Journal of Applied Mathematics and Physics, 2024, 12, 3163-3184
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Legendre-Weighted Residual Methods for System of Fractional Order Differential Equations

Umme Ruman[1], Md. Shafiqul Islam[2*]

[1]Department of Computer Science & Engineering, Green University of Bangladesh, Dhaka, Bangladesh
[2]Department of Applied Mathematics, University of Dhaka, Dhaka, Bangladesh
Email: ruman@cse.green.edu.bd, *mdshafiqul@du.ac.bd

How to cite this paper: Ruman, U. and Islam, Md.S. (2024) Legendre-Weighted Residual Methods for System of Fractional Order Differential Equations. *Journal of Applied Mathematics and Physics*, **12**, 3163-3184.
https://doi.org/10.4236/jamp.2024.129190

Received: August 9, 2024
Accepted: September 24, 2024
Published: September 27, 2024

Abstract

The numerical approach for finding the solution of fractional order systems of boundary value problems (BPVs) is derived in this paper. The implementation of the weighted residuals such as Galerkin, Least Square, and Collocation methods are included for solving fractional order differential equations, which is broadened to acquire the approximate solutions of fractional order systems with differentiable polynomials, namely Legendre polynomials, as basis functions. The algorithm of the residual formulations of matrix form can be coded efficiently. The interpretation of Caputo fractional derivatives is employed here. We have demonstrated these methods numerically through a few examples of linear and nonlinear BVPs. The results in absolute errors show that the present method efficiently finds the numerical solutions of fractional order systems of differential equations.

Keywords

Fractional Differential Equations, System of Fractional Order BVPs, Weighted Residual Methods, Modified Legendre Polynomials

1. Introduction

Since the systems of ordinary differential Equations (ODEs) are applied in computational science, engineering, physics and biology, the approximate solution of the linear and nonlinear systems of boundary value problems (BVPs) is derived for better accuracy. For instance, solving BVPs with the finite difference method in linear cases is widely accepted by Cheng and Zhong [1] in which they have deduced the positive solutions for ODEs in second order. A new method has been introduced by using series in reproducing kernel space for acquiring the solution for the second order differential equation by Geng and Cui [2]. Exploiting variational iteration method for solving a nonlinear system of second-order BVPs has been established in [3], and the Galerkin method has been developed in [4] for the numerical solution of systems of second-order BVPs.

As and when the order of ODEs is fractional, then the generalization turns to fractional differential Equations (FDEs), firm effects to acquire variational changes more accurately than the regular ODEs, which are used to model and process of anomalous diffusion, viscoelastic materials, engineering control systems, signal processing, modeling biological systems, finance option pricing and hydrology groundwater flow. For this, researchers from several fields of science and engineering are paying attention to the FDEs that deal with dynamic systems for more convergence. The fractional difference integrals played a vital role in the time domain analysis of fractional dynamical systems and were used to solve problems of control theory [5].

The Galerkin method was recently implemented in [6] to find the numerical solutions to linear fractional order two-point BVPs with homogeneous and nonhomogeneous boundary conditions using differentiable polynomials. Chebyshev collocation method incorporated by Khader *et al.* [7] [8] for solving high-order FDEs. The analytic study of the existence and uniqueness solution of initial value problems for fractional order systems was extensively reported in [9]-[11]. The approximate solutions of linear and nonlinear systems of fractional differential equations with the initial value problems developed by different methods such as the homotopy analysis method [12], the fractional finite difference method with Chebyshev polynomial, shifted fractional order Jacobi orthogonal functions [13], Adomian decomposition method [14], differential transform method [15], Haar wavelet collocation methods in [16], and so on.

Azizi [17] has established the Chebyshev finite difference method for a system of fractional BVPs. A coupled system of nonlinear FDEs has been derived by Xinwei in [18]. Bernstein polynomials have been used to find the approximate analytical solution for nonlinear systems of FDEs with boundary conditions by Alipour and Baleanu in [19]. The existence and uniqueness of solution have been studied for the system of periodic fractional BVPs by Dhaigude *et al.* [20]. Very recently, an efficient matrix method for a couple of systems of fractional ODEs has been discussed in [21]. Adomian decomposition method for solving nonlinear system of fractional differential equations was established by Ziada in [22]. The oscillatory theory for two classes of fractional neutral differential equations was derived in [23]. Mu'lla [24] has discussed about the existence and uniqueness of the solution in an alternative way for FDEs elaborately using Appell and Lauricella hypergeometric functions.

Thus, from the above literature review, we may observe that some methods provide poor accuracy, and some are costlier in computation. Thus, we are motivated to find an efficient numerical technique to find approximate solutions for the

FDES system. However, in this research work, we consider the linear systems of fractional order differential equations in two unknown functions, namely, $u(x)$ and $v(x)$ in the following systems [4]:

$$\left.\begin{aligned} a_2 D^{\alpha} u(x) + a_1 u(x) + a_0 v(x) = f_1(x) \\ b_2 D^{\beta} v(x) + b_1 v(x) + b_0 u(x) = f_2(x) \end{aligned}\right\} \quad (1)$$

where $f_1(x), f_2(x)$ are given functions, and a_i, b_i are coefficients for $i = 0,1,2$ and $0 < \alpha, \beta \le 2$.

We also consider the nonlinear systems of fractional order differential equations in two unknown functions: $u(x)$ and $v(x)$ in a system of the form:

$$\left.\begin{aligned} a_2 D^{\alpha} u(x) + a_1 u(x) + a_0 v(x) + N_1(u(x), v(x)) = f_1(x) \\ b_2 D^{\beta} v(x) + b_1 v(x) + b_0 u(x) + N_2(u(x), v(x)) = f_2(x) \end{aligned}\right\} \quad (2)$$

where, $f_1(x), f_2(x)$ are given functions, N_1, N_2 are nonlinear functions, and a_i, b_i are coefficients for $i = 0,1,2$ and $0 < \alpha, \beta \le 2$.

The proposed research work in this study is appraised numerically by the weighted residual methods, such as Galerkin, Least Square and Collocation, for solving linear and nonlinear systems of fractional order initial and boundary value problems. However, the organization of this paper is as follows.

Section 2 describes some essential ingredients, such as definitions of fractional order derivatives and integrals of fractional calculus, including modified Legendre polynomials. The mathematical formulations of the proposed methods for systems of FDEs are explained in Section 3. Section 4 is reserved for validating the proposed techniques. The obtained numerical solutions to the specific problems and the corresponding absolute errors using different approaches are presented in tabular and graphic form. Finally, the conclusion and references are included.

2. Some Preliminaries and Notations

Mittag-Leffler function: The Mittag-Leffler function is first introduced as a one-parameter generalized function of the exponential form by the series [5]:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha > 0, \alpha \in \mathbb{R}, z \in \mathbb{C}.$$

The two-parameter generalization is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta > 0, \ \alpha, \beta \in \mathbb{R}, \ z \in \mathbb{C}$$

Definition: The Caputo fractional derivative D_*^{α} of order α of $u(x)$ is defined in the following form:

$$D_*^{\alpha} u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-y)^{m-\alpha-1} f^{(m)}(y) \mathrm{d}y,$$

where $\alpha > 0$.

Linearity: Caputo fractional order derivative operator is a linear operation:

$$D_*^\alpha\left(a_1u(x)+a_2v(x)\right)=a_1D_*^\alpha u(x)+a_2D_*^\alpha v(x),$$

where a_i is constant for $i=1,2$.

For Caputo fractional derivative we have,

1) $D_*^\alpha C=0$,

2) $D_*^\alpha x^m=\begin{cases}0, & \text{for } m\in N\\ \dfrac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}x^{m-\alpha}, & \text{for } m\in N\end{cases}.$

Exponential function: Let $\alpha\in\mathbb{R}, n-1<\alpha<n, n\in\mathbb{N}, \lambda\in\mathbb{C}$, then the Caputo fractional derivative of the exponential function has the form:

$$D^\alpha e^{\lambda t}=\lambda^n t^{n-\alpha}E_{1,n-\alpha+1}(\lambda t).$$

Other frequently used functions:

Let $\alpha\in\mathbb{R}, n-1<\alpha<n, n\in\mathbb{N}, \lambda\in\mathbb{C}$ then

$$D^\alpha\sin\lambda t=-\frac{1}{2}i(i\lambda)^n t^{n-\alpha}\left(E_{1,n-\alpha+1}(i\lambda t)-(-1)^n\left(E_{1,n-\alpha+1}(-i\lambda t)\right)\right).$$

$$D^\alpha\cos\lambda t=\frac{1}{2}(i\lambda)^n t^{n-\alpha}\left(E_{1,n-\alpha+1}(i\lambda t)+(-1)^n\left(E_{1,n-\alpha+1}(-i\lambda t)\right)\right).$$

Modified Legendre Polynomials: The analogue of Rodrigues formula for the Legendre polynomials is given by [6]:

$$L_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}\left(x^2-x\right)^n.$$

To satisfy the condition $L_n(0)=L_n(1)=0, n\geq 1$, we modify the Legendre polynomials: $L_n(x)=\left[\frac{1}{n!}\frac{d^n}{dx^n}\left(x^2-x\right)^n-(-1)^n\right](x-1)$.

We can write the first three ($n=3$) modified Legendre polynomials over the interval [0, 1] which are used throughout this paper:

$$L_1(x)=-2x+2x^2$$

$$L_2(x)=6x-12x^2+6x^3$$

$$L_3(x)=-12x+42x^2-50x^3+20x^4$$

The special properties of the modified Legendre polynomials are:

$$L_n(0)=0 \quad\text{and}\quad L_n(1)=0, n\geq 1.$$

The modified Legendre polynomials are smooth, continuous, differentiable and integrable. Thus, the set of basis functions satisfying the corresponding homogeneous boundary conditions are exploited in the matrix formulation of fractional order boundary value problems over the interval $[0,1]$.

3. Mathematical Formulations

Galerkin formulation for the system of fractional order BVPs:

Let us consider three-term linear fractional order systems:

$$\left.\begin{aligned} D^{\alpha}u(x)+p(x)u(x)+q(x)v(x)=f(x) \\ D^{\beta}v(x)+r(x)v(x)+s(x)u(x)=g(x) \end{aligned}\right\} \quad (3)$$

with the boundary conditions:

$$u(0)=u(1)=v(0)=v(1)=0 \text{ and } 0<x<1. \quad (4)$$

We assume two approximate solutions for the pair of functions $u(x)$ and $v(x)$ of the system (3) are given by:

$$\left.\begin{aligned} \tilde{u}(x)=\sum\nolimits_{j=1}^{n} a_j L_j(x),\, n\geq 1 \\ \tilde{v}(x)=\sum\nolimits_{j=1}^{n} b_j L_j(x),\, n\geq 1 \end{aligned}\right\} \quad (5)$$

where a_j and b_j are parameter, $L_j(x)$ are polynomial functions which satisfy boundary conditions (4).

Now we apply Galerkin formulation described in system (3), and we obtain the system of residual equations:

$$\left.\begin{aligned} \int_0^1 \left(D^{\alpha}u(x)+p(x)u(x)+q(x)v(x)\right)L_i(x)\mathrm{d}x=\int_0^1 f(x)L_i(x)\mathrm{d}x \\ \int_0^1 \left(D^{\beta}v(x)+r(x)v(x)+s(x)u(x)\right)L_i(x)\mathrm{d}x=\int_0^1 g(x)L_i(x)\mathrm{d}x \end{aligned}\right\} \quad (6)$$

On using integration by parts we can obtain the following system:

$$\left.\begin{aligned} \int_0^1 \left(D^{\alpha}\tilde{u}(x)L_i(x)+p(x)\tilde{u}(x)L_i(x)+q(x)\tilde{v}(x)L_i(x)\right)\mathrm{d}x=\int_0^1 f(x)L_i(x)\mathrm{d}x \\ \int_0^1 \left(D^{\beta}\tilde{v}(x)L_i(x)+r(x)\tilde{v}(x)L_i(x)+s(x)\tilde{u}(x)L_i(x)\right)\mathrm{d}x=\int_0^1 g(x)L_i(x)\mathrm{d}x \end{aligned}\right\} \quad (7)$$

Now putting the Equation (5) into (7) to get

$$\int_0^1 \left(\sum_{j=1}^{n} a_j L_j(x)^{\alpha} L_i(x)+p(x)\sum_{j=1}^{n} a_j L_j(x)L_i(x)+q(x)\sum_{j=1}^{n} b_j L_j(x)L_i(x)\right)\mathrm{d}x$$
$$=\int_0^1 f(x)L_i(x)\mathrm{d}x$$

$$\int_0^1 \left(\sum_{j=1}^{n} b_j L_j(x)^{\beta} L_i(x)+r(x)\sum_{j=1}^{n} b_j L_j(x)L_i(x)+s(x)\sum_{j=1}^{n} a_j L_j(x)L_i(x)\right)\mathrm{d}x$$
$$=\int_0^1 g(x)L_i(x)\mathrm{d}x$$

We can write the above equations as

$$\sum_{j=1}^{n}\left(\int_0^1 \left[\left(L_j(x)^{\alpha} L_i(x)\right)a_j+\left(p(x)L_j(x)L_i(x)\right)a_j+\left(q(x)L_j(x)L_i(x)\right)b_j\right]\mathrm{d}x\right)$$
$$=\int_0^1 f(x)L_i(x)\mathrm{d}x$$

$$\sum_{j=1}^{n}\left(\int_0^1 \left[\left(L_j(x)^{\beta} L_i(x)\right)b_j+\left(r(x)L_j(x)L_i(x)\right)b_j+\left(s(x)L_j(x)L_i(x)\right)a_j\right]\mathrm{d}x\right)$$
$$=\int_0^1 g(x)L_i(x)\mathrm{d}x$$

where $i=1,2,3,\cdots,n$.

Equivalently, the matrix formulation is given by

$$\left.\begin{aligned}\sum_{j=1}^{n}\left(A_{j,i}a_j+B_{j,i}b_j\right)=F_i\\ \sum_{j=1}^{n}\left(C_{j,i}b_j+D_{j,i}a_j\right)=G_i\end{aligned}\right\} \tag{8}$$

where,

$$A_{j,i}=\int_0^1\left[\left(L_j(x)^{\alpha}L_i(x)\right)a_j+\left(p(x)L_j(x)L_i(x)\right)a_j\right]\mathrm{d}x$$

$$C_{j,i}=\int_0^1\left[\left(L_j(x)^{\beta}L_i(x)\right)b_j+\left(r(x)L_j(x)L_i(x)\right)b_j\right]\mathrm{d}x$$

$$B_{j,i}=\int_0^1\left[\left(q(x)L_j(x)L_i(x)\right)b_j\right]\mathrm{d}x,\ F_i=\int_0^1 f(x)L_i(x)\mathrm{d}x$$

$$D_{j,i}=\int_0^1\left[\left(s(x)L_j(x)L_i(x)\right)a_j\right]\mathrm{d}x,\ G_i=\int_0^1 g(x)L_i(x)\mathrm{d}x \quad \text{for } i,j=1,2,3,\cdots,n.$$

Once we get the system of linear Equations (8), a_j and b_j can be obtained easily.

Least Square formulation for the system of fractional order BVPs:

In this case, weighting function is chosen to be

$$W_j=\frac{\partial R_1(x)}{\partial m_j} \text{ and } Z_j=\frac{\partial R_2(x)}{\partial n_j},\ j=1,2,\cdots,n$$

Now this choice of W_j and Z_j corresponds to minimize the mean square residual

$$WR_1=\frac{1}{2}\int R_1^2\mathrm{d}x = \text{minimum and } ZR_2=\frac{1}{2}\int R_2^2\mathrm{d}x = \text{minimum}$$

The necessary condition for WR to be minimum are given by

$$\left.\begin{aligned}\frac{\partial WR_1}{\partial m_j}=\int_0^1 R_1(x)\frac{\partial R_1}{\partial m_j}\mathrm{d}x=0,\ j=1,2,\cdots,n\\ \frac{\partial ZR_2}{\partial n_j}=\int_0^1 R_2(x)\frac{\partial R_2}{\partial n_j}\mathrm{d}x=0,\ j=1,2,\cdots,n\end{aligned}\right\} \tag{9}$$

which is clearly the matrix form of a system of n linear equations with the coefficients m_j and n_j. Solving the system (9), we find the values of some parameters and substitute into Equation (5); we are able to find the approximate solution of the desired FBVP (3).

Collocation formulation for the system of fractional order BVPs:

We assume that the boundary condition $[a,b]$ such as boundary conditions $u(a)=a_0$, $u(b)=b_0$ and $v(a)=a_1$, $v(b)=b_1$. In this case, we choose n parameters and the grid points as:

$$x_j=\frac{a+j}{n+1},$$

where $j=1,2,3,\cdots,n$.

Since the residual functions of (3) are:

$$\left.\begin{aligned}R_1(x)=p(x)D\tilde{u}(x)+q(x)D^{\alpha}\tilde{u}(x)+\tilde{v}(x)-f(x)\\ R_2(x)=r(x)D\tilde{v}(x)+s(x)D^{\beta}\tilde{v}(x)+\tilde{u}(x)-g(x)\end{aligned}\right\} \tag{10}$$

Setting $R(x_j)=0$, we obtain the system of unknown parameter a_j and b_j. Using the values of parameters, we get the approximate solution of system of fractional order boundary value problems.

4. Numerical Simulations

To demonstrate the effectiveness of the proposed methods in this literature, four examples are validated in this section. Compare the results of several fractional orders α and β, we will establish that the present method is very effective and convenient for all orders of α and β.

The efficiency and reliability of the proposed method are validated by computing the maximum absolute error L_∞ as:

$$L_\infty = \max\left|u(x)-\tilde{u}(x)\right| \text{ and } L_\infty = \max\left|v(x)-\tilde{v}(x)\right|,$$

where $u(x), v(x)$ and $\tilde{u}(x), \tilde{v}(x)$ are the exact and approximate solutions, respectively. Throughout this section, we use the symbols as follows:

$\tilde{u}_G(x)$ and $\tilde{v}_G(x)$: approximate solutions of $\tilde{u}(x)$ and $\tilde{v}(x)$ using Galerkin formulation;

$\tilde{u}_L(x)$ and $\tilde{v}_L(x)$: approximate solutions of $\tilde{u}(x)$ and $\tilde{v}(x)$ using Least square formulation;

$\tilde{u}_C(x)$ and $\tilde{v}_C(x)$: approximate solutions of $\tilde{u}(x)$ and $\tilde{v}(x)$ using Collocation formulation.

Problem 1: Consider the linear system of fractional differential equations [21]:

$$\left.\begin{aligned} D^\alpha u(x) &= u+v+\frac{120x^{5-\alpha}}{\Gamma(6-\alpha)}-x^5-x^4 \\ D^\alpha v(x) &= u+v+\frac{24x^{4-\alpha}}{\Gamma(5-\alpha)}-x^5-x^4 \end{aligned}\right\} \tag{11}$$

with the initial conditions $u(0)=v(0)=0$, $u'(0)=v'(0)=0$.

For $\alpha=2$, the exact solution is $u(x)=x^5$ and $v(x)=x^4$.

Using Galerkin formulation in (8) the approximate solutions of $\tilde{u}_G(x)$ and $\tilde{v}_G(x)$ of (11) with three modified Legendre polynomials are given by, respectively:

$$\tilde{u}_G(x) = -0.04453x+0.56018x^2-1.85860x^3+2.34296x^4,$$

$$\tilde{v}_G(x) = -0.00445x+0.01557x^2-0.02120x^3+1.01008x^4.$$

Now we evaluate the absolute errors of $\tilde{u}_G(x)$ and $\tilde{v}_G(x)$ for different values of $\alpha=0.6, 0.7, 0.8, 0.9, 1.5, 2$ for the systems of (11) which are displayed in Table 1 and Table 2, respectively.

Table 1. Absolute errors of $\tilde{u}_G(x)$ of problem 1 for $n=3$.

x	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.5$	$\alpha=2$
0.1	2.97×10^{-3}	3.50×10^{-3}	4.78×10^{-3}	8.79×10^{-3}	4.85×10^{-4}	1.92×10^{-3}
0.2	9.24×10^{-4}	1.50×10^{-3}	2.94×10^{-3}	7.50×10^{-3}	2.06×10^{-3}	7.96×10^{-4}

Continued

0.3	1.03×10^{-3}	6.18×10^{-4}	4.95×10^{-4}	4.08×10^{-3}	3.42×10^{-3}	2.79×10^{-3}
0.4	1.21×10^{-3}	9.79×10^{-4}	2.34×10^{-4}	2.24×10^{-3}	2.60×10^{-3}	2.31×10^{-3}
0.5	1.45×10^{-4}	7.56×10^{-6}	6.02×10^{-4}	2.66×10^{-3}	6.37×10^{-4}	0
0.6	5.93×10^{-4}	7.71×10^{-4}	1.56×10^{-3}	4.15×10^{-3}	6.27×10^{-4}	2.31×10^{-3}
0.7	3.97×10^{-4}	7.43×10^{-5}	1.12×10^{-3}	4.91×10^{-3}	2.87×10^{-4}	2.79×10^{-3}
0.8	3.14×10^{-3}	2.67×10^{-3}	1.13×10^{-3}	3.65×10^{-3}	3.27×10^{-3}	7.96×10^{-4}
0.9	5.11×10^{-3}	4.67×10^{-3}	3.31×10^{-3}	8.71×10^{-4}	5.46×10^{-3}	1.92×10^{-3}
L_∞	5.11×10^{-3}	4.67×10^{-3}	4.78×10^{-3}	8.79×10^{-3}	5.46×10^{-3}	2.79×10^{-3}

Table 2. Absolute errors of $\tilde{v}_G(x)$ of problem 1 for $n=3$.

x	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.5$	$\alpha=2$
0.1	3.23×10^{-3}	4.20×10^{-3}	5.71×10^{-3}	9.69×10^{-3}	3.09×10^{-4}	1.94×10^{-5}
0.2	3.09×10^{-3}	4.24×10^{-3}	6.02×10^{-3}	1.06×10^{-2}	4.21×10^{-4}	2.59×10^{-5}
0.3	1.72×10^{-3}	2.68×10^{-3}	4.15×10^{-3}	7.88×10^{-3}	4.25×10^{-4}	2.27×10^{-5}
0.4	6.04×10^{-4}	1.30×10^{-3}	2.35×10^{-3}	4.98×10^{-3}	3.89×10^{-4}	1.29×10^{-5}
0.5	5.68×10^{-4}	1.13×10^{-3}	1.96×10^{-3}	4.05×10^{-3}	3.54×10^{-4}	0
0.6	1.79×10^{-3}	2.41×10^{-3}	3.31×10^{-3}	5.73×10^{-3}	3.39×10^{-4}	1.29×10^{-5}
0.7	3.80×10^{-3}	4.63×10^{-3}	5.82×10^{-3}	9.20×10^{-3}	3.39×10^{-4}	2.27×10^{-5}
0.8	5.46×10^{-3}	6.47×10^{-3}	7.93×10^{-3}	1.21×10^{-2}	3.22×10^{-4}	2.59×10^{-5}
0.9	5.01×10^{-3}	5.87×10^{-3}	7.14×10^{-3}	1.08×10^{-2}	2.35×10^{-4}	1.94×10^{-5}
L_∞	5.46×10^{-3}	6.47×10^{-3}	7.93×10^{-3}	1.21×10^{-2}	4.25×10^{-4}	2.59×10^{-5}

From Table 1 and Table 2, we may observe that our proposed method is comparatively easier and the accuracy is reasonable.

The graphical representation of absolute errors for considering the value of fractional order $\alpha = 0.6, 0.8, 0.9, 1.5$ can be shown in Figure 1 and Figure 2. From the above discussion, it is clear that our method is more straightforward than other methods in the existing literature.

Problem 2: Consider the linear system of fractional differential equations [16]:

$$\left.\begin{aligned} D^\alpha u(x) + Dv(x) + x^2 Du(x) = f(x) \\ D^\alpha v(x) + (x-1) D^2 u(x) = g(x) \end{aligned}\right\} \tag{12}$$

with the boundary conditions $u(0)=0$, $u(1)=1$, $v(0)=v(1)=0$ and $0 \le x \le 1$, when $\alpha = 1.5$ the right-hand side function becomes

$f(x) = \frac{4}{\Gamma(0.5)}\sqrt{x} + 2x^3 + 3x^2 - 1$ and $g(x) = \frac{8}{\Gamma(0.5)} x^{\frac{3}{2}} + 2(x-1)$. The exact solutions are $u(x) = x^2$ and $v(x) = x^3 - x$.

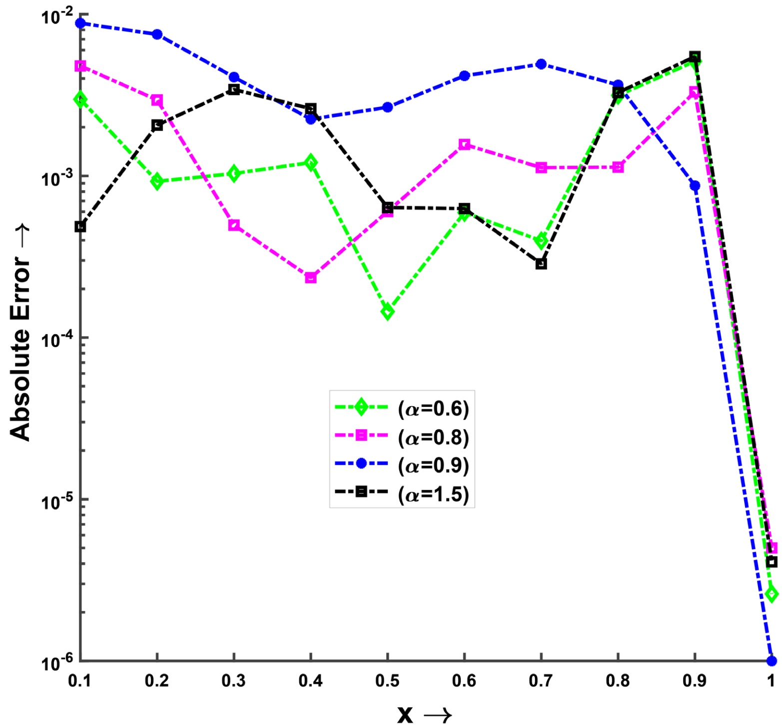

Figure 1. Absolute errors of $u(x)$ for problem 1.

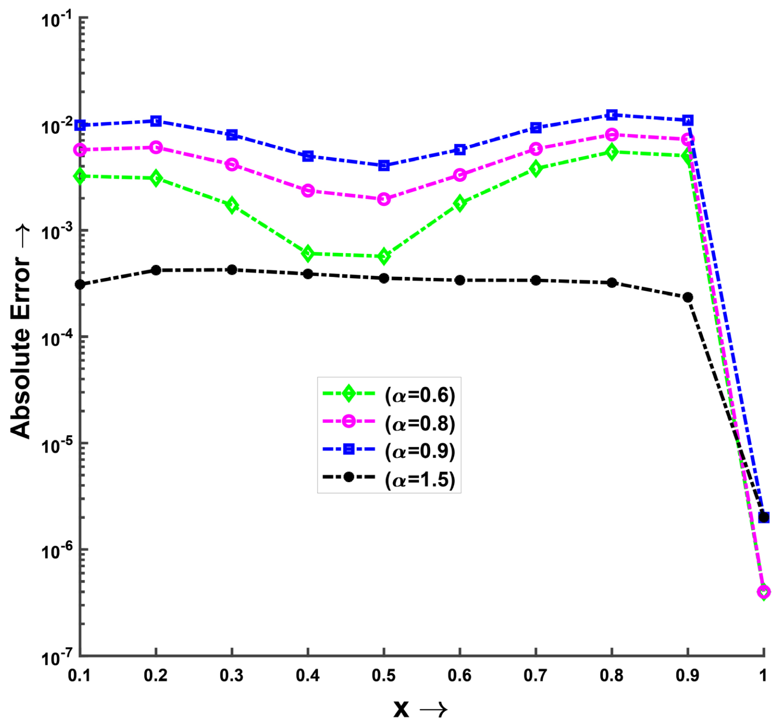

Figure 2. Absolute errors of $v(x)$ for problem 1.

For $\alpha = 1.5$, three weighted residual meth od give the approximate solution $\tilde{u}(x)$ and $\tilde{v}(x)$ of the given system (12) are as follows:

$$\tilde{u}_G(x) = 5.561\times10^{-14}x + 0.999x^2 + 4.897\times10^{-13}x^3 - 2.638\times10^{-13}x^4,$$

$$\tilde{u}_L(x) = -3.382\times10^{-15}x + 1.00x^2 - 1.214\times10^{-13}x^3 + 6.483\times10^{-14}x^4,$$

$$\tilde{u}_C(x) = -1.857\times10^{-16}x + 1.00x^2 - 1.49\times10^{-15}x^3 + 6.864\times10^{-16}x^4$$

and

$$\tilde{v}_G(x) = -0.999x - 6.947\times10^{-13}x^2 + 1.00x^3 - 6.039\times10^{-13}x^4,$$

$$\tilde{v}_L(x) = -0.999x + 3.780\times10^{-14}x^2 + 0.999x^3 + 5.839\times10^{-14}x^4,$$

$$\tilde{v}_C(x) = -1.00x + 1.896\times10^{-15}x^2 + 0.999x^3 + 1.749\times10^{-15}x^4.$$

Now we compute the approximate solutions of $\tilde{u}(x)$ and $\tilde{v}(x)$ in tabular form by three weighted residual methods: Galerkin, Least Square and Collocation, and are displayed in **Table 3** and **Table 4**, respectively. From **Table 3** and **Table 4** we may observe that the approximate solutions by the proposed three methods perform excellently, converge to the exact solutions, and coincide with the solutions of [16]. The exact and approximate solutions of the problem 2 are shown in **Figure 3**, while absolute errors are displayed in **Figure 4**.

Table 3. Absolute errors of $\tilde{u}(x)$ for the problem 2.

x	Galerkin	Least Square	Collocation
0.1	3.21×10^{-15}	1.56×10^{-16}	3.46×10^{-18}
0.2	3.36×10^{-15}	8.67×10^{-16}	6.93×10^{-18}
0.3	2.44×10^{-15}	1.62×10^{-15}	0
0.4	1.80×10^{-15}	2.13×10^{-15}	0
0.5	2.16×10^{-15}	2.16×10^{-15}	0
0.6	3.60×10^{-15}	1.72×10^{-15}	0
0.7	5.60×10^{-15}	8.88×10^{-16}	0
0.8	6.99×10^{-15}	1.11×10^{-16}	0
0.9	5.88×10^{-15}	4.44×10^{-16}	0
L_∞	6.99×10^{-15}	2.16×10^{-15}	6.93×10^{-18}

Table 4. Absolute errors of $\tilde{v}(x)$ for the problem 2.

x	Galerkin	Least Square	Collocation
0.1	7.95×10^{-15}	9.43×10^{-16}	1.38×10^{-17}
0.2	8.07×10^{-15}	2.10×10^{-15}	2.77×10^{-17}
0.3	5.21×10^{-15}	2.99×10^{-15}	1.11×10^{-16}
0.4	2.88×10^{-15}	3.60×10^{-15}	5.55×10^{-17}
0.5	2.55×10^{-15}	3.55×10^{-15}	5.55×10^{-17}
0.6	5.05×10^{-15}	2.88×10^{-15}	0
0.7	9.21×10^{-15}	1.94×10^{-15}	0
0.8	1.27×10^{-14}	8.32×10^{-16}	1.11×10^{-16}
0.9	1.13×10^{-14}	5.55×10^{-17}	2.77×10^{-17}
L_∞	1.27×10^{-14}	3.60×10^{-15}	1.11×10^{-16}

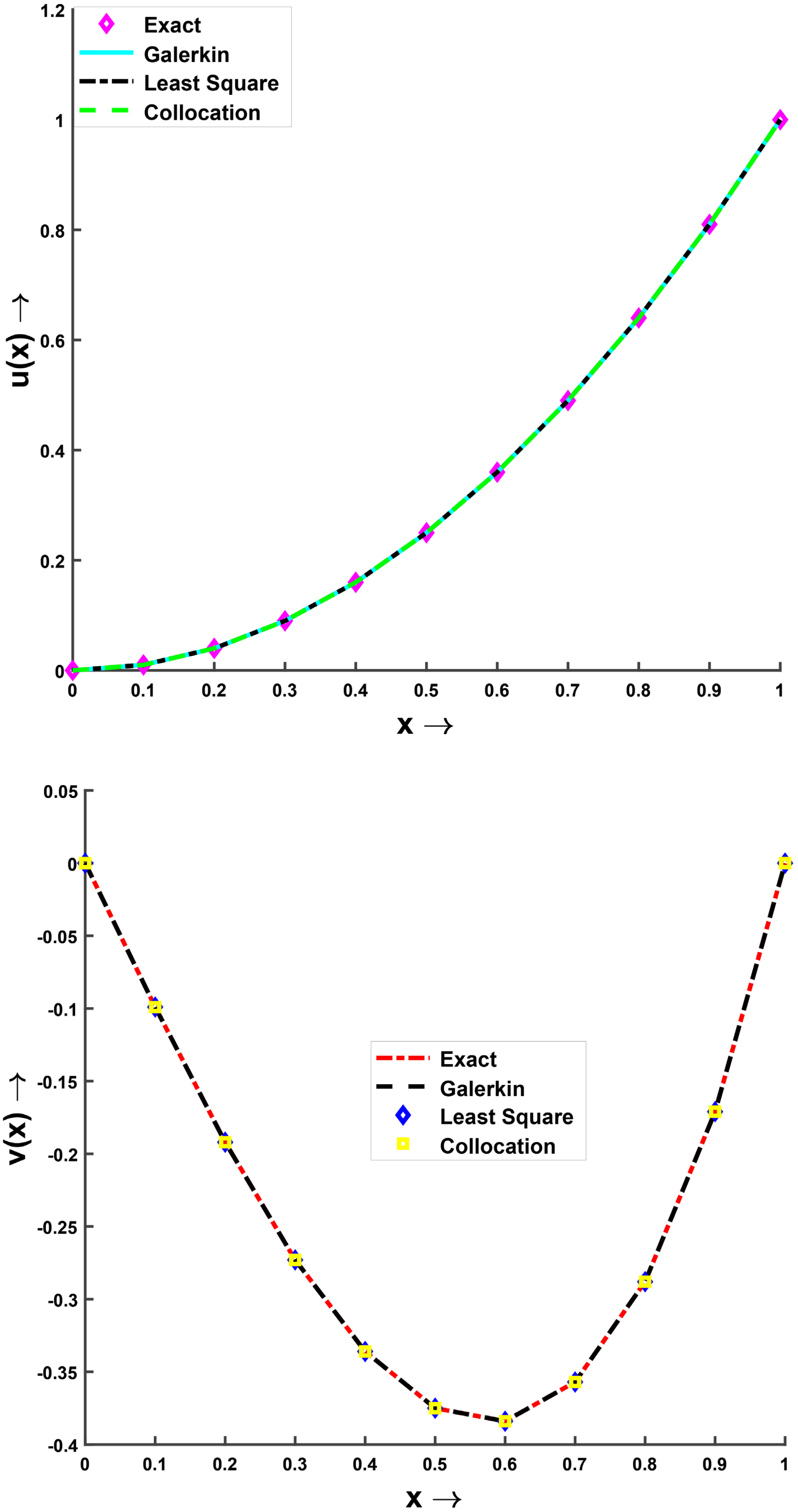

Figure 3. Exact and approximate solutions of $u(x)$ and $v(x)$ for problem 2.

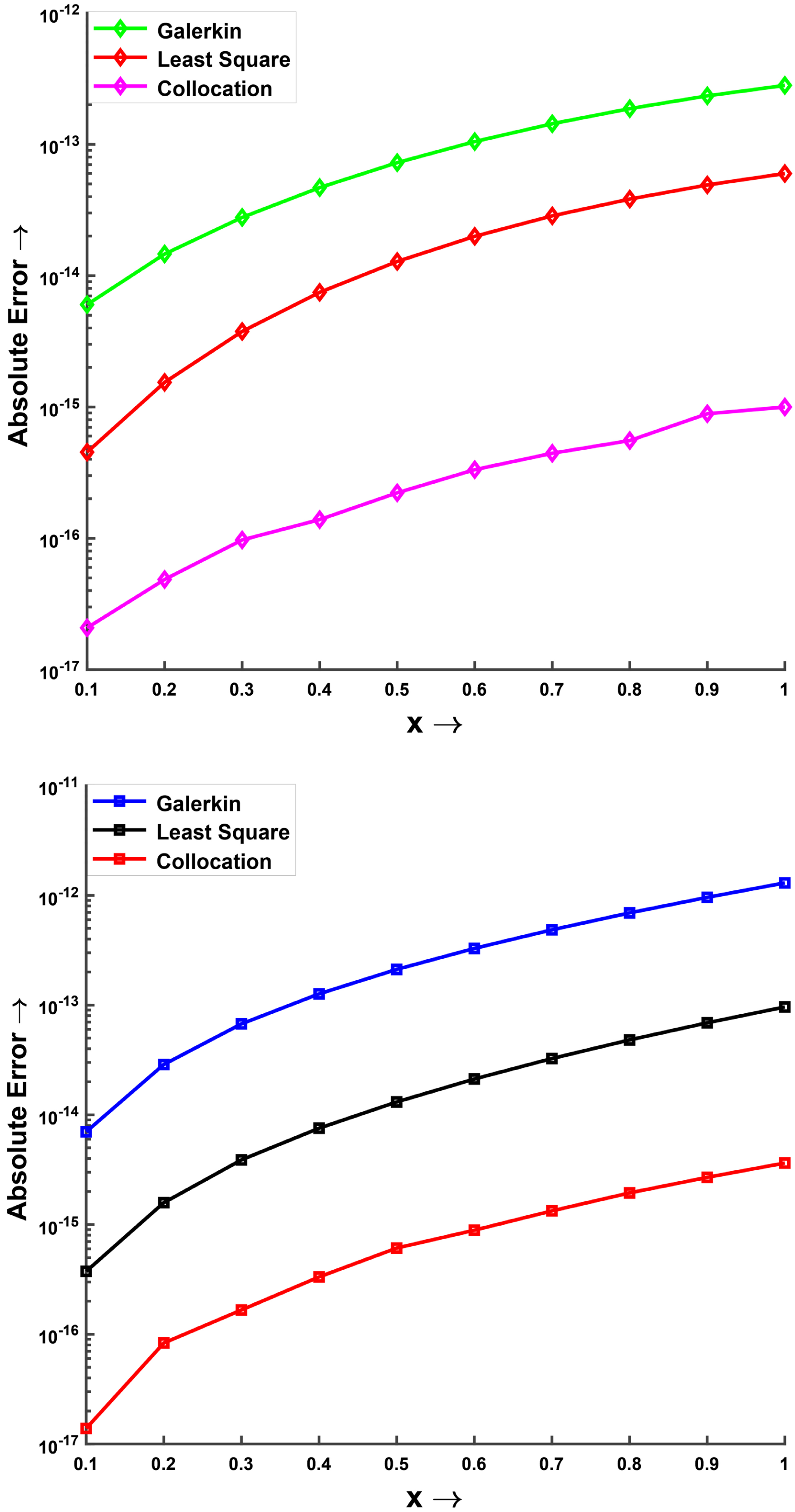

Figure 4. Absolute errors of $u(x)$ and $v(x)$ for problem 2.

We observe from the graphical representation of **Figure 3** and **Figure 4** for both $u(x)$ and $v(x)$ that the more accurate results can be obtained by Collocation method for this particular problem.

Problem 3: Consider the linear system of fractional differential equations [3] [4] [17]:

$$\left.\begin{aligned} &D^{\alpha}u(x)+(2x-1)Du(x)+\cos(\pi x)Dv(x)=f(x)\\ &D^{\beta}v(x)+xu(x)=g(x)\end{aligned}\right\} \tag{13}$$

with the boundary conditions $u(0)=0$, $u(1)=0$, $v(0)=v(1)=0$ and $0\le x\le 1$, and where,

$$\begin{aligned} f(x)=&\Bigg(\pi*\frac{0.564}{x^{0.5}}-\frac{\pi^3*4.513x^{1.5}}{6}+\frac{\pi^5*10.316x^{3.5}}{120}\\ &-\frac{\pi^7*17.506x^{5.5}}{5040}+\frac{\pi^9*25.856x^{7.5}}{362880}-\frac{\pi^{11}*35.222x^{9.5}}{39916800}\Bigg)\\ &+(2x-1)\pi\cos(\pi x)+\cos(\pi x)(2x-1)\end{aligned}$$

and

$$g(x)=2.14734x^{0.8}-\frac{0.85893}{x^{0.19999}}+x\sin(\pi x).$$

For $\alpha=\beta=2$ the system gives the exact solution $u(x)=\sin(\pi x)$ and $v(x)=x^2-x$. This problem was solved in [4] using the Galerkin method with Legendre and Bernstein polynomials. In this paper, we consider the values of $\alpha=1.5$ and $\beta=1.2$ and obtain the approximate solution $\tilde{u}(x)$ of the given system (13) by three weighted residual methods with the modified Legendre polynomials as:

$$\tilde{u}_G(x)=3.109x+0.443x^2-7.107x^3+3.555x^4,$$

$$\tilde{u}_L(x)=3.138x+0.355x^2-7.00x^3+3.506x^4,$$

$$\tilde{u}_C(x)=3.050x+0.681x^2-7.494x^3+3.763x^4.$$

Similarly, the approximate solution $\tilde{v}(x)$ are:

$$\tilde{v}_G(x)=-1.00x+1.00x^2-0.002x^3+0.001x^4,$$

$$\tilde{v}_L(x)=-0.999x+1.001x^2-0.004x^3+0.002x^4,$$

$$\tilde{v}_C(x)=-1.023x+1.069x^2-0.087x^3+0.041x^4.$$

The absolute errors are obtained for $\alpha=1.5$, $\beta=1.2$, and also for $\alpha=\beta=2$, which are displayed in **Table 5** and **Table 6**, respectively for $\tilde{u}(x)$ and $\tilde{v}(x)$. The exact and approximate solutions, and absolute errors are depicted in **Figure 5** and **Figure 6**, respectively. By comparing the results of our proposed methods with the variational iteration method [16], it is evident that our method also gives the better results. In **Table 5** and **Table 6**, we observe that Galerkin method gives the more accurate and better results than that of other residual methods and the reference results for this example.

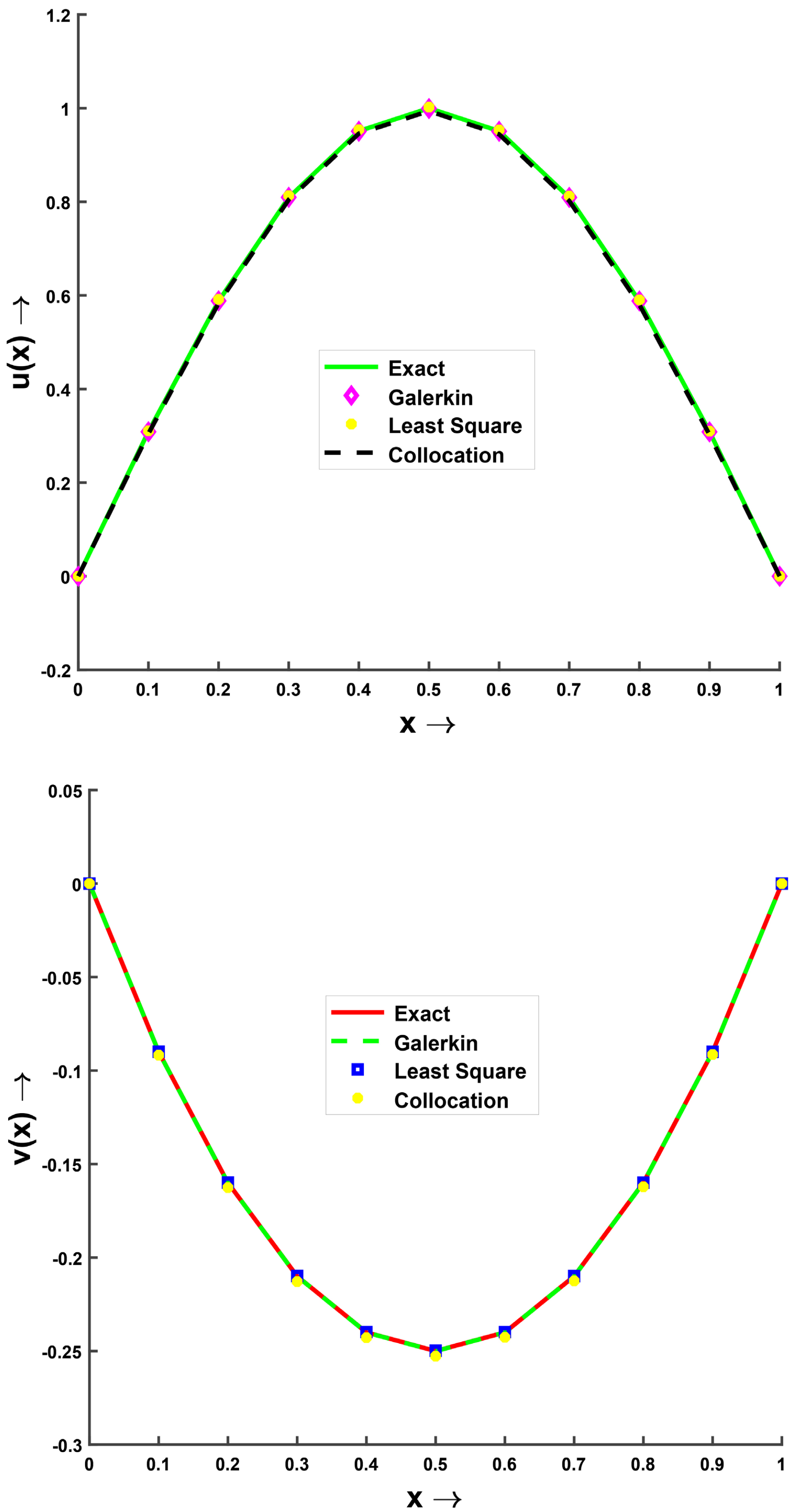

Figure 5. Exact and approximate solutions of $u(x)$ and $v(x)$ of problem 3.

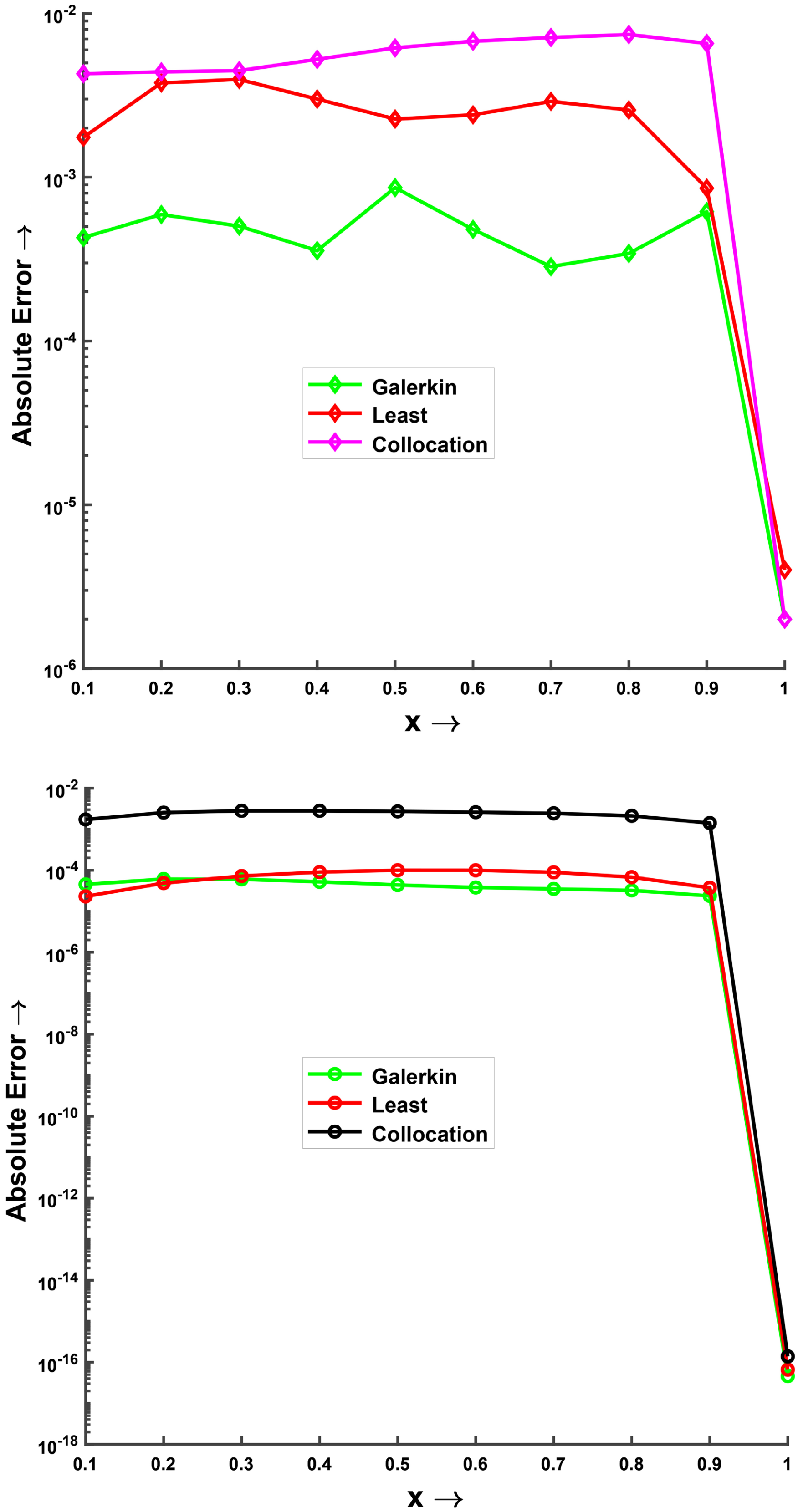

Figure 6. Absolute errors of $u(x)$ and $v(x)$ of problem 3.

Table 5. Absolute errors of $\tilde{u}(x)$ for the problem 3.

x	$\alpha=1.5, \beta=1.2$ and $n=3$			$\alpha=\beta=2$ and $n=3$			
	Galerkin	Least Square	Collocation	Galerkin	Least Square	Collocation	Reference [3]
0.1	4.28×10^{-4}	1.75×10^{-3}	4.28×10^{-3}	2.53×10^{-4}	1.13×10^{-3}	7.26×10^{-3}	3.00×10^{-4}
0.2	5.93×10^{-4}	3.77×10^{-3}	4.40×10^{-3}	7.33×10^{-4}	1.55×10^{-3}	8.11×10^{-3}	2.50×10^{-3}
0.3	5.03×10^{-4}	3.95×10^{-3}	4.47×10^{-3}	5.43×10^{-4}	6.87×10^{-5}	7.68×10^{-3}	7.80×10^{-3}
0.4	3.55×10^{-4}	3.00×10^{-3}	5.25×10^{-3}	3.83×10^{-4}	1.99×10^{-3}	7.47×10^{-3}	1.66×10^{-2}
0.5	8.62×10^{-4}	2.26×10^{-3}	6.16×10^{-3}	8.75×10^{-4}	2.91×10^{-3}	7.45×10^{-3}	2.77×10^{-2}
0.6	4.79×10^{-4}	2.40×10^{-3}	6.77×10^{-3}	3.83×10^{-4}	1.99×10^{-3}	7.46×10^{-3}	3.87×10^{-2}
0.7	2.86×10^{-4}	2.90×10^{-3}	7.14×10^{-3}	5.43×10^{-4}	6.95×10^{-5}	7.68×10^{-3}	4.59×10^{-2}
0.8	3.44×10^{-4}	2.57×10^{-3}	7.44×10^{-3}	7.33×10^{-4}	1.55×10^{-3}	8.11×10^{-3}	4.49×10^{-2}
0.9	6.14×10^{-4}	8.53×10^{-4}	6.56×10^{-3}	2.53×10^{-4}	1.14×10^{-3}	7.25×10^{-3}	3.09×10^{-2}
L_∞	8.62×10^{-4}	3.95×10^{-3}	7.44×10^{-3}	8.75×10^{-4}	2.91×10^{-3}	8.11×10^{-3}	4.59×10^{-2}

Table 6. Absolute errors of $\tilde{v}(x)$ for the problem 3.

x	$\alpha=1.5, \beta=1.2$ and $n=3$			$\alpha=\beta=2$ and $n=3$			
	Galerkin	Least Square	Collocation	Galerkin	Least Square	Collocation	Reference [4]
0.1	4.56×10^{-5}	5.98×10^{-5}	1.74×10^{-3}	3.44×10^{-7}	1.19×10^{-5}	1.31×10^{-4}	2.53×10^{-4}
0.2	6.13×10^{-5}	1.22×10^{-4}	2.56×10^{-3}	6.10×10^{-7}	2.54×10^{-5}	2.51×10^{-4}	5.42×10^{-4}
0.3	5.99×10^{-5}	1.70×10^{-4}	2.83×10^{-3}	1.71×10^{-6}	3.65×10^{-5}	3.54×10^{-4}	5.42×10^{-4}
0.4	5.15×10^{-5}	1.94×10^{-4}	2.83×10^{-3}	2.22×10^{-6}	4.26×10^{-5}	4.34×10^{-4}	3.84×10^{-4}
0.5	4.25×10^{-5}	1.89×10^{-4}	2.73×10^{-3}	1.77×10^{-6}	4.27×10^{-5}	4.84×10^{-4}	8.76×10^{-4}
0.6	3.65×10^{-5}	1.57×10^{-4}	2.62×10^{-3}	3.96×10^{-7}	3.67×10^{-5}	4.97×10^{-4}	3.83×10^{-4}
0.7	3.38×10^{-5}	1.05×10^{-4}	2.46×10^{-3}	1.48×10^{-6}	2.60×10^{-5}	4.65×10^{-4}	5.45×10^{-4}
0.8	3.14×10^{-5}	4.75×10^{-5}	2.13×10^{-3}	3.05×10^{-6}	1.35×10^{-5}	3.77×10^{-4}	7.36×10^{-4}
0.9	2.32×10^{-5}	3.78×10^{-6}	1.42×10^{-3}	3.09×10^{-6}	3.02×10^{-6}	2.26×10^{-4}	2.50×10^{-4}
L_∞	6.13×10^{-5}	1.94×10^{-4}	2.83×10^{-3}	3.09×10^{-6}	4.27×10^{-5}	4.97×10^{-4}	8.76×10^{-4}

Problem 4: Consider the nonlinear system of fractional differential equations [3] [4]

$$\left.\begin{aligned} D^{\alpha}u(x)+xDu(x)+\cos(\pi x)Dv(x)&=f(x)\\ D^{\beta}v(x)+xDu(x)+u^{2}(x)&=g(x)\end{aligned}\right\} \tag{14}$$

subject to the boundary condition

$$u(0)=u(1)=0,\ v(0)=v(1)=0 \tag{15}$$

with $\alpha=\beta=1.5$ and $0<x<1$, where

$$f(x)=\left(2.256x^{0.5}-\frac{7.221x^{2.5}}{6}+\frac{13.755x^{4.5}}{120}-\frac{21.547x^{6.5}}{5040}+\frac{30.419x^{8.5}}{362880}\right.$$
$$-\frac{40.254x^{10.5}}{39916800}-\frac{0.564}{x^{0.5}}+\frac{4.513x^{1.5}}{6}-\frac{10.316x^{3.5}}{120}+\frac{17.506x^{5.5}}{5040}$$
$$\left.-\frac{25.856x^{7.5}}{362880}+\frac{35.222x^{9.5}}{39916800}\right)+\left(x^2-x\right)\cos x+x\sin x+(1-2x)\cos(\pi x)$$

and

$$g(x)=\left(\frac{0.564}{x^{0.5}}-2.256x^{0.5}\right)+x\sin x+\left(x^2-x\right)\cos x+(x-1)^2(\sin x)^2.$$

For $\alpha=\beta=2$ the system gives the exact solution $u(x)=(x-1)\sin(x)$ and $v(x)=x-x^2$. This problem was solved in [4] using the Galerkin method with Legendre and Bernstein polynomials.

We use modified Legendre polynomials as trial approximate solution to solve the systems (14). Consider the solution of the form:

$$\left.\begin{aligned}\tilde{u}(x)&=\sum_{j=1}^{n}a_jN_j(x),\ n\ge 1\\ \tilde{v}(x)&=\sum_{j=1}^{n}b_jN_j(x),\ n\ge 1\end{aligned}\right\} \tag{16}$$

where a_j and b_j are parameter, $N_j(x)$ are polynomial functions which satisfy boundary conditions (15).

Now using the Galerkin method, we get the equation of the form:

$$\sum_{j=1}^{n}\left(\int_0^1\left[\left(N_j(x)^{\alpha}N_i(x)\right)a_j+\left(xN_j(x)N_i'(x)\right)a_j+\left(\cos(\pi x)N_j(x)N_i'(x)\right)b_j\right]\mathrm{d}x\right)$$
$$=\int_0^1 f(x)N_i(x)\mathrm{d}x$$

or

$$\sum_{j=1}^{n}\left(\int_0^1\left[\left(N_j(x)^{\beta}N_i(x)\right)b_j+\left(xN_j(x)N_i'(x)\right)a_j+\left(N_j(x)N_i^2(x)\right)a_j\right]\mathrm{d}x\right)$$
$$=\int_0^1 g(x)N_i(x)\mathrm{d}x,\ i,j=1,2,3,\cdots,n.$$

Equivalently, in matrix form:

$$\left.\begin{aligned}&\sum_{i=1}^{n}\left(A_{j,i}a_j+B_{j,i}b_j\right)=F_i\\ &\sum_{i=1}^{n}\left(C_{j,i}b_j+\left(D_{j,i}+E_{j,i}\right)a_j\right)=G_i\end{aligned}\right\} \tag{17}$$

where,

$$A_{j,i}=\int_0^1\left[\left(N_j(x)^{\alpha}N_i(x)\right)a_j+\left(xN_j(x)N_i'(x)\right)a_j\right]\mathrm{d}x$$

$$B_{j,i}=\int_0^1\left[\left(\cos(\pi x)N_j(x)N_i'(x)\right)b_j\right]\mathrm{d}x$$

$$F_i = \int_0^1 \left[\left(2.256x^{0.5} - \frac{7.221x^{2.5}}{6} + \frac{13.755x^{4.5}}{120} - \frac{21.547x^{6.5}}{5040} + \frac{30.419x^{8.5}}{362880} - \frac{40.254x^{10.5}}{39916800} - \frac{0.564}{x^{0.5}} + \frac{4.513x^{1.5}}{6} - \frac{10.316x^{3.5}}{120} + \frac{17.506x^{5.5}}{5040} - \frac{25.856x^{7.5}}{362880} + \frac{35.222x^{9.5}}{39916800} \right) + \left(x^2 - x\right)\cos x + x\sin x + (1-2x)\cos(\pi x) \right] N_i(x)\mathrm{d}x$$

$$C_{j,i} = \int_0^1 \left[\left(N_j(x)^{\beta} N_i(x) \right) b_j \right] \mathrm{d}x$$

$$D_{j,i} = \int_0^1 \left[\left(x N_j(x) N_i'(x) \right) a_j \right] \mathrm{d}x$$

$$E_{j,i} = \int_0^1 \left[\left(N_j(x) N_i^2(x) \right) a_j \right] \mathrm{d}x$$

$$G_i = \int_0^1 \left[\left(\frac{0.564}{x^{0.5}} - 2.256x^{0.5} \right) + x\sin x + \left(x^2 - x\right)\cos x + (x-1)^2 (\sin x)^2 \right] N_i(x)\mathrm{d}x$$

for all $i, j = 1, 2, 3, \cdots, n$.

Once we get the system of linear Equations (17), the parameter a_j and b_j can be then obtained easily. In this problem, we consider $\alpha = \beta = 1.5$, and obtain the approximate solutions $\tilde{u}(x)$ as:

$$\tilde{u}_G(x) = -1.00x + 1.00x^2 + 0.150x^3 - 0.156x^4,$$

$$\tilde{u}_L(x) = -1.00x + 1.00x^2 + 0.146x^3 - 0.152x^4,$$

$$\tilde{u}_C(x) = -0.999x + 1.00x^2 + 0.158x^3 - 0.160x^4.$$

Similarly, for $\alpha = \beta = 1.5$ the approximations of $\tilde{v}(x)$ are as follows:

$$\tilde{v}_G(x) = 1.00x - 1.00x^2 + 0.0001x^3 - 0.0001x^4,$$

$$\tilde{v}_L(x) = 0.999x - 1.00x^2 + 0.0005x^3 - 0.0003x^4,$$

$$\tilde{v}_C(x) = 1.00x - 1.00x^2 + 0.001x^3 - 0.0006x^4.$$

Now the graphical previews of the approximate and exact solutions are displayed in Figure 7. The absolute errors are summarized in Table 7 and Table 8, and the corresponding graphical representations are depicted in Figure 8. The absolute errors are compared with the results obtained by Variational iteration method [3]. From the tables and figures, we may emphasise that our methods perform the better accuracy.

Table 7. Absolute errors of $\tilde{u}(x)$ for the problem 4.

x	$\alpha = \beta = 1.5$ and $n = 3$			$\alpha = \beta = 2$ and $n = 3$			
	Galerkin	Least Square	Collocation	Galerkin	Least Square	Collocation	Reference [3]
0.1	6.26×10^{-6}	3.13×10^{-5}	1.04×10^{-4}	2.90×10^{-5}	4.55×10^{-5}	3.46×10^{-4}	1.42×10^{-3}
0.2	4.59×10^{-5}	9.75×10^{-5}	1.97×10^{-4}	7.53×10^{-6}	1.14×10^{-4}	4.36×10^{-4}	7.68×10^{-4}
0.3	7.04×10^{-5}	1.34×10^{-4}	2.59×10^{-4}	3.99×10^{-5}	1.43×10^{-4}	4.05×10^{-5}	6.09×10^{-4}

Continued

0.4	6.34×10^{-5}	1.16×10^{-4}	2.91×10^{-4}	3.74×10^{-5}	1.07×10^{-4}	3.49×10^{-5}	1.76×10^{-3}
0.5	3.55×10^{-5}	5.38×10^{-5}	3.09×10^{-4}	3.48×10^{-6}	2.02×10^{-5}	3.25×10^{-4}	2.16×10^{-3}
0.6	1.38×10^{-5}	1.91×10^{-5}	3.35×10^{-4}	3.56×10^{-5}	7.84×10^{-5}	3.48×10^{-5}	1.64×10^{-3}
0.7	2.43×10^{-5}	6.20×10^{-5}	3.77×10^{-4}	4.73×10^{-5}	1.40×10^{-4}	3.99×10^{-4}	4.37×10^{-4}
0.8	7.06×10^{-5}	4.92×10^{-5}	4.09×10^{-4}	1.49×10^{-5}	1.32×10^{-4}	4.22×10^{-5}	8.87×10^{-4}
0.9	1.05×10^{-4}	2.04×10^{-6}	3.44×10^{-4}	3.49×10^{-5}	5.90×10^{-5}	3.28×10^{-4}	1.43×10^{-3}
L_∞	1.05×10^{-4}	1.34×10^{-4}	4.09×10^{-4}	4.73×10^{-5}	1.43×10^{-4}	3.99×10^{-4}	2.16×10^{-3}

Table 8. Absolute errors of $\tilde{v}(x)$ for the problem 4.

x	$\alpha=\beta=1.5$ and $n=3$			$\alpha=\beta=2$ and $n=3$			
	Galerkin	Least Square	Collocation	Galerkin	Least Square	Collocation	Reference [3]
0.1	1.26×10^{-6}	2.27×10^{-6}	1.33×10^{-5}	2.49×10^{-7}	5.83×10^{-6}	1.39×10^{-4}	8.61×10^{-4}
0.2	2.28×10^{-6}	5.92×10^{-6}	1.86×10^{-5}	7.25×10^{-7}	1.24×10^{-5}	2.03×10^{-5}	6.27×10^{-4}
0.3	3.49×10^{-6}	8.94×10^{-6}	2.05×10^{-5}	1.79×10^{-6}	1.77×10^{-5}	2.07×10^{-5}	3.61×10^{-4}
0.4	5.05×10^{-6}	1.00×10^{-5}	2.21×10^{-5}	2.21×10^{-6}	2.04×10^{-5}	1.68×10^{-4}	1.76×10^{-2}
0.5	6.90×10^{-6}	8.74×10^{-6}	2.48×10^{-5}	1.64×10^{-6}	1.98×10^{-5}	1.05×10^{-4}	3.25×10^{-3}
0.6	8.70×10^{-6}	5.24×10^{-6}	2.86×10^{-5}	1.57×10^{-7}	1.62×10^{-5}	3.30×10^{-5}	4.47×10^{-2}
0.7	9.87×10^{-6}	5.19×10^{-7}	3.20×10^{-5}	1.80×10^{-6}	1.04×10^{-5}	3.05×10^{-5}	5.10×10^{-2}
0.8	9.58×10^{-6}	3.69×10^{-6}	3.18×10^{-5}	3.38×10^{-6}	4.11×10^{-6}	6.84×10^{-5}	4.79×10^{-3}
0.9	6.73×10^{-6}	4.94×10^{-6}	2.32×10^{-5}	3.32×10^{-6}	4.13×10^{-7}	6.39×10^{-5}	3.20×10^{-3}
L_∞	9.87×10^{-6}	1.00×10^{-5}	3.20×10^{-5}	3.38×10^{-6}	2.04×10^{-5}	1.68×10^{-4}	5.10×10^{-2}

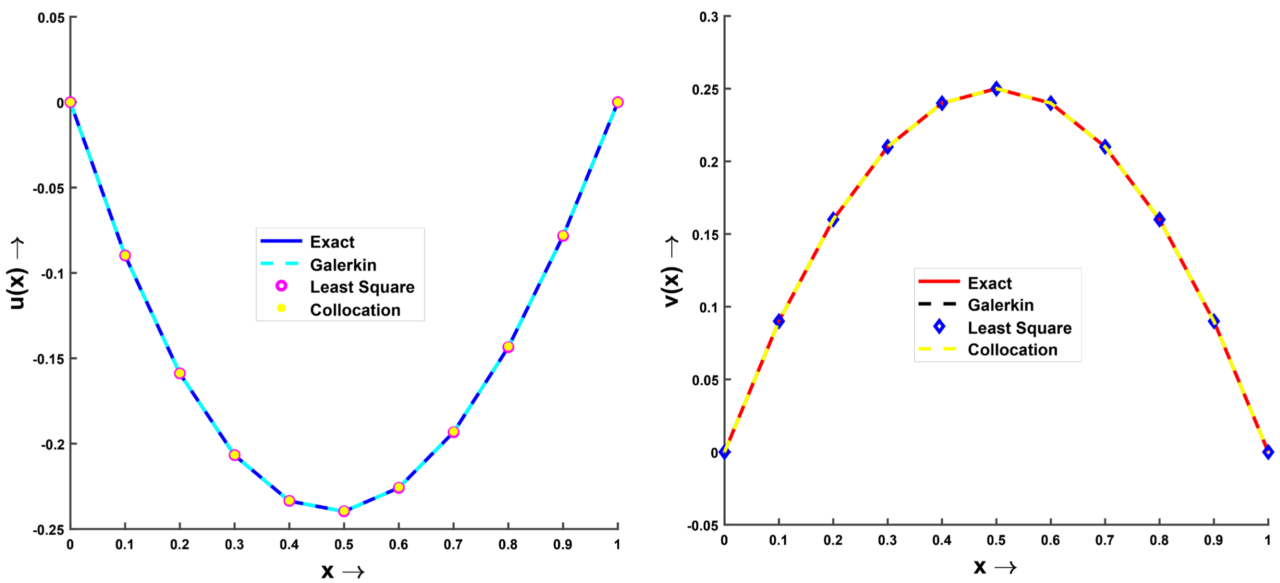

Figure 7. Exact and approximate solutions of $u(x)$ and $v(x)$ of problem 4.

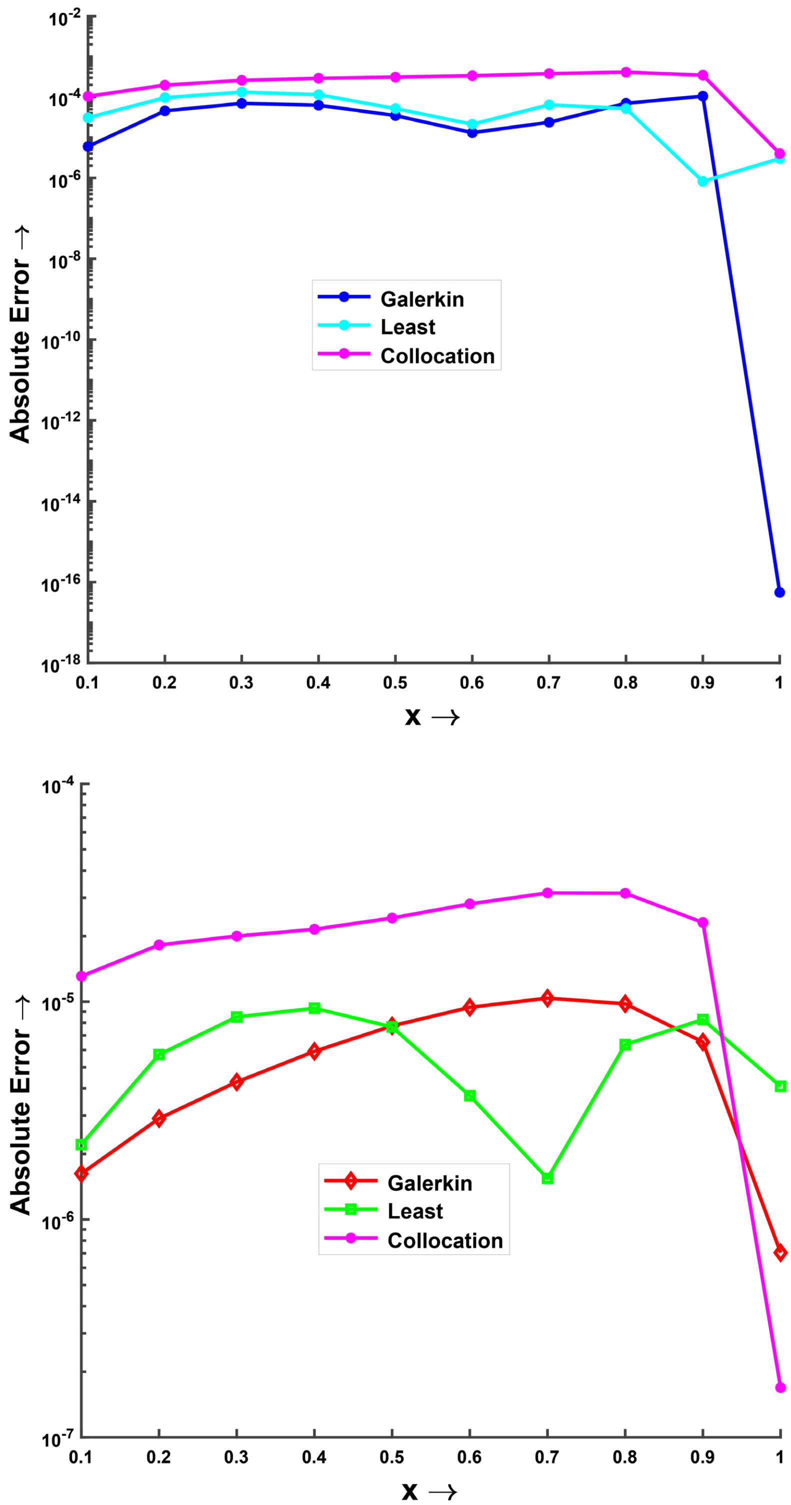

Figure 8. Absolute errors of $u(x)$ and $v(x)$ of problem 4.

5. Conclusion

We have derived three weighted residual methods in this paper, namely Galerkin, Least Square and Collocation—to solve the fractional order differential equations system. The algorithm of the rigorous matrix formulations can be coded efficiently. The approximate results converge monotonically to the exact solutions. In most cases, the three methods show their output's closeness, and provide accurate and satisfactory results. Upon using some examples and comparing the results of these methods, it is concluded that the results differ based on the order of the fractional differential equations. Finally, we may conclude that the approximate solutions of any coupled system with initial and boundary conditions can be generated by the present techniques with differentiable piecewise polynomials.

Acknowledgements

The authors are thankful to the peer reviewer for invaluable feedback and insightful suggestions, which significantly enhanced the quality of the original manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Cheng, X. and Zhong, C. (2005) Existence of Positive Solutions for a Second-Order Ordinary Differential System. *Journal of Mathematical Analysis and Applications*, **312**, 14-23. https://doi.org/10.1016/j.jmaa.2005.03.016

[2] Geng, F. and Cui, M. (2007) Solving a Nonlinear System of Second Order Boundary Value Problems. *Journal of Mathematical Analysis and Applications*, **327**, 1167-1181. https://doi.org/10.1016/j.jmaa.2006.05.011

[3] Lu, J. (2007) Variational Iteration Method for Solving a Nonlinear System of Second-Order Boundary Value Problems. *Computers & Mathematics with Applications*, **54**, 1133-1138. https://doi.org/10.1016/j.camwa.2006.12.060

[4] Rupa, M.J. and Islam, M.S. (2018) Numerical Solutions of System of Second Order Boundary Value Problems Using Galerkin Method. *Journal of Bangladesh Mathematical Society*, **37**, 161-174. https://doi.org/10.3329/ganit.v37i0.35734

[5] Podlubny, I. (1999) Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press.

[6] Ruman, U. and Shafiqul Islam, M. (2020) Numerical Solutions of Linear Fractional Order BVP by Galerkin Residual Method with Differentiable Polynomials. *Applied and Computational Mathematics*, **9**, 20-25. https://doi.org/10.11648/j.acm.20200902.11

[7] Khader, M.M., El Danaf, T.S. and Hendy, A.S. (2013) A Computational Matrix Method for Solving Systems of High Order Fractional Differential Equations. *Applied Mathematical Modelling*, **37**, 4035-4050. https://doi.org/10.1016/j.apm.2012.08.009

[8] Khader, M.M., Sweilam, N.H. and Mahdy, A.M.S. (2015) Two Computational Algorithms for the Numerical Solution for System of Fractional Differential Equations. *Arab Journal of Mathematical Sciences*, **21**, 39-52. https://doi.org/10.1016/j.ajmsc.2013.12.001

[9] Daftardar-Gejji, V. and Babakhani, A. (2004) Analysis of a System of Fractional Differential Equations. *Journal of Mathematical Analysis and Applications*, **293**, 511-

522. https://doi.org/10.1016/j.jmaa.2004.01.013

[10] Odibat, Z.M. (2010) Analytic Study on Linear Systems of Fractional Differential Equations. *Computers & Mathematics with Applications*, **59**, 1171-1183. https://doi.org/10.1016/j.camwa.2009.06.035

[11] Bonilla, B., Rivero, M. and Trujillo, J.J. (2007) On Systems of Linear Fractional Differential Equations with Constant Coefficients. *Applied Mathematics and Computation*, **187**, 68-78. https://doi.org/10.1016/j.amc.2006.08.104

[12] Zurigat, M., Momani, S., Odibat, Z. and Alawneh, A. (2010) The Homotopy Analysis Method for Handling Systems of Fractional Differential Equations. *Applied Mathematical Modelling*, **34**, 24-35. https://doi.org/10.1016/j.apm.2009.03.024

[13] Bhrawy, A.H. and Zaky, M.A. (2016) Shifted Fractional-Order Jacobi Orthogonal Functions: Application to a System of Fractional Differential Equations. *Applied Mathematical Modelling*, **40**, 832-845. https://doi.org/10.1016/j.apm.2015.06.012

[14] Daftardar-Gejji, V. and Jafari, H. (2005) Adomian Decomposition: A Tool for Solving a System of Fractional Differential Equations. *Journal of Mathematical Analysis and Applications*, **301**, 508-518. https://doi.org/10.1016/j.jmaa.2004.07.039

[15] Ertürk, V.S. and Momani, S. (2008) Solving Systems of Fractional Differential Equations Using Differential Transform Method. *Journal of Computational and Applied Mathematics*, **215**, 142-151. https://doi.org/10.1016/j.cam.2007.03.029

[16] Abdeljawad, T., Amin, R., Shah, K., Al-Mdallal, Q. and Jarad, F. (2020) Efficient Sustainable Algorithm for Numerical Solutions of Systems of Fractional Order Differential Equations by Haar Wavelet Collocation Method. *Alexandria Engineering Journal*, **59**, 2391-2400. https://doi.org/10.1016/j.aej.2020.02.035

[17] Azizi, H. (2015) Chebyshev Finite Difference Method for Fractional Boundary Value Problems, *Journal of Mathematical Extension*, **9**, 57-71.

[18] Su, X. (2009) Boundary Value Problem for a Coupled System of Nonlinear Fractional Differential Equations. *Applied Mathematics Letters*, **22**, 64-69. https://doi.org/10.1016/j.aml.2008.03.001

[19] Alipour, M. and Baleanu, D. (2013) Approximate Analytical Solution for Nonlinear System of Fractional Differential Equations by Bps Operational Matrices. *Advances in Mathematical Physics*, **2013**, 1-9. https://doi.org/10.1155/2013/954015

[20] Dhaigude, D.B., Jadhav, N.B. and Nanware, J.A. (2018) Method of Upper Lower Solutions for Nonlinear System of Fractional Differential Equations and Applications. *Malaya Journal of Matematik*, **6**, 467-472. https://doi.org/10.26637/mjm0603/0001

[21] Shah, K., Abdalla, B., Abdeljawad, T. and Suwan, I. (2023) An Efficient Matrix Method for Coupled Systems of Variable Fractional Order Differential Equations. *Thermal Science*, **27**, 195-210. https://doi.org/10.2298/tsci23s1195s

[22] Ziada, E.A.A. (2021) Analytical Solution of Nonlinear System of Fractional Differential Equations. *Journal of Applied Mathematics and Physics*, **9**, 2544-2557. https://doi.org/10.4236/jamp.2021.910164

[23] Wang, X., Long, S. and Liu, A. (2022) Oscillation Theorems for Two Classes of Fractional Neutral Differential Equations. *Journal of Applied Mathematics and Physics*, **10**, 3037-3052. https://doi.org/10.4236/jamp.2022.1010203

[24] Mu'lla, M.A.M. (2020) Fractional Calculus, Fractional Differential Equations and Applications. *Open Access Library Journal*, **7**, e6244.