,\infty}} \lesssim \delta. \tag{4.2}$$

The following are a series of lemmas for energy estimation. First, we perform zero order energy estimation.

Lemma 4.1. Under the suppositions of Theorem 1.1 meanwhile (4.1) are valid, then

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|Z\|_{L^2}^2 + \|w\|_{L^2}^2 + \|n\|_{L^2}^2\right\} + \left(\kappa_1\|\nabla w\|_{L^2}^2 + \kappa_2\|\operatorname{div} w\|_{L^2}^2\right)\\ &+\left(\varkappa_1\|\nabla n\|_{L^2}^2 + \varkappa_2\|\operatorname{div} n\|_{L^2}^2\right) \le C\delta\left(\|\nabla w\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right).\end{aligned} \tag{4.3}$$

Proof. Multiplying $(1.14)_3$, $(1.14)_4$, $(1.14)_5$ by $\frac{\beta_3}{\beta_1}Z$, w, n, respectively, we integrate and sum the resulting equation over $\mathbb{R}^3$, then using $(1.14)_1$ and $(1.14)_2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|Z\|_{L^2}^2 + \|w\|_{L^2}^2 + \|n\|_{L^2}^2\right\} + \left(\kappa_1\|\nabla w\|_{L^2}^2 + \kappa_2\|\operatorname{div} w\|_{L^2}^2\right)\\ &+\left(\varkappa_1\|\nabla n\|_{L^2}^2 + \varkappa_2\|\operatorname{div} n\|_{L^2}^2\right)\\ &=\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}G_1 Z\mathrm{d}x + \int_{\mathbb{R}^3} w\left(H_1 + R_1\right)\mathrm{d}x + \int_{\mathbb{R}^3} n\left(H_2 + R_2\right)\mathrm{d}x =: \sum_{i=1}^{3} I_3.\end{aligned} \tag{4.4}$$

By employing integration by parts, (4.1), Lemma 2.1, and related Sobolev inequality, then

$$
\begin{aligned}
|I_1| \le & C\left\|\left(m^+,Z\right)\right\|_{L^3}\|\theta\|_{L^6}\|\nabla w\|_{L^2}+\left\|\left(m^+,Z\right)\right\|_{L^3}\|Z\|_{L^6}\|\nabla n\|_{L^2}+\|\nabla Z\|_{L^2}\|w\|_{L^6}\left\|m^+\right\|_{L^3} \\
& +\|\nabla Z\|_{L^2}\|n\|_{L^6}\left\|m^-\right\|_{L^3}+\|Z\|_{L^6}\left\|\nabla\left(m^+,Z\right)\right\|_{L^3}\|w\|_{L^6}\left\|m^+\right\|_{L^3} \\
& +\|\theta\|_{L^6}\left\|\nabla\left(m^+,Z\right)\right\|_{L^3}\|n\|_{L^6}\left\|m^-\right\|_{L^3}+\|Z\|_{L^6}\|\nabla w\|_{L^2}\left\|m^+\right\|_{L^3}+\|Z\|_{L^6}\|\nabla n\|_{L^2}\left\|m^-\right\|_{L^3} \\
\le & C\left(\delta\|\nabla Z\|_{L^2}\|\nabla w\|_{L^2}+\delta\|\nabla Z\|_{L^2}\|\nabla n\|_{L^2}+\delta\|\nabla Z\|_{L^2}\|\nabla w\|_{L^2}+\delta\|\nabla Z\|_{L^2}\|\nabla n\|_{L^2}\right) \\
\le & C\delta\left(\|\nabla w\|_{L^2}^2+\|\nabla Z\|_{L^2}^2+\|\nabla n\|_{L^2}^2\right).
\end{aligned} \tag{4.5}
$$

For the term I_2, in the same way, we obtain

$$
\begin{aligned}
|I_2| \le & \int_{\mathbb{R}^3}\Big(|w||w||\nabla w|+|Z||\nabla Z||w|+|\nabla Z||\nabla w||w|+|Z||\nabla w|^2+|\nabla Z||\operatorname{div} w||w| \\
& +|Z||\operatorname{div} w|^2+|Z|\left|\nabla m^+\right||\nabla w||w|+|\nabla Z|\left|m^+\right||\nabla w||w|\Big)\mathrm{d}x \\
\le & C\Big(\|\nabla w\|_{L^2}\|w\|_{L^3}\|w\|_{L^6}+\|Z\|_{L^6}\|\nabla Z\|_{L^2}\|w\|_{L^3}+\|\nabla Z\|_{L^2}\|\nabla w\|_{L^2}\|w\|_{L^\infty} \\
& +\|Z\|_{L^\infty}\|\nabla w\|_{L^2}^2+\|Z\|_{L^6}\left\|\nabla m^+\right\|_{L^3}\|\nabla w\|_{L^2}\|w\|_{L^\infty} \\
& +\|Z\|_{L^2}\|\nabla w\|_{L^2}\|w\|_{L^\infty}\left\|m^+\right\|_{L^\infty}\Big) \\
\le & C\delta\left(\|\nabla w\|_{L^2}^2+\|\nabla Z\|_{L^2}^2\right).
\end{aligned} \tag{4.6}
$$

We also obtain

$$
|I_3| \le C\delta\left(\|\nabla n\|_{L^2}^2+\|\nabla Z\|_{L^2}^2\right). \tag{4.7}
$$

Substituting (4.5)-(4.7) into (4.4), when δ is small enough, we can obtain (4.3) immediately. Lemma 4.1 is proved. □

Lemma 4.2. Under the suppositions of Theorem 1.1 meanwhile (4.1) are valid, then

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|\nabla Z\|_{H^1}^2+\|\nabla w\|_{H^1}^2+\|\nabla n\|_{H^1}^2\right\}+\left(\kappa_1\left\|\nabla^2 w\right\|_{H^1}^2+\kappa_2\|\nabla \operatorname{div} w\|_{H^1}^2\right) \\
& +\left(\varkappa_1\left\|\nabla^2 n\right\|_{H^1}^2+\varkappa_2\|\nabla \operatorname{div} n\|_{H^1}^2\right) \le C\delta\left(\|\nabla Z\|_{H^1}^2+\|\nabla w\|_{H^2}^2+\|\nabla n\|_{H^2}^2\right).
\end{aligned} \tag{4.8}
$$

Proof. First, we consider $k=1$,

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|\nabla Z\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla n\|_{L^2}^2\right\}+\left(\kappa_1\left\|\nabla^2 w\right\|_{L^2}^2+\kappa_2\|\nabla \operatorname{div} w\|_{L^2}^2\right) \\
& +\left(\varkappa_1\left\|\nabla^2 n\right\|_{L^2}^2+\varkappa_2\|\nabla \operatorname{div} n\|_{L^2}^2\right) \le C\delta\left(\|\nabla Z\|_{L^2}^2+\|\nabla w\|_{H^1}^2+\|\nabla n\|_{H^1}^2\right).
\end{aligned} \tag{4.9}
$$

Multiplying ∇ (1.14)$_3$, ∇ (1.14)$_4$, ∇ (1.14)$_5$ by $\frac{\beta_3}{\beta_1}\nabla Z$, ∇w, ∇n, respectively, we integrate and sum then the resulting equation over $\mathbb{R}^3$, we have

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|\nabla Z\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla n\|_{L^2}^2\right\}+\left(\kappa_1\left\|\nabla^2 w\right\|_{L^2}^2+\kappa_2\|\nabla \operatorname{div} w\|_{L^2}^2\right) \\
& +\left(\varkappa_1\left\|\nabla^2 n\right\|_{L^2}^2+\varkappa_2\|\nabla \operatorname{div} n\|_{L^2}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}\nabla G_1\cdot\nabla Z\mathrm{d}x+\int_{\mathbb{R}^3}\left(\nabla H_1+\nabla R_1\right)\cdot\nabla w\mathrm{d}x+\int_{\mathbb{R}^3}\left(\nabla H_2+\nabla R_2\right)\cdot\nabla n\mathrm{d}x\\
&=:\sum_{i=1}^{3}J_i.
\end{aligned}
\tag{4.10}
$$

Employing relevant inequalities, we obtain

$$
\left|J_1\right|\le C\delta\left(\|\nabla Z\|_{L^2}^2+\|\nabla n\|_{H^1}^2+\|\nabla w\|_{H^1}^2\right).
\tag{4.11}
$$

We calculate

$$
\begin{aligned}
\left|J_2\right|\le{}& C\Big(\|w\|_{L^\infty}\|\nabla w\|_{L^2}\left\|\nabla^2 w\right\|_{L^2}+\|Z\|_{L^\infty}\|\nabla Z\|_{L^2}\left\|\nabla^2 w\right\|_{L^2}\\
&+\|Z\|_{L^\infty}\left\|\nabla^2 w\right\|_{L^2}\left\|\nabla^2 w\right\|_{L^2}+\|\nabla w\|_{L^2}\left\|\nabla\left(m^+,Z\right)\right\|_{L^\infty}\left\|\nabla^2 w\right\|_{L^2}\|Z\|_{L^\infty}\\
&+\|\nabla w\|_{L^2}\|\nabla Z\|_{L^\infty}\left\|\left(m^+,Z\right)\right\|_{L^\infty}\left\|\nabla^2 w\right\|_{L^2}\Big)\\
\le{}& C\delta\left(\|\nabla w\|_{L^2}^2+\|\nabla Z\|_{L^2}^2+\left\|\nabla^2 w\right\|_{L^2}^2\right)\\
\le{}& C\delta\left(\|\nabla Z\|_{L^2}^2+\|\nabla w\|_{H^1}^2\right).
\end{aligned}
\tag{4.12}
$$

Similarly, we obtain

$$
\left|J_3\right|\le C\delta\left(\|\nabla Z\|_{L^2}^2+\|\nabla n\|_{H^1}^2\right).
\tag{4.13}
$$

Substituting (4.11)-(4.13), into (4.10), then, we obtain (4.9) if δ is small enough.

Next we consider $k=2$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\left\|\nabla^2 Z\right\|_{L^2}^2+\left\|\nabla^2 w\right\|_{L^2}^2+\left\|\nabla^2 n\right\|_{L^2}^2\right\}+\left(\kappa_1\left\|\nabla^3 w\right\|_{L^2}^2+\kappa_2\left\|\nabla^2\mathrm{div}w\right\|_{L^2}^2\right)\\
&+\left(\varkappa_1\left\|\nabla^3 n\right\|_{L^2}^2+\varkappa_2\left\|\nabla^2\mathrm{div}n\right\|_{L^2}^2\right)\le C\delta\left(\|\nabla Z\|_{H^1}^2+\|\nabla n\|_{H^2}^2+\|\nabla w\|_{H^2}^2\right).
\end{aligned}
\tag{4.14}
$$

Multiplying ∇^2 (1.14)$_3$, ∇^2 (1.14)$_3$, ∇^2 (1.14)$_4$ by $\frac{\beta_3}{\beta_1}\nabla^2 Z$, $\nabla^2 w$, $\nabla^2 n$, respectively, we integrate and sum up the resulting equation over $\mathbb{R}^3$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\left\|\nabla^2 Z\right\|_{L^2}^2+\left\|\nabla^2 w\right\|_{L^2}^2+\left\|\nabla^2 n\right\|_{L^2}^2\right\}+\left(\kappa_1\left\|\nabla^3 w\right\|_{L^2}^2+\kappa_2\left\|\nabla^2\mathrm{div}n\right\|_{L^2}^2\right)\\
&+\left(\varkappa_1\left\|\nabla^3 n\right\|_{L^2}^2+\varkappa_2\left\|\nabla^2\mathrm{div}n\right\|_{L^2}^2\right)\\
&=\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}\nabla^2 G_1\cdot\nabla^2 Z\mathrm{d}x+\int_{\mathbb{R}^3}\left(\nabla^2 H_1+\nabla^2 R_1\right)\cdot\nabla^2 w\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\left(\nabla^2 H_2+\nabla^2 R_2\right)\cdot\nabla^2 n\mathrm{d}x\\
&=:J_4+J_5+J_6.
\end{aligned}
\tag{4.15}
$$

Similar to (4.11), we obtain

$$
\left|J_4\right|\le C\delta\left(\left\|\nabla^2 Z\right\|_{L^2}^2+\|\nabla w\|_{H^2}^2+\|\nabla n\|_{H^2}^2\right).
\tag{4.16}
$$

Similar to (4.12), we obtain

$$
\left|J_5\right|\le C\delta\left(\|\nabla w\|_{L^2}^2+\left\|\nabla^2 w\right\|_{L^2}^2+\left\|\nabla^3 w\right\|_{L^2}^2+\|\nabla Z\|_{L^2}^2+\left\|\nabla^2 Z\right\|_{L^2}^2\right)
$$

$$\leq C\delta\left(\|\nabla Z\|_{H^1}^2+\|\nabla w\|_{H^2}^2\right). \tag{4.17}$$

We also have

$$|J_6|\leq C\delta\left(\|\nabla Z\|_{H^1}^2+\|\nabla n\|_{H^2}^2\right). \tag{4.18}$$

Substituting (4.16)-(4.18) into (4.15), we obtain (4.14). From (4.9) and (4.14), if δ is little sufficiently, we acquire (4.8). The proof is accomplished. □

Lemma 4.3. Under the suppositions of Theorem 1.1 and (4.1) is valid, then

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|Z_t\|_{L^2}^2+\|w_t\|_{L^2}^2+\|n_t\|_{L^2}^2\right\}+\left(\kappa_1\|\nabla w_t\|_{L^2}^2+\kappa_2\|\mathrm{div}w_t\|_{L^2}^2\right)\\&+\left(\varkappa_1\|\nabla n_t\|_{L^2}^2+\varkappa_2\|\mathrm{div}n_t\|_{L^2}^2\right)\\&\leq C\delta\left(\|\nabla w(t)\|_{H^1}^2+\|\nabla n(t)\|_{H^1}^2+\|\nabla w_t(t)\|_{L^2}^2+\|\nabla n_t(t)\|_{L^2}^2\right),\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\sqrt{\rho^{\mu_1}}\nabla w_t\right\|_{L^2}^2+\left\|\sqrt{\rho^{\mu_1}+\rho^{\lambda_1}}\mathrm{div}w_t\right\|_{L^2}^2+\left\|\sqrt{\eta^{\mu_2}}\nabla n_t\right\|_{L^2}^2\right.\\&\left.+\left\|\sqrt{\eta^{\mu_2}+\eta^{\lambda_2}}\mathrm{div}n_t\right\|_{L^2}^2\right\}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{\rho}w_{tt}\right\|_{L^2}^2+\left\|\sqrt{\eta}n_{tt}\right\|_{L^2}^2\right)\\&\leq C\delta\left(\|\nabla w\|_{H^1}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla n\|_{H^1}^2+\|\nabla n_t\|_{L^2}^2\right).\end{aligned} \tag{4.20}$$

Proof. Calculating the derivative of $(1.14)_{2,3,4}$ in terms of t, then

$$\begin{cases}\partial_{tt}Z+\beta_1\mathrm{div}w_t+\beta_2\mathrm{div}n_t=(G_1)_t,\\ \partial_{tt}w+\beta_3\nabla Z_t-\kappa_1\Delta w_t-\kappa_2\nabla\mathrm{div}w_t=(H_1)_t+(R_1)_t,\\ \partial_{tt}n+\beta_4\nabla Z_t-\varkappa_1\Delta n_t-\varkappa_2\nabla\mathrm{div}n_t=(H_1)_t+(R_2)_t.\end{cases} \tag{4.21}$$

We multiply $(4.21)_1$, $(4.21)_2$, $(4.21)_3$, by $\frac{\beta_3}{\beta_1}Z_t, w_t, n_t$ respectively, integrate and sum the equality over $\mathbb{R}^3$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}|Z_t|^2+|w_t|^2+|n_t|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}\kappa_1|\nabla w_t|^2+\kappa_2|\mathrm{div}\,w_t|^2\,\mathrm{d}x\\&+\int_{\mathbb{R}^3}\varkappa_1|\nabla n_t|^2+\varkappa_2|\mathrm{div}\,n_t|^2\,\mathrm{d}x\\&=\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}(G_1)_t Z_t+\left[(H_1)_t+(R_1)_t\right]w_t+\left[(H_2)_t+(R_2)_t\right]n_t\mathrm{d}x,\end{aligned} \tag{4.22}$$

from (1.14),

$$\begin{aligned}\|Z_t\|_{L^2}&\leq\|G_1\|_{L^2}+\|\beta_1\mathrm{div}w\|_{L^2}+\|\beta_1\mathrm{div}n\|_{L^2}\\&\leq C\left(\|\nabla w\|_{L^2}+\|\nabla n\|_{L^2}\right).\end{aligned} \tag{4.23}$$

Similar to the previous process, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\beta_3}{\beta_1}(G_1)_t Z_t\mathrm{d}x&\leq C\|(G_1)_t\|_{L^2}\|Z_t\|_{L^2}\\&\leq C\left(\|m_t^+\|_{L^3}\|\nabla w\|_{L^6}+\|m^+\|_{L^\infty}\|\nabla w_t\|_{L^2}+\|\nabla m^+\|_{L^3}\|w_t\|_{L^6}\right.\end{aligned}$$

$$
\begin{aligned}
&+\left\|\nabla m_t^+\right\|_{L^2}\|w\|_{L^\infty}+\|Z_t\|_{L^2}\|\nabla w\|_{L^\infty}+\|Z\|_{L^\infty}\|\nabla w_t\|_{L^2}\\
&+\left\|m_t^-\right\|_{L^3}\|\nabla n\|_{L^6}+\left\|m^-\right\|_{L^\infty}\|\nabla n_t\|_{L^2}+\left\|\nabla m^-\right\|_{L^3}\|n_t\|_{L^6}\\
&+\left\|\nabla m_t^-\right\|_{L^2}\|n\|_{L^\infty}+\|Z_t\|_{L^2}\|\nabla n\|_{L^\infty}+\|Z\|_{L^\infty}\|\nabla n_t\|_{L^2}\Big)\|Z_t\|_{L^2}\\
&\le C\delta\left(\|\nabla w\|_{H^1}^2+\|\nabla n\|_{H^1}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla n_t\|_{L^2}^2\right),
\end{aligned}
\tag{4.24}
$$

and

$$
\int_{\mathbb{R}^3}\left[\left(H_1\right)_t+\left(R_1\right)_t\right]w_t\mathrm{d}x\le C\delta\left(\|\nabla w\|_{H^1}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla n\|_{H^1}^2+\|\nabla n_t\|_{L^2}^2\right).\tag{4.25}
$$

Similarly, we have

$$
\int_{\mathbb{R}^3}\left[\left(H_2\right)_t+\left(R_2\right)_t\right]n_t\mathrm{d}x\le C\delta\left(\|\nabla w\|_{H^1}^2+\|\nabla w_t\|_{L^2}^2+\|\nabla n\|_{H^1}^2+\|\nabla n_t\|_{L^2}^2\right).\tag{4.26}
$$

Substituting (4.24)-(4.26) into (4.22), we obtain (4.19). We restate the equation $(4.21)_2$, $(4.21)_3$, and get the following form:

$$
\begin{cases}
\rho\partial_{tt}w-\rho^{\mu_1}\Delta w_t-\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\nabla\mathrm{div}w_t\\
=-\rho_t w_t-\nabla Z_t-\Bigg[\rho w\cdot\nabla w+\nabla\left(\rho^{\mu_1}\right)\left(\nabla w+\nabla^t w\right)+\nabla\left(\rho^{\lambda_1}\right)\mathrm{div}w\\
\quad+\dfrac{\nabla\left(\rho^{\mu_1}\right)\left(\nabla w+\nabla^t w\right)\nabla\sigma^+}{\sigma^+}+\dfrac{\nabla\left(\rho^{\lambda_1}\right)\mathrm{div}w\nabla\sigma^+}{\sigma^+}\Bigg]_t\\
\eta\partial_{tt}n-\eta^{\mu_2}\Delta n_t-\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\nabla\mathrm{div}n_t\\
=-\eta_t n_t-\nabla Z_t-\Bigg[\eta n\cdot\nabla n+\nabla\left(\eta^{\mu_2}\right)\left(\nabla n+\nabla^t n\right)+\nabla\left(\eta^{\lambda_2}\right)\mathrm{div}n\\
\quad+\dfrac{\nabla\left(\eta^{\mu_2}\right)\left(\nabla n+\nabla^t n\right)\nabla\sigma^-}{\sigma^-}+\dfrac{\nabla\left(\eta^{\lambda_2}\right)\mathrm{div}n\nabla\sigma^-}{\sigma^-}\Bigg]_t
\end{cases}
\tag{4.27}
$$

Multiplying $(4.27)_1$ by w_{tt} respectively, integrating over $\mathbb{R}^3$, then

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\rho\left|w_{tt}\right|^2\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho^{\mu_1}\left|\nabla w_t\right|^2+\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\left|\mathrm{div}w_t\right|^2\mathrm{d}x\\
&\le C\Big(\|\rho_t\|_{L^3}\|w_t\|_{L^6}\|w_{tt}\|_{L^2}+\|\nabla Z_t\|_{L^2}\|w_{tt}\|_{L^2}+\|\nabla Z\|_{L^3}\|\nabla w_t\|_{L^2}\|w_{tt}\|_{L^6}\\
&\quad+\|\nabla Z_t\|_{L^\infty}\|\nabla w_t\|_{L^2}^2+\|w\|_{L^\infty}\|\nabla w\|_{L^2}\|w_{tt}\|_{L^2}+\|\rho_t\|_{L^2}\|w\|_{L^\infty}\|\nabla w\|_{L^\infty}\|w_{tt}\|_{L^2}\\
&\quad+\|w_t\|_{L^6}\|\nabla w\|_{L^3}\|w_t\|_{L^2}+\|w\|_{L^\infty}\|\nabla w_t\|_{L^2}\|w_{tt}\|_{L^2}+\|\nabla Z_t\|_{L^2}\|\nabla w\|_{L^\infty}\|Z_{tt}\|_{L^2}\\
&\quad+\|Z_t\|_{L^2}\|\nabla w\|_{L^\infty}\left\|\nabla\sigma^+\right\|_{L^\infty}\|w_{tt}\|_{L^2}+\|Z\|_{L^\infty}\|\nabla w_t\|_{L^2}\left\|\nabla\sigma^+\right\|_{L^\infty}\|w_{tt}\|_{L^2}\\
&\quad+\|Z\|_{L^\infty}\|\nabla w\|_{L^\infty}\left\|\nabla\sigma_t^+\right\|_{L^2}\|w_{tt}\|_{L^2}+\left\|\nabla\sigma^+\right\|_{L^6}\|\sigma_t\|_{L^6}\|\nabla w\|_{L^6}\|w_{tt}\|_{L^2}\\
&\quad+\|\rho_t\|_{L^2}\left\|\nabla\sigma^+\right\|_{L^\infty}\|\nabla u\|_{L^\infty}\|w_{tt}\|_{L^2}\\
&\le C\delta\left(\|\nabla w\|_{H^1}^2+\|\nabla w_t\|_{L^2}^2\right)+\frac{1}{2}\int_{\mathbb{R}^3}\rho\left|w_{tt}(t)\right|^2\mathrm{d}x.
\end{aligned}
\tag{4.28}
$$

We multiply $(4.27)_2$ by n_{tt} respectively, and integrate the resulting equality over $\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}\eta|n_{tt}|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\eta^{\mu_2}|\nabla n_t|^2+(\eta^{\mu_2}+\eta^{\lambda_2})|\mathrm{div}n_t|^2\,dx \\ \le C\delta\left(\|\nabla n\|_{H^1}^2+\|\nabla n_t\|_{L^2}^2\right)+\frac{1}{2}\int_{\mathbb{R}^3}\eta|n_{tt}(t)|^2\,dx. \tag{4.29}$$

By using (4.28) and (4.29), we obtain (4.20). □

5. The Time Decay Rates for (Z,w,n)

Theorem 5.1. If $C>0$, C is not less than 0. Under the conditions of Theorem 1.1 and (4.1), then

$$\|(Z,w,n)(t)\|_{L^2}\le CN_0(1+t)^{-\frac{3}{4}}, \tag{5.1}$$

$$\|\nabla Z(t)\|_{H^1}+\|\nabla w(t)\|_{H^2}+\|\nabla n(t)\|_{H^2} \\ +\|(Z_t,w_t,n_t,\nabla w_t,\nabla n_t)\|_{L^2}^2\le CN_0(1+t)^{-\frac{5}{4}} \tag{5.2}$$

and

$$\int_0^t(1+\tau)^{\frac{9}{8}}\left(\|w_{tt}(\tau)\|_{L^2}^2+\|n_{tt}(\tau)\|_{L^2}^2\right)d\tau\le CN_0. \tag{5.3}$$

Proof. From (1.14), we see that

$$\|w_t\|_{L^2}\le C\left(\|(\nabla^2w,\nabla Z)\|_{L^2}+\delta\|\nabla w\|_{H^1}+\delta\|\nabla^2Z\|_{L^2}\right), \tag{5.4}$$

$$\|n_t\|_{L^2}\le C\left(\|(\nabla^2n,\nabla Z)\|_{L^2}+\delta\|\nabla n\|_{H^1}+\delta\|\nabla^2Z\|_{L^2}\right). \tag{5.5}$$

From (1.14), (4.19) and (4.20), we have

$$\|\nabla^2Z\|_{L^2}\le C\|\nabla((1+m^+)w_t)\|_{L^2}+C\|\nabla^3w\|_{L^2}+C\|\nabla A_1+\nabla B_1\|_{L^2} \\ \le C\left(\|(\nabla w_t,\nabla^3w)\|_{L^2}+\delta\|\nabla^2w\|_{L^2}\right). \tag{5.6}$$

Similarly, we also have

$$\|\nabla^3w\|_{L^2}\le C\left(\|(\nabla w_t,\nabla^2Z)\|_{L^2}+\delta\|\nabla^2w\|_{L^2}\right),$$

$$\|\nabla^3n\|_{L^2}\le C\left(\|(\nabla n_t,\nabla^2Z)\|_{L^2}+\delta\|\nabla^2n\|_{L^2}\right). \tag{5.7}$$

Assuming that the constant $D>0$. From (5.4)-(5.7), summing up D × ((4.8) + (4.19)) + (4.20), then there exists an energy functional, designated as $W(w,Z,n)$, which is equivalent to $\|\nabla Z\|_{H^1}^2+\|\nabla w\|_{H^2}^2+\|\nabla n\|_{H^2}^2+\|(Z_t,w_t,n_t,\nabla w_t,\nabla n_t)\|_{L^2}^2$, then for some positive constant C_1, we can get

$$\frac{d}{dt}W(t)+C_1\left(W(t)+\|w_{tt}\|_{L^2}^2+\|n_{tt}\|_{L^2}^2\right)\le C\|\nabla(Z^l,w^l,n^l)\|_{L^2}^2. \tag{5.8}$$

In addition, Define the time-weighted energy functional:

$$\mathcal{I}(t)=\sup_{0\le\tau\le t}\left\{(1+\tau)^{\frac{5}{2}}W(\tau)\right\}, \tag{5.9}$$

Denoting

$$\mathcal{G}=\left(\widehat{G_1},\widehat{H_1+R_1},\widehat{H_2+R_2}\right),$$

by Duhamel principle, we have

$$\chi^l=\mathrm{e}^{t\mathcal{K}}\chi^l(0)+\int_0^t \mathrm{e}^{(t-\tau)\mathcal{K}}\mathcal{G}^l(\tau)\,\mathrm{d}\tau,$$

So we can derive

$$\left\|\nabla\left(Z^l,w^l,n^l\right)(t)\right\|_{L^2}\lesssim(1+t)^{-\frac{5}{4}}\left\|(Z,w,n)(0)\right\|_{L^1}+\int_0^t\left\|\nabla\mathrm{e}^{(t-\tau)\mathcal{K}}\mathcal{G}(\tau)\right\|_{L^2}\mathrm{d}\tau. \quad (5.10)$$

Because of the non-dissipative nature of $m^{\pm}$, we need to devise innovative approaches to address the final two terms of G_1 in $(1.14)_3$. The core concept is to treat $-\mathcal{L}\eta w\cdot\nabla m^+-\mathcal{L}\rho n\cdot\nabla m^-$ as a whole, and then restate cleverly. If we notice that $\eta r_-^2-\rho r_+^2\neq 0$, and use the subtle relation between the variables, it is somewhat unexpected that $-\mathcal{L}\eta u\cdot\nabla m^+-\mathcal{L}\rho v\cdot\nabla m^-$ can be rewritten as

$$\begin{aligned}
&-\mathcal{L}\eta w\cdot\nabla m^+-\mathcal{L}\rho n\cdot\nabla m^-\\
&=-\mathcal{L}\eta u\cdot\nabla m^+-\mathcal{L}\rho v\cdot\left(\frac{\nabla Z}{\mathcal{L}\rho}-\frac{\eta\nabla m^+}{\rho}\right)\\
&=-n\cdot\nabla Z+\mathcal{L}\eta(n-w)\cdot\nabla m^+\\
&=-n\cdot\nabla Z+\frac{r_+^2r_-^2\rho\eta(n-w)}{(\rho r_+)^2+\left(\eta r_-^2-\rho r_+^2\right)M^+}\cdot\nabla m^+\\
&=-n\cdot\nabla Z+\frac{r_+^2r_-^2\rho\eta(n-w)}{\eta r_-^2-\rho r_+^2}\cdot\nabla\ln\left((\rho r_+)^2+\left(\eta r_-^2-\rho r_+^2\right)M^+\right)\\
&\quad+\frac{r_+^2r_-^2\rho\eta(n-w)}{\eta r_-^2-\rho r_+^2}\cdot\left(\nabla(\rho r_+)^2+M^+\nabla\left(\eta r_-^2-\rho r_+^2\right)\right)\\
&=-n\cdot\nabla Z+\operatorname{div}\left[\frac{r_+^2r_-^2\rho\eta(n-w)}{\eta r_-^2-\rho r_+^2}\ln\left(\frac{(\rho r_+)^2+\left(\eta r_-^2-\rho r_+^2\right)M^+}{(\bar\rho\bar r_+)^2+\left(\bar\eta\bar r_-^2-\bar\rho\bar r_+^2\right)\bar M^+}\right)\right]\\
&\quad+\ln\left(\frac{(\rho r_+)^2+\left(\eta r_-^2-\rho r_+^2\right)M^+}{(\bar\rho\bar r_+)^2+\left(\bar\eta\bar r_-^2-\bar\rho\bar r_+^2\right)\bar M^+}\right)\operatorname{div}\left[\frac{r_+^2r_-^2\rho\eta(n-w)}{\eta r_-^2-\rho r_+^2}\right]\\
&\quad+\frac{r_+^2r_-^2\rho\eta(w-n)}{\eta r_-^2-\rho r_+^2}\cdot\left(\nabla(\rho r_+)^2+M^+\nabla\left(\eta r_-^2-\rho r_+^2\right)\right).
\end{aligned} \quad (5.11)$$

From Proposition 3.6, (5.9) and (5.10), we have

$$\begin{aligned}
&\left\|\nabla\left(Z^l,w^l,n^l\right)(t)\right\|_{L^2}\\
&\lesssim(1+t)^{-\frac{5}{4}}N_0+\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(m^{\pm},Z\right)(\tau)\right\|_{L^\infty}\left\|(\nabla w,\nabla n)(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\quad+\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(m^+,Z,w,\nabla m^+,\nabla Z\right)(\tau)\right\|_{L^\infty}\left\|\left(\nabla Z,\nabla w,\nabla^2w\right)(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\quad+\int_0^t(1+t-\tau)^{-\frac{5}{4}}\left\|\left(m^-,Z,n,\nabla m^-,\nabla Z\right)(\tau)\right\|_{L^\infty}\left\|\left(\nabla Z,\nabla n,\nabla^2n\right)(\tau)\right\|_{L^2}\mathrm{d}\tau\\
&\lesssim(1+t)^{-\frac{5}{4}}N_0+\delta\int_0^t(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}\sqrt{\mathcal{I}(t)}\,\mathrm{d}\tau\\
&\lesssim(1+t)^{-\frac{5}{4}}N_0+\delta(1+t)^{-\frac{5}{4}}\sqrt{\mathcal{I}(t)},
\end{aligned} \quad (5.12)$$

so we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t)+C_1\left(W(t)+\|w_{tt}\|_{L^2}^2+\|n_{tt}\|_{L^2}^2\right)\le C(1+t)^{-\frac{5}{2}}N_0^2+C\delta^2(1+t)^{-\frac{5}{2}}\mathcal{I}(t). \tag{5.13}$$

By Gronwall inequality, we obtain

$$W(t)\le C(1+t)^{-\frac{5}{2}}N_0^2+C\delta^2(1+t)^{-\frac{5}{2}}\mathcal{I}(t),$$

that is,

$$\mathcal{I}(t)\le CN_0^2+C\delta^2\mathcal{I}(t),$$

when δ is small enough, (5.2) holds.

Employing Parseval's theorem, we have

$$\begin{aligned}\left\|\left(Z^l,w^l,n^l\right)(t)\right\|_{L^2}&\lesssim(1+t)^{-\frac{3}{4}}N_0+\delta^2\int_0^t(1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{5}{4}}\mathrm{d}\tau\\&\lesssim(1+t)^{-\frac{3}{4}}N_0+\delta^2(1+t)^{-\frac{3}{4}}.\end{aligned} \tag{5.14}$$

Substituting (5.14) into (4.3), if $C_2>0$, C_2 a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\beta_3}{\beta_1}\|Z\|_{L^2}^2+\|w\|_{L^2}^2+\|n\|_{L^2}^2\right\}+C_2\left(\frac{\beta_3}{\beta_1}\|Z\|_{L^2}^2+\|w\|_{L^2}^2+\|n\|_{L^2}^2\right)\lesssim(1+t)^{-\frac{3}{2}}N_0^2.$$

From the aforesaid inequality, we acquire (5.1). Multiplying (5.13) by $(1+t)^{\frac{9}{8}}$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^{\frac{9}{8}}W(t)+C_1(1+t)^{\frac{9}{8}}\left(W(t)+\|w_{tt}\|_{L^2}^2+\|n_{tt}\|_{L^2}^2\right)\\&\le C\left((1+t)^{\frac{9}{8}}(1+t)^{-\frac{5}{2}}N_0^2+(1+t)^{\frac{1}{8}}W(t)\right)\\&\le C\left((1+t)^{\frac{9}{8}}(1+t)^{-\frac{5}{2}}N_0^2+(1+t)^{\frac{1}{8}}(1+t)^{-\frac{5}{2}}N_0^2\right)\\&\le C(1+t)^{-\frac{11}{8}}N_0^2.\end{aligned} \tag{5.15}$$

(5.14) is integrated from 0 to t, we obtain (5.3). The proof is completed. □

Lemma 5.2. Under the assumptions of Theorem 1.1 meanwhile (4.1) are valid, then

$$\left\|\left(m^+,m^-\right)\right\|_{H^3}\le CN_0. \tag{5.16}$$

Proof. The new semi-linearized system of (1.1) is as follows:

$$\begin{cases}\partial_t m^+ +\left(1+m^+\right)\mathrm{div}w+w\cdot\nabla m^+=0,\\ \rho\partial_t w+\mathcal{L}\left(\eta\nabla m^+ +\rho\nabla m^-\right)-\rho^{\mu_1}\Delta w-\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\nabla\mathrm{div}w=E_1,\\ \partial_t m^- +\left(1+m^-\right)\mathrm{div}n+n\cdot\nabla m^-=0,\\ \eta\partial_t n+\mathcal{L}\left(\eta\nabla m^+ +\rho\nabla m^-\right)-\eta^{\mu_2}\Delta n-\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\nabla\mathrm{div}n=E_2,\end{cases} \tag{5.17}$$

where

$$E_1 = -\rho w\cdot\nabla w + \frac{\rho^{\mu_1}\left(\nabla w+\nabla^t w\right)\nabla\sigma^+}{\sigma^+} + \nabla\left(\rho^{\mu_1}\right)\left(\nabla w+\nabla^t w\right) + \frac{\nabla\sigma^+\rho^{\lambda_1}\mathrm{div}w}{\sigma^+} + \nabla\left(\rho^{\lambda_1}\right)\mathrm{div}w, \tag{5.18}$$

$$E_2 = -\eta v\cdot\nabla n + \frac{\eta^{\mu_2}\left(\nabla n+\nabla^t n\right)\nabla\sigma^-}{\sigma^-} + \nabla\left(\eta^{\mu_2}\right)\left(\nabla n+\nabla^t n\right) + \frac{\nabla\sigma^-\eta^{\lambda_2}\mathrm{div}n}{\sigma^-} + \nabla\left(\eta^{\lambda_2}\right)\mathrm{div}n. \tag{5.19}$$

Multiplying $\nabla^\ell(5.17)_1$, $\nabla^\ell(5.17)_3$ by $\nabla^\ell m^+$, $\nabla^\ell m^-$ separately, for $0\le\ell\le 2$, integrating over $\mathbb{R}^3$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left|\nabla^\ell m^\pm\right|^2\mathrm{d}x = -\int_{\mathbb{R}^3}\nabla^\ell\left(\mathrm{div}w+m^+\mathrm{div}w+w\cdot\nabla m^+\right)\nabla^\ell m^+\mathrm{d}x - \int_{\mathbb{R}^3}\nabla^\ell\left(\mathrm{div}n+m^-\mathrm{div}n+n\cdot\nabla m^-\right)\nabla^\ell m^-\mathrm{d}x. \tag{5.20}$$

Integrating by parts, we have

$$\int_{\mathbb{R}^3} w\cdot\nabla\nabla^\ell m^+\nabla^\ell m^+\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}w\left|\nabla^\ell m^+\right|^2\mathrm{d}x.$$

Similarly, we have

$$\int_{\mathbb{R}^3} n\cdot\nabla\nabla^\ell m^-\nabla^\ell m^-\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}n\left|\nabla^\ell m^-\right|^2\mathrm{d}x.$$

Then we deal with (5.20)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left|\nabla^\ell m^+\right|^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left|\nabla^\ell m^-\right|^2\mathrm{d}x \le C\|\nabla w\|_{H^2}\left\|m^+\right\|_{H^2} + C\|\nabla n\|_{H^2}\left\|m^-\right\|_{H^2}.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|m^+\right\|_{H^2} + \frac{\mathrm{d}}{\mathrm{d}t}\left\|m^-\right\|_{H^2} \le C\left(\|\nabla w\|_{H^2}+\|\nabla n\|_{H^2}\right). \tag{5.21}$$

By Theorem 5.1, we conclude that $\int_0^t\|\nabla w(\tau)\|_{H^2}\mathrm{d}\tau\le CN_0$, $\int_0^t\|\nabla n(\tau)\|_{H^2}\mathrm{d}\tau\le CN_0$. We integrate inequality from 0 to t, so $\left\|m^\pm\right\|_{H^2}\le CN_0$.

However, the uniform boundedness of the $L^1(0,t)$-norm of $\left\|\nabla^4 w\right\|$ and $\left\|\nabla^4 n\right\|$ is uncertain. Therefore, we need another way. The classic L^p-estimate of elliptic systems can be applied to (4.27), which yields

$$\begin{aligned}\left\|\nabla^2 w_t\right\|_{L^2} \lesssim{}& \left\|m_t^+\right\|_{L^3}\|w_t\|_{L^6} + \left\|\sigma^+\right\|_{L^6}\|\nabla Z_t\|_{L^3} + \left\|\sigma_t^+\right\|_{L^2}\|\nabla Z\|_{L^\infty} + \|w\|_{L^\infty}\|\nabla w\|_{L^2} \\ &+\|\nabla w\|_{L^3}\|w_t\|_{L^6} + \left\|\nabla\sigma^+\right\|_{L^\infty}\|Z_t\|_{L^2}\|\nabla w\|_{L^\infty} + \left\|\sigma_t^+\right\|_{L^6}\|\sigma\|_{L^\infty}\|\nabla w\|_{L^3} \\ &+\left\|\sigma_t^+\right\|_{L^6}\|\nabla Z\|_{L^\infty}\|\nabla w\|_{L^3} + \left\|\sigma^+\right\|_{L^\infty}\|\nabla Z\|_{L^\infty}\|\nabla w_t\|_{L^2} \\ &+\left\|\sigma^+\right\|_{L^\infty}\|\nabla\sigma_t\|_{L^2}\|\nabla w\|_{L^\infty} + \left\|\sigma_t^+\right\|_{L^\infty}\|Z\|_{L^\infty}\left\|\nabla^2 w\right\|_{L^2} \\ &+\left\|\sigma^+\right\|_{L^\infty}\|Z_t\|_{L^\infty}\left\|\nabla^2 w\right\|_{L^2} + \left\|\nabla\sigma^+\right\|_{L^\infty}\|Z\|_{L^\infty}\|\nabla w_t\|_{L^2} \\ &+\left\|\sigma^+\right\|_{L^\infty}\|\nabla Z\|_{L^\infty}\|\nabla w_t\|_{L^2} + \left(\left\|\sigma^+\right\|_{L^\infty}\|Z_t\|_{L^\infty} + \left\|\sigma_t^+\right\|_{L^\infty}\|Z\|_{L^\infty}\right)\|\nabla w_t\|_{L^2}^2 \\ \lesssim{}& \left\|\left(w_{tt},w_t,\nabla w_t\right)\right\|_{L^2} + \|\nabla w\|_{H^1}.\end{aligned} \tag{5.22}$$

Similarly,

$$\left\|\nabla^2 n_t\right\|_{L^2} \lesssim \left\|\left(n_{tt}, n_t, \nabla n_t\right)\right\|_{L^2} + \left\|\nabla n\right\|_{H^1}. \tag{5.23}$$

Multiplying ∇^3 (5.17)₂ and ∇^3 (5.17)₄ by $\nabla^3 w$ and $\nabla^3 n$, integrating over $\mathbb{R}^3$, then

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho\left|\nabla^3 w\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\rho^{\mu_1}\left|\nabla^4 w\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\left|\nabla^3\mathrm{div}w\right|^2\mathrm{d}x\\
&+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\eta\left|\nabla^3 n\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\eta^{\mu_2}\left|\nabla^4 n\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\left|\nabla^3\mathrm{div}n\right|^2\mathrm{d}x\\
&=\frac{1}{2}\int_{\mathbb{R}^3}\rho_t\left|\nabla^3 w\right|^2\mathrm{d}x+\eta_t\left|\nabla^3 n\right|^2\mathrm{d}x-\sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}\rho\nabla^z w_t\nabla^3 w\mathrm{d}x\\
&-\sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}\eta\nabla^z n_t\nabla^3 n\mathrm{d}x+\sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}\rho^{\mu_1}\cdot\nabla^z\Delta w\cdot\nabla^3 w\mathrm{d}x\\
&+\sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}\eta^{\mu_2}\cdot\nabla^z\Delta n\cdot\nabla^3 n\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla\left(\rho^{\mu_1}\right)\left|\nabla^4 w\right|\left|\nabla^3 w\right|+\nabla\left(\eta^{\mu_2}\right)\left|\nabla^4 n\right|\left|\nabla^3 n\right|\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\sum_{z=0}^{2}\nabla^{3-z}\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\nabla^z\nabla\mathrm{div}w\nabla^3 w\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\sum_{z=0}^{2}\nabla^{3-z}\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\nabla^z\nabla\mathrm{div}n\nabla^3 n\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\left|\nabla^3 w\right|\left|\nabla^3\mathrm{div}w\right|\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\left|\nabla^3 n\right|\left|\nabla^3\mathrm{div}n\right|\mathrm{d}x-\int_{\mathbb{R}^3}\nabla^3\left[\mathcal{L}\left(\eta\nabla m^+ +\rho\nabla m\right)\right]\nabla^3 w\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla^3\left[\mathcal{L}\left(\eta\nabla m^+ +\rho\nabla m\right)\right]\nabla^3 n\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^3 E_1\nabla^3 w\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^3 E_2\nabla^3 n\mathrm{d}x\\
&=\sum_{i=1}^{14} I_i.
\end{aligned}\tag{5.24}$$

For (1.14)₁, we have

$$\begin{aligned}
\left|I_1\right| &= \frac{1}{2}\int_{\mathbb{R}^3}\rho_t\left|\nabla^3 w\right|^2\mathrm{d}x+\eta_t\left|\nabla^3 n\right|^2\mathrm{d}x\\
&\lesssim \left\|\rho_t\right\|_{L^\infty}\left\|\nabla^3 w\right\|_{L^2}^2+\left\|\eta_t\right\|_{L^\infty}\left\|\nabla^3 n\right\|_{L^2}^2\\
&\lesssim \left\|w\right\|_{W^{1,\infty}}\left\|\nabla^3 w\right\|_{L^2}^2+\left\|n\right\|_{W^{1,\infty}}\left\|\nabla^3 n\right\|_{L^2}^2\\
&\lesssim \delta\left(\left\|\nabla^3 u\right\|_{L^2}^2+\left\|\nabla^3 v\right\|_{L^2}^2\right).
\end{aligned}\tag{5.25}$$

From (5.22), we have

$$\begin{aligned}
\left|I_2\right| &= -\sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}\rho\nabla^z w_t\nabla^3 w\mathrm{d}x\\
&\lesssim\left(\left\|\nabla^3\rho\right\|_{L^2}\left\|w_t\right\|_{L^\infty}+\left\|\nabla^2\rho\right\|_{L^3}\left\|\nabla w_t\right\|_{L^6}+\left[\nabla\rho\right]_{L^\infty}\left\|\nabla^2 w_t\right\|_{L^2}\right)\left\|\nabla^3 w\right\|_{L^2}\\
&\lesssim\delta\left(\left\|\left(w_{tt}, w_t, \nabla w_t\right)\right\|_{L^2}^2+\left\|\nabla w\right\|_{H^2}\right).
\end{aligned}\tag{5.26}$$

Similarly, we have

$$I_3 \lesssim \delta\left(\left\|\left(n_{tt}, v_t, \nabla n_t\right)\right\|_{L^2}^2+\left\|\nabla n\right\|_{H^2}\right), \tag{5.27}$$

$$
\begin{aligned}
|I_4| &\lesssim \|\nabla^2 Z\|_{L^2}\|\nabla^2 w\|_{L^3}\|\nabla^3 w\|_{L^6} + \|\nabla^2 Z\|_{L^2}\|\nabla^3 w\|_{L^6}\|\nabla^3 u\|_{L^3} \\
&\quad + \|\nabla Z\|_{L^\infty}\|\nabla^4 w\|_{L^2}\|\nabla^3 w\|_{L^2} \\
&\lesssim \delta\left(\|\nabla w_t\|_{L^2} + \|\nabla w\|_{H^2} + \|\nabla^4 w\|_{L^2}\right),
\end{aligned} \tag{5.28}
$$

$$
\begin{aligned}
|I_5| &= \sum_{z=0}^{2}\int_{\mathbb{R}^3} \nabla^{3-z}\eta^{\mu_2}\cdot\nabla^z \Delta n\cdot\nabla^3 n\mathrm{d}x \\
&\lesssim \delta\left(\|\nabla n_t\|_{L^2} + \|\nabla n\|_{H^2} + \|\nabla^4 n\|_{L^2}\right),
\end{aligned} \tag{5.29}
$$

$$
\begin{aligned}
I_6 &= -\int_{\mathbb{R}^3} \nabla\rho^{\mu_1}|\nabla^4 w||\nabla^3 w| + \nabla\left(\eta^{\mu_2}\right)|\nabla^4 n||\nabla^3 n|\mathrm{d}x \\
&\lesssim \|\nabla Z\|_{L^\infty}\|\nabla^4 w\|_{L^2}\|\nabla^3 w\|_{L^2} + \|\nabla Z\|_{L^\infty}\|\nabla^4 n\|_{L^2}\|\nabla^3 n\|_{L^2} \\
&\lesssim \delta\left(\|\nabla^4 w\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2\right).
\end{aligned} \tag{5.30}
$$

Using the similar process to (5.28) and (5.29), we have

$$
\begin{aligned}
I_7 &= -\int_{\mathbb{R}^3}\sum_{z=0}^{2}\nabla^{3-z}\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\nabla^z\nabla\mathrm{div}w\nabla^3 w\mathrm{d}x \\
&\lesssim \delta\left(\|\nabla w_t\|_{L^2} + \|\nabla w\|_{H^2} + \|\nabla^4 w\|_{L^2}\right),
\end{aligned} \tag{5.31}
$$

$$
\begin{aligned}
I_8 &= -\int_{\mathbb{R}^3}\sum_{z=0}^{2}\nabla^{3-z}\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\nabla^z\nabla\mathrm{div}n\nabla^3 n\mathrm{d}x \\
&\lesssim \delta\left(\|\nabla n_t\|_{L^2} + \|\nabla n\|_{H^2} + \|\nabla^4 n\|_{L^2}\right),
\end{aligned} \tag{5.32}
$$

$I_9 + I_{10}$ is similar to I_6, we have

$$
I_9 + I_{10} \lesssim \delta\left(\|\nabla^4 w\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2 + \|\nabla^4 n\|_{L^2}^2 + \|\nabla^3 n\|_{L^2}^2\right), \tag{5.33}
$$

$$
\begin{aligned}
I_{11} &= -\int_{\mathbb{R}^3}\nabla^3\left[\mathcal{L}\left(\eta\nabla m^+ + \rho\nabla m^-\right)\right]\nabla^3 w\mathrm{d}x \\
&= -\int_{\mathbb{R}^3}\mathcal{L}\eta\nabla^3 m^+\nabla^3\left(\frac{m_t^+ + w\cdot\nabla m^+}{1+m^+}\right)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\mathcal{L}\rho\nabla^3 m^-\nabla^3\left(\frac{m_t^- + n\cdot\nabla m^-}{1+m^-}\right)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\left[\nabla(\mathcal{L}\eta)\nabla^3 m^+ + \nabla(\mathcal{C}\rho)\nabla^3 m^-\right]\nabla^3 w\mathrm{d}x \\
&\quad - \sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}(\mathcal{L}\eta)\nabla\nabla^z m^+\nabla^3 w\mathrm{d}x \\
&\quad - \sum_{z=0}^{2}\int_{\mathbb{R}^3}\nabla^{3-z}(\mathcal{L}\rho)\nabla\nabla^z m^-\nabla^3 w\mathrm{d}x \\
&\le -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\mathcal{L}\eta\left|\nabla^3 m^+\right|^2}{1+m^+}\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\mathcal{L}\rho\left|\nabla^3 m^-\right|^2}{1+m^-}\mathrm{d}x \\
&\quad + C\delta\left(\|(w_{tt}, n_{tt}, w_t, n_t, \nabla w_t, \nabla n_t)\|_{L^2} + \|(\nabla w, \nabla n)\|_{H^2}\right).
\end{aligned} \tag{5.34}
$$

Using the similar process to (5.34), we obtain

$$
\begin{aligned}
I_{12} &\le -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\mathcal{L}\rho\left|\nabla^3 m^-\right|^2}{1+m^-}\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{\mathcal{L}\eta\left|\nabla^3 m^+\right|^2}{1+m^+}\mathrm{d}x \\
&\quad + C\delta\left(\|(w_{tt}, n_{tt}, w_t, n_t, \nabla w_t, \nabla n_t)\|_{L^2} + \|(\nabla w, \nabla n)\|_{H^2}\right).
\end{aligned} \tag{5.35}
$$

By applying integration by parts, we have

$$
\begin{aligned}
|I_{13}| &\le C\left\|\nabla^2 E_1\right\|_{L^2}\left\|\nabla^4 w\right\|_{L^2} \\
&\le C\Big(\left\|\nabla^2 w\right\|_{L^2}\left\|\nabla w\right\|_{L^\infty}+\left\|\nabla^3 w\right\|_{L^2}\left\|\nabla w\right\|_{L^\infty}+\left\|\nabla^2 Z\right\|_{L^2}\left\|\nabla^2 w\right\|_{L^\infty}\left\|\nabla m^+\right\|_{L^\infty} \\
&\quad+\left\|\nabla^2 Z\right\|_{L^2}\left\|\nabla w\right\|_{L^\infty}\left\|\nabla^2 m^+\right\|_{L^\infty}+\left\|Z\right\|_{L^\infty}\left\|\nabla^3 w\right\|_{L^2}\left\|\nabla^2 m^+\right\|_{L^\infty} \\
&\quad+\left\|Z\right\|_{L^\infty}\left\|\nabla w\right\|_{L^2}\left\|\nabla m^+\right\|_{L^\infty}\Big)\left\|\nabla^4 w\right\|_{L^2} \\
&\le C\delta\left(\left\|\nabla w_t\right\|_{L^2}^2+\left\|\nabla w\right\|_{H^2}^2+\left\|\nabla^4 w\right\|_{L^2}^2\right).
\end{aligned}
\tag{5.36}
$$

Similarly, for I_{14}, we have

$$
I_{14} \le C\delta\left(\left\|\nabla n_t\right\|_{L^2}^2+\left\|\nabla n\right\|_{H^2}^2+\left\|\nabla^4 n\right\|_{L^2}^2\right). \tag{5.37}
$$

Substituting (5.25)-(5.37) into (5.24) yields,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho\left|\nabla^3 w\right|^2+\eta\left|\nabla^3 n\right|^2+\frac{\mathcal{L}\eta\left|\nabla^3 m^+\right|^2}{1+m^+}+\int_{\mathbb{R}^3}\frac{\mathcal{L}\rho\left|\nabla^3 m^-\right|^2}{1+m^-}+\int_{\mathbb{R}^3}\frac{\mathcal{L}\rho\left|\nabla^3 m^-\right|^2}{1+m^-} \\
&+\int_{\mathbb{R}^3}\frac{\mathcal{L}\eta\left|\nabla^3 m^+\right|^2}{1+m^+}\mathrm{d}x+\int_{\mathbb{R}^3}\rho^{\mu_1}\left|\nabla^4 w\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\rho^{\mu_1}+\rho^{\lambda_1}\right)\left|\nabla^3\mathrm{div}w\right|^2\mathrm{d}x \\
&+\int_{\mathbb{R}^3}\eta^{\mu_2}\left|\nabla^4 n\right|^2\mathrm{d}x+\int_{\mathbb{R}^3}\left(\eta^{\mu_2}+\eta^{\lambda_2}\right)\left|\nabla^3\mathrm{div}n\right|^2\mathrm{d}x \\
&\le C\delta\left(\left\|\left(w_{tt},w_t,\nabla w_t\right)\right\|_{L^2}+\left\|\nabla w\right\|_{H^2}+\left\|\left(n_{tt},n_t,\nabla n_t\right)\right\|_{L^2}+\left\|\nabla n\right\|_{H^2}\right).
\end{aligned}
\tag{5.38}
$$

In virtue of (5.2) and (5.3), for any $t \ge 0$, then

$$
\int_0^t\left(\left\|\left(w_{tt},n_{tt},w_t,n_t,\nabla w_t,\nabla n_t\right)(\tau)\right\|_{L^2}+\left\|\left(\nabla w,\nabla n\right)(\tau)\right\|_{H^2}\right)\mathrm{d}\tau \le CN_0.
$$

we deduce that

$$
\left\|\nabla^3 m^\pm\right\|_{L^2} \le CN_0. \tag{5.39}
$$

The proof is completed. □

Proof of Theorem 1.1 and 1.2. We can immediately derive Theorem 1.1 and Theorem 1.2 by Theorem 5.1 and Lemma 5.2.

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Conflicts of Interest

All authors declare no competing interests.

References

[1] Gao, D., Zhang, S., Ning, Z., Chen, W. and Li, H. (2021) On the Coupling Mechanism of Rain-Wind Two-Phase Flow Induced Cable Vibration: A Wake-Dynamics Perspective. *Physics of Fluids*, **33**, Article 117102. https://doi.org/10.1063/5.0065535

[2] Rastello, M. and Hopfinger, E.J. (2004) Sediment-Entraining Suspension Clouds: A Model of Powder-Snow Avalanches. *Journal of Fluid Mechanics*, **509**, 181-206.

https://doi.org/10.1017/s0022112004009322

[3] Dai, W., Wu, P. and Liu, G. (2020) A Two-Phase Flow Approach for Modeling Blood Stasis and Estimating the Thrombosis Potential of a Ventricular Assist Device. *The International Journal of Artificial Organs*, **44**, 471-480. https://doi.org/10.1177/0391398820975405

[4] Garcia Escorcia, A.E. (2013) Mathematical Modeling of Two-Phase Arterial Blood Flow. The University of Texas-Pan American, State of Texas.

[5] Dong, P. and Zhang, K. (1999) Two-Phase Flow Modelling of Sediment Motions in Oscillatory Sheet Flow. *Coastal Engineering*, **36**, 87-109. https://doi.org/10.1016/s0378-3839(98)00052-0

[6] Muste, M., Yu, K., Fujita, I. and Ettema, R. (2005) Two-Phase versus Mixed-Flow Perspective on Suspended Sediment Transport in Turbulent Channel Flows. *Water Resources Research*, **41**, W10402. https://doi.org/10.1029/2004wr003595

[7] Li, Y., Li, C., Chen, E. and Ying, Y. (2011) Pressure Wave Propagation Characteristics in a Two-Phase Flow Pipeline for Liquid-Propellant Rocket. *Aerospace Science and Technology*, **15**, 453-464. https://doi.org/10.1016/j.ast.2010.09.011

[8] Wang, J. and Yang, Z. (2019) Effect of Non-Spherical Particles on Nozzle Two-Phase Flow Loss in Nano-Iron Powder Metal Fuel Motor. *Aerospace Science and Technology*, **91**, 372-381. https://doi.org/10.1016/j.ast.2019.05.022

[9] Babbs, C. and Steiner, M. (1990) Simulation of Free Radical Reactions in Biology and Medicine: A New Two-Compartment Kinetic Model of Intracellular Lipid Peroxidation. *Free Radical Biology and Medicine*, **8**, 471-485. https://doi.org/10.1016/0891-5849(90)90060-v

[10] Bannari, R., Kerdouss, F., Selma, B., Bannari, A. and Proulx, P. (2008) Three-Dimensional Mathematical Modeling of Dispersed Two-Phase Flow Using Class Method of Population Balance in Bubble Columns. *Computers & Chemical Engineering*, **32**, 3224-3237. https://doi.org/10.1016/j.compchemeng.2008.05.016

[11] Breward, C.J.W., Byrne, H.M. and Lewis, C.E. (2002) The Role of Cell-Cell Interactions in a Two-Phase Model for Avascular Tumour Growth. *Journal of Mathematical Biology*, **45**, 125-152. https://doi.org/10.1007/s002850200149

[12] Du, Z., Yu, Y. and Wang, J. (2007) Extraction of Proteins from Biological Fluids by Use of an Ionic Liquid/Aqueous Two-Phase System. *Chemistry—A European Journal*, **13**, 2130-2137. https://doi.org/10.1002/chem.200601234

[13] Huh, D., Tung, Y., Wei, H., Grotberg, J.B., Skerlos, S.J., Kurabayashi, K., *et al.* (2002) Use of Air-Liquid Two-Phase Flow in Hydrophobic Microfluidic Channels for Disposable Flow Cytometers. *Biomedical Microdevices*, **4**, 141-149. https://doi.org/10.1023/a:1014691416614

[14] Azbel, D. (1982) Two Phase Flows in Chemical Engineering. Cambridge University Press.

[15] Cheng, L.Y., Drew, D.A. and Lahey, R.T. (1985) An Analysis of Wave Propagation in Bubbly Two-Component, Two-Phase Flow. *Journal of Heat Transfer*, **107**, 402-408. https://doi.org/10.1115/1.3247429

[16] Kiambi, S.L., Kiriamiti, H.K. and Kumar, A. (2011) Characterization of Two Phase Flows in Chemical Engineering Reactors. *Flow Measurement and Instrumentation*, **22**, 265-271. https://doi.org/10.1016/j.flowmeasinst.2011.03.006

[17] Li, Y. and Sun, Y. (2021) On Global-in-Time Weak Solutions to a Two-Dimensional Full Compressible Nonresistive MHD System. *SIAM Journal on Mathematical Analysis*, **53**, 4142-4177. https://doi.org/10.1137/20m1379940

[18] Bresch, D., Huang, X. and Li, J. (2011) Global Weak Solutions to One-Dimensional Non-Conservative Viscous Compressible Two-Phase System. *Communications in Mathematical Physics*, **309**, 737-755. https://doi.org/10.1007/s00220-011-1379-6

[19] Yao, L., Zhang, T. and Zhu, C. (2010) Existence and Asymptotic Behavior of Global Weak Solutions to a 2D Viscous Liquid-Gas Two-Phase Flow Model. *SIAM Journal on Mathematical Analysis*, **42**, 1874-1897. https://doi.org/10.1137/100785302

[20] Vasseur, A., Wen, H. and Yu, C. (2019) Global Weak Solution to the Viscous Two-Fluid Model with Finite Energy. *Journal de Mathématiques Pures et Appliquées*, **125**, 247-282. https://doi.org/10.1016/j.matpur.2018.06.019

[21] Cui, H., Wang, W., Yao, L. and Zhu, C. (2016) Decay Rates for a Nonconservative Compressible Generic Two-Fluid Model. *SIAM Journal on Mathematical Analysis*, **48**, 470-512. https://doi.org/10.1137/15m1037792

[22] Wu, G., Yao, L. and Zhang, Y. (2023) Global Well-Posedness and Large Time Behavior of Classical Solutions to a Generic Compressible Two-Fluid Model. *Mathematische Annalen*, **389**, 3379-3415. https://doi.org/10.1007/s00208-023-02732-5

[23] Bresch, D. and Noble, P. (2007) Mathematical Justification of a Shallow Water Model. *Methods and Applications of Analysis*, **14**, 87-118. https://doi.org/10.4310/maa.2007.v14.n2.a1

[24] Bresch, D. and Noble, P. (2011) Mathematical Derivation of Viscous Shallow-Water Equations with Zero Surface Tension. *Indiana University Mathematics Journal*, **60**, 1137-1170. https://doi.org/10.1512/iumj.2011.60.4273

[25] Lions P.L. (1998) Mathematical Topics in Fluid Mechanics. Oxford University Press.

[26] Marche F. (2007) Derivation of A New Two-Dimensional Viscous Shallow Water Model with Varying Topography, Bottom Friction and Capillary Effects. *European Journal of Mechanics-B/Fluids*, **26**, 49-63. https://doi.org/10.1016/j.euromechflu.2006.04.007

[27] Chen, M., Xu, X. and Zhang, J. (2015) Global Weak Solutions of 3D Compressible Micropolar Fluids with Discontinuous Initial Data and Vacuum. *Communications in Mathematical Sciences*, **13**, 225-247. https://doi.org/10.4310/cms.2015.v13.n1.a11

[28] Zodji, S.M. (2023) A Well-Posedness Result for the Compressible Two-Fluid Model with Density Dependent Viscosity. https://arxiv.org/abs/2310.12525

[29] Li, H.L., Wang, Y. and Zhang, Y. (2024) Non-Existence of Classical Solutions to a Two-Phase Flow Model with Vacuum. https://arxiv.org/abs/2402.02531

[30] Bresch, D., Desjardins, B., Ghidaglia, J.-M. and Grenier, E. (2009) Global Weak Solutions to a Generic Two-Fluid Model. *Archive for Rational Mechanics and Analysis*, **196**, 599-629. https://doi.org/10.1007/s00205-009-0261-6

[31] Evje, S., Wang, W. and Wen, H. (2016) Global Well-Posedness and Decay Rates of Strong Solutions to a Non-Conservative Compressible Two-Fluid Model. *Archive for Rational Mechanics and Analysis*, **221**, 1285-1316. https://doi.org/10.1007/s00205-016-0984-0

[32] Danchin, R. (2000) Global Existence in Critical Spaces for Compressible Navier-Stokes Equations. *Inventiones Mathematicae*, **141**, 579-614. https://doi.org/10.1007/s002220000078

[33] Chen, Q. and Tan, Z. (2010) Global Existence and Convergence Rates of Smooth Solutions for the Compressible Magnetohydrodynamic Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **72**, 4438-4451. https://doi.org/10.1016/j.na.2010.02.019

[34] Nirenberg, L. (1959) On Elliptic Partial Differential Equations. *Annali della Scuola*

Normale Superiore di Pisa-Scienze Fisiche e Matematiche, **13**, 115-162.

[35] Kato, T. and Ponce, G. (1988) Commutator Estimates and the Euler and Navier-Stokes Equations. *Communications on Pure and Applied Mathematics*, **41**, 891-907. https://doi.org/10.1002/cpa.3160410704

[36] Kenig, C.E., Ponce, G. and Vega, L. (1991) Well-Posedness of the Initial Value Problem for the Korteweg-De Vries Equation. *Journal of the American Mathematical Society*, **4**, 323-347. https://doi.org/10.1090/s0894-0347-1991-1086966-0

[37] Matsumura, A. and Nishida, T. (1979) The Initial Value Problem for the Equations of Motion of Compressible Viscous and Heat-Conductive Fluids. *Proceedings of the Japan Academy*, *Series A*, *Mathematical Sciences*, **55**, 337-342. https://doi.org/10.3792/pjaa.55.337

[38] Matsumura, A. and Nishida, T. (1980) The Initial Value Problem for the Equations of Motion of Viscous and Heat-Conductive Gases. *Kyoto Journal of Mathematics*, **20**, 67-104. https://doi.org/10.1215/kjm/1250522322

[39] Kobayashi, T. and Shibata, Y. (1999) Decay Estimates of Solutions for the Equations of Motion of Compressible Viscous and Heat-Conductive Gases in An Exterior Domain in $\mathbb{R}^3$. *Communications in Mathematical Physics*, **200**, 621-659.